Integral expressions for Hilbert–type infinite multilinear form and related multiple Hurwitz–Lerch Zeta functions

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Abstract
The article deals with different kinds integral expressions concerning multiple Hurwitz–Lerch Zeta function (introduced originally by Barnes [4]), Hilbert–type infinite multilinear form and its power series extension. Here Laplace integral forms and multiple Mellin–Barnes type integral representation are derived for these special functions. As a special cases of our investigations we deduce the integral expressions for the Matsumoto’s multiple Mordell–Tornheim Zeta function, that is, for Tornheim’s double sum i.e. Mordell–Witten Zeta, for the multiple Hurwitz Zeta and for the multiple Hurwitz–Euler Eta function, recently studied by Choi and Srivastava [7].

Keywords: Multiple Hurwitz–Lerch Zeta function; Hilbert–type infinite multilinear form; Multiple Hurwitz–Lerch Zeta power series; Tornheim’s double sum; Mordell–Witten Zeta function; Matsumoto’s multiple Mordell–Tornheim Zeta function; Dirichlet–series; Cahen’s Laplace integral formula; Mellin–Barnes type integral.

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1 Introduction

The general Hurwitz–Lerch Zeta function (HLZ), denoted by \( \Phi(z, s, a) \), is defined as \([10]\), also see \([22]\)
\[
\Phi(z, s, a) = \sum_{n \geq 0} \frac{z^n}{(n + a)^s}, \tag{1.1}
\]
where \( a \in \mathbb{C} \setminus \mathbb{Z} \setminus \{0\} \); \( \Re(s) > 1 \) when \( |z| = 1 \) and \( s \in \mathbb{C} \) when \( |z| < 1 \) and continues meromorphically to the complex \( s \)–plane, except for the simple pole at \( s = 1 \), with its residue equal to 1.

The function \( \Phi(z, s, a) \) has many special cases such as Riemann Zeta \([10]\), Hurwitz Zeta \([22]\) and Lerch Zeta function \([28]\). Some other special cases involve the polylogarithm (or Jonqui`ere’s function) \( \text{Li}_s(z) = z \Phi(z, s, 1) \) \([28]\), and the Lipschitz–Lerch Zeta function \( \Phi(e^{2\pi i \xi}, s, 1) \) \([22]\). A multiple HLZ is studied by Choi et al. \([8]\). Other generalizations of HLZ are discussed by Lin–Srivastava \([15]\), Garg–Jain–Kalla \([12, 13]\), Lin–Srivastava–Wang \([16]\), Goyal–Laddha \([14]\), Srivastava–Saxena–Pogá`ny–Saxena \([23]\) and others. The generalized \( d \)–ple Hurwitz Zeta function \( \zeta_d(s; a|z_1, \ldots, z_d) \) has been introduced and studied by Barnes \([4]\) (see also the earlier papers \([1, 2, 3]\)), for \( \Re(s) > d \); \( d \in \mathbb{N} \), by the multiple series
\[
\zeta_d(s; a|\nu) := \sum_{n \in \mathbb{N}_0^d} \frac{(-1)^{n \circ 1}}{(a + n \circ \nu)^s};
\]
here and in what follows \( x = (x_1, \ldots, x_d) \), \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\} \), and \( x \circ y = \sum_{j=1}^d x_j y_j \) stands for the inner product of \( d \)–dimensional vectors \( x, y \). Choi and Srivastava devoted their article \([7]\) to different type integral representations either for \( \zeta_d(s; a|1) \) or to the so–called alternating multiple generalized Hurwitz Zeta (or, equivalently Hurwitz–Euler Eta function)
\[
\eta_d(s, a) := \sum_{n \in \mathbb{N}_0^d} \frac{(-1)^{n \circ 1}}{(a + n \circ 1)^s};
\]
where \( \Re(s) > 0; a > 0; d \in \mathbb{N} \) \([7]\). Obviously \( \eta_1(s, 1) \equiv \eta(s) \) is the Dirichlet Eta function. Inspiring by \( \eta_d(s, a) \) we will consider the extended variant of Hurwitz–Euler Eta (at the same time the alternating variant of \( \zeta_d \)), defined by
\[
\eta_d(s; a|\nu) := \sum_{n \in \mathbb{N}_0^d} \frac{(-1)^{n \circ 1}}{(a + n \circ \nu)^s}, \quad \eta_d(s; a|1) \equiv \eta_d(s, a).
\]
Another kind of special function closely connected to HLZ and/or their expansions is the Hilbert–type infinite bilinear form \([19]\)
\[
\mathcal{B}_{\lambda, \rho}^{a,b}(s) := \sum_{m,n \geq 1} \frac{a_m b_n}{(\lambda_n + \rho_m)^s},
\]
to derive deeper general Hilbert’s double series theorems with respect to the non–homogeneous kernel \( (\lambda_n + \rho_m)^{-s} \). Here \( \lambda|_\mathbb{N} = \lambda = (\lambda_n)_{n \geq 1}, \rho|_\mathbb{N} = \rho = (\rho_n)_{n \geq 1} \) are restrictions coming from monotone increasing functions \( \lambda, \rho: \mathbb{R}_+ \mapsto \mathbb{R}_+ \) which behave
\[
\lim_{x \to \infty} \left\{ \begin{array}{c} \lambda \\ \rho \end{array} \right\}(x) = \infty. \tag{1.2}
\]
Also, Hilbert–type infinite $d$–linear form $\mathcal{H}(a; s; \alpha, \lambda)$ was introduced by Draščić Ban–Pečarić–Perić–Pogány [9], by the following $d$–ple series:

$$\mathcal{H}(a; s; \alpha, \lambda) := \sum_{n \in \mathbb{N}^d} \frac{1}{\prod_{j=1}^{d} (a + \lambda_i(n_1) + \cdots + \lambda_d(n_d))^s},$$

(1.3)

for all $a, s > 0, d \in \mathbb{N}_2 = \{2, 3, \ldots\}$. Here $\alpha_j = (\alpha_j(n_j)), n_j \geq 1, j = \overline{1, d}$ are nonnegative sequences and $\lambda_1, \ldots, \lambda_d: \mathbb{R}_+ \to \mathbb{R}_+$ are monotone increasing positive functions possessing property (1.2). Let us remark that the main authors’ goal in [9] was to derive a sharp discrete multiple Hilbert–type inequalities for (1.3), by finding sharp upper bounds for the related Laplace–integral expressions obtained for $\mathcal{H}(a; s; \alpha, \lambda)$.

It is straightforward that $\mathcal{H}(a; s; \alpha, \lambda)$ significantly generalizes $\zeta_d(s; a|\nu)$, as

$$\mathcal{H}(a; s; 1, I \circ \nu) + a^{-s} = \zeta_d(s; a|\nu);$$

(1.4)

here $I$ denotes the identity transformation applied to summation indices $d$–ple sequence $n$. On the other hand integral representations achieved for $\mathcal{H}(a; s; 1, I \circ \nu)$ in [19] and [9] can be easily extended to less restrictive real sequences $\alpha_j, j = \overline{1, d}$ covered by this Hurwitz–Euler Eta function as well:

$$\mathcal{H}(a; s; -1, I \circ \nu) + a^{-s} = \eta_d(s; a|\nu),$$

(1.5)

writing $(-1) := (-1)^{n_01}$.

In 1950 Tornheim [25] introduced the double series

$$T(p, q, r) = \sum_{n \in \mathbb{N}^2} \frac{1}{n_1^p n_2^q (n_1 + n_2)^r} \quad p, q, r \geq 0, r + \min\{p, q\} > 1,$$

In honour to the author it is called Tornheim double series (also known in literature as Witten Zeta function or Mordell series). Later, various continuations of the domain for the function $T(p, q, r)$ are given [5, 11, 24, 26, 27]. We point out that

$$T(p, q, r) = \mathcal{H}(0, r; N^{-p}N^{-q}, I \circ 1),$$

where $N^{-p}N^{-q}$ denotes the coefficients sequence $(n_1^{-p}n_2^{-q}), n_1, n_2 \geq 1$. Matsumoto [17] introduced the related, so–called Mordell-Tornheim $d$–ple Zeta function in the form

$$\zeta_{MT,d}(r; s) = \sum_{n \in \mathbb{N}^d} \frac{1}{n_1^{r_1} \cdots n_d^{r_d}(n_1 + \cdots + n_d)^s},$$

where $r := (r_1, \ldots, r_d); r_j, s \in \mathbb{C}$. The series (1.5) is absolutely convergent for $\Re(r_j) > 1, j = \overline{1, d}$ and $\Re(s) > 0$. It is obvious that by taking $d = 2$ and real parameters (1.5) it reduces to (1.4). Convergence conditions for $\zeta_{MT,d}(r; s)$ are prescribed in [17]. It is interesting to mention that

$$\zeta_{MT,d}(r; s) = \mathcal{H}(0, s; N^{-r}, I \circ 1),$$

(1.6)

where $N^{-r} := (n_1^{-r_1} \cdots n_d^{-r_d}), n_j \geq 1, j = \overline{1, d}$. 
Obviously, the most general special function here considered is the Hilbert–type infinite \(d\)–linear form \(\mathcal{H}_d(a, s; \alpha, \lambda)\) defined by (1.3).

Finally, it is known that [10]

\[
\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{t^{s-1}e^{-(a-1)t}}{e^t - z} \, dt.
\]

Another integral representation results are derived by Draščić Ban–Pečarić–Perić–Pogány [9]

\[
\zeta_{MT,d}(r; s) = \frac{1}{2^d \Gamma(s)} \prod_{j=1}^{d} \Gamma(r_j) \int_{\mathbb{R}_+^d} \frac{x^{s-1}t_1^{s-1}t_2^{s-1} \cdots t_d^{s-1}}{\sinh \frac{x+1}{2} \sinh \frac{x+i}{2}} \, dx \, dt,
\]

where \(r_1 \geq 0, \ldots, r_d \geq 0, \min_{1 \leq j \leq d}(r_j) + d(s - 1) > 0\) and \(dt = \prod_{j=1}^{d} dt_j\). In the case of Mordell–Tornheim Zeta function this result reduces to [9]

\[
T(p, q, r) = \frac{1}{4 \Gamma(p) \Gamma(q) \Gamma(r)} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} x^{p-1}y^{q-1}z^{r-1} \frac{e^{-(1+i)t}e^{-(1+i)t}}{\sinh \frac{x+1}{2} \sinh \frac{x+i}{2}} \, dx \, dt \, dt_2.
\]

Laplace integral representation formulae, established by the associated Dirichlet series’ have been presented also in [9] (see also [19] for the case \(d = 2, a = 0\)).

At this point let us introduce the \(d\)–variate Hilbert type infinite \(d\)–linear form function

\[
e_{H_d}(a; s; |z|) := \sum_{n \in \mathbb{N}^d} \prod_{j=1}^{d} \frac{\alpha_j(\lambda_j^n)}{(a + \lambda_j(1) + \cdots + \lambda_d(n_d))^s}.
\]

(1.7)

Clearly \(\widetilde{\mathcal{H}}_d(a, s; \alpha, \lambda | z) \equiv \mathcal{H}_d(a, s; \alpha, \lambda)\). So, the associated Zeta and Eta functions become

\[
\zeta_d(s; a | \nu, z) := \sum_{n \in \mathbb{N}^d} \frac{\prod_{j=1}^{d} \nu_j^n}{(a + n \circ \nu)^s}; \quad \eta_d(s; a | \nu, z) := \sum_{n \in \mathbb{N}^d} \frac{\prod_{j=1}^{d} (-\nu_j^n)}{(a + n \circ \nu)^s}.
\]

(1.8)

In this research article we establish various integral representation formulae for this type of multiple series and some of its special cases as \(\zeta_d\) and \(\eta_d\): 1. multiple Laplace–integral expression and 2. Mellin–Barnes type of integral formulae. Finally, multidimensional Mellin transforms of \(\zeta_d\) and \(\eta_d\) are also discussed.

## 2 Laplace–integral representation formula for \(\widetilde{\mathcal{H}}_d(s, a; \alpha, \lambda | z)\)

In the beginning, we recall the widely known Gamma function property

\[
\int_{\mathbb{R}^+} x^{s-1}e^{-Ax} \, dx = \frac{\Gamma(s)}{A^s}, \quad \min\{\Re(A), \Re(s)\} > 0.
\]

(2.9)
Theorem 2.1. Let monotone \( \lambda = (\lambda_1, \cdots, \lambda_d) : \mathbb{R}^d_+ \to \mathbb{R}^d_+ \) have property (1.2) and possess restrictions \( \lambda_j \big|_{\mathbb{N}} = (\lambda_j(n_j))_{n_j \geq 1}, j = \overline{1,d} \). Parameters \( a > 0, s > 0, \nu = (\nu_1, \cdots, \nu_d) > 0 \) and variable \( z = (z_1, \cdots, z_d) \in \mathbb{C}^d \) satisfy convergence condition

\[
\lim_{m \to \infty} \sup_{\lambda_j} \left| \frac{\alpha_j(m)^{1/m} z_j^{\lambda_j(m)/m}}{\lambda_j(m)^{s/(dm)}} \right| < 1, \quad j = \overline{1,d}.
\] (2.10)

Then we have the Laplace–integral representation formula

\[
\tilde{H}_d(a, s; \alpha, \lambda | z) = \int_{\mathbb{R}^d_+} \left\{ \prod_{j=1}^d \frac{|\lambda_j^{-1}(t_j)|}{(a + \lambda_j(n_j))^{s/d}} \right\} \frac{dt}{(a + t \circ 1)^{s+d}},
\] (2.11)

where \([x]\) denotes the integer part of some \( x \) and \( \lambda^{-1} \) stands for the inverse of \( \lambda \).

Proof. First, we establish the convergence of the Hilbert’s \( d \)-linear form (1.3). By virtue of the arithmetic mean–geometric mean (A–G) inequality applied to the positive denominator of \( \tilde{H}_d \), we obtain

\[
|\tilde{H}_d(a, s; \alpha, \lambda | z)| \leq \prod_{j=1}^d \sum_{n_j \geq 1} \left| \frac{\alpha_j(n_j)|z_j|^{\lambda_j(n_j)}}{(a + \lambda_j(n_j))^{s/d}} \right| \sim \prod_{j=1}^d \sum_{n_j \geq 1} \frac{\alpha_j(n_j)|z_j|^{\lambda_j(n_j)}}{\lambda_j(n_j)^{s/d}}.
\]

by virtue of the Cauchy’s convergence test \( \tilde{H}_d \) absolutely converges.

On the other hand let us express the denominator using the Gamma function formula (2.9):

\[
\tilde{H}_d(a, s; \alpha, \lambda | z) = \frac{1}{\Gamma(s)} \int_{\mathbb{R}^d_+} x^{s-1} e^{-ax} \prod_{j=1}^d D_j(z_j, x) \, dx,
\] (2.12)

where the Dirichlet series

\[ D_j(z_j, x) := \sum_{n_j \geq 1} \alpha_j(n_j) z_j^{\lambda_j(n_j)} e^{-\lambda_j(n_j)x} \]

has the Cahen’s Laplace integral representation formula [6], which has been exactly proved by Perron [18] (also see e.g. [21]):

\[ D_j(z_j, x) = x \int_{\mathbb{R}^d_+} e^{-xt_j} \left\{ \sum_{n: \lambda_j(n) \leq t_j} \alpha_j(n) z_j^{\lambda_j(n)} \right\} \, dt_j \], \quad j = \overline{1,d}.

As \( \lambda_j \) is monotone increasing, it possesses an unique inverse \( \lambda_j^{-1} \), so \( 1 \leq n \leq [\lambda_j^{-1}(t_j)] \). Substituting the previous Cahen’s integrals into (2.12):

\[
\tilde{H}_d(s, a; \alpha, \lambda | z) = \int_{\mathbb{R}^d_+} x^{s+d-1} e^{-(a + t \circ 1)x} \left\{ \prod_{j=1}^d \sum_{n_j=1}^{[\lambda_j^{-1}(t_j)]} \alpha_j(n_j) z_j^{\lambda_j(n_j)} \right\} \, dx \, dt \.
\]

Calculating the inner \( x \)-integral by (2.9) with \( A = a + t \circ 1 \), we complete the proof. ☐

Remark 2.1. The second author studied Hilbert’s bilinear form \( \tilde{H}_2(0, s; 1, \lambda | (x, x)) \) in deriving new Hilbert-type double series theorems [20].
Theorem 2.2. Assume that $a > 0, \nu > 0; d \in \mathbb{N}_2$ and $\Re(s) > d$ when $|z_j| = 1$, while $s \in \mathbb{C}$ when $|z_j|^{\nu_j} < 1, j = 1, d$. Then for all $|z_j| < 1, j = 1, d$ we have

$$
\zeta_d(s; a|\nu, z) = a^{-s} + \prod_{j=1}^d z_j^{\nu_j} \frac{\prod_{j=1}^d (z_j^{\nu_j t_j/\nu_j} - 1)}{(a + t \circ 1)^{s+d}} \mathrm{d}t
$$
(2.13)

$$
\tilde{\eta}_d(s; a|\nu, z) = a^{-s} + \prod_{j=1}^d \frac{z_j^{\nu_j} + 1}{(a + t \circ 1)^{s+d}} \mathrm{d}t.
$$
(2.14)

Proof. By applying the A–G inequality to $\tilde{\zeta}_d$ we deduce the inequality

$$
|\tilde{\zeta}_d(s; a|\nu, z)| \leq d^{-s} \prod_{j=1}^d \nu_j^{-s/d} \sum_{n \in \mathbb{N}_d} \prod_{j=1}^d |z_j|^{\nu_j n_j} \left( \frac{a}{\nu_j} + n_j \right)^{s/d}
$$

$$
= \left\{ d^{-s} \prod_{j=1}^d \nu_j^{1/d} \right\}^{s/d} \prod_{j=1}^d \Phi(|z_j|^{\nu_j}, s \frac{a}{d'}, d). \quad \Phi(x, s) = \frac{x^s}{(1 + x)^s}\frac{d}{ds} \frac{d}{ds}
$$

The resulting HLZ functions converge for $a/(d \cdot \nu_j) \notin \mathbb{Z}_0; \Re(s) > d$ when $|z_j| = 1$ and $s \in \mathbb{C}$ when $|z_j|^{\nu_j} < 1, j = 1, d$, (see(1.1)). However, these constraints are guaranteed by the assumptions of the Theorem.

Next, having in mind (1.4), (1.8) and (2.11), and because $\lambda_j^{-1}(x) = x/\nu_j, j = 2, d$, we get

$$
\tilde{\zeta}_d(s; a|\nu, z) = a^{-s} + \tilde{\zeta}_d(a, s; 1, 1 \circ \nu \mid z) = a^{-s} + \int_{\mathbb{R}_+^d} \frac{\prod_{j=1}^d |z_j|^{\nu_j n_j}}{(a + t \circ 1)^{s+d}} \mathrm{d}t,
$$
(2.15)

which is equivalent to the first assertion (2.13) of the Theorem 2.2. Relation (2.14) we prove in completely similar manner.

By taking $z_j^{\nu_j} = 1, j = 1, d$ in Theorem 2.2, we deduce from (2.15) the following.

Corollary 2.1. Let $a > 0, s \in \mathbb{C}, \Re(s) > d$ and $\nu > 0$. Then it holds true

$$
\zeta_d(s; a|\nu) = a^{-s} + \int_{\mathbb{R}_+^d} \frac{\prod_{j=1}^d \left| t_j \right|}{\nu_j} \frac{\mathrm{d}t}{(a + t \circ 1)^{s+d}}
$$
(2.16)

$$
\eta_d(s; a|\nu) = a^{-s} + \int_{\mathbb{R}_+^d} \frac{\prod_{j=1}^d \sin^2 \left( \frac{\pi}{2} \left| t_j \right| \nu_j \right)}{(a + t \circ 1)^{s+d}} \frac{\mathrm{d}t}{(a + t \circ 1)^{s+d}}
$$
(2.17)

Proof. Relation (2.16) is straightforward. Since

$$
\sum_{n_j=1}^{\left| t_j/\nu_j \right|} (-1)^{n_j} = \frac{1}{2} \left( 1 - (-1)^{\left| t_j/\nu_j \right|} \right) = \sin^2 \left( \frac{\pi}{2} \left| t_j \right| \nu_j \right)
$$
(2.17) follows too.

\( \Box \)

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Let us establish the convergence domain of $1$

Proof. The resulting polylogarithmic/de Jonqui`ere’s functions converge according to the theorem’s associated parameter space, that is for

Theorem 2.3. Let $d \in \mathbb{N}_2, r_j > 0, s \in \mathbb{C}$ when $|z_j| < 1, j = \frac{1}{\nu}, d$ and $r_j + \Re(s)/d > 1$ when $|z_j| = 1, j = \frac{1}{\nu}, d$. Then we have

$$\tilde{\zeta}_{MT,d}(r; s \mid z) = \prod_{j=1}^{d} \frac{z_j}{\Gamma(r_j)} \cdot \int_{\mathbb{R}^d_+} \prod_{j=1}^{d} x_j^{r_j-1} \left\{ 1 - (e^{-x_j z_j})^{t_j} \right\} \frac{dt \, dx}{(t \circ 1)^{s+d}}. \quad (2.18)$$

Proof. First of all, let us establish the convergence domain of $\tilde{\zeta}_{MT,d}(r; s \mid z)$. By the A–G inequality we conclude:

$$\left| \tilde{\zeta}_{MT,d}(r; s \mid z) \right| \leq d^{-s} \prod_{j=1}^{d} \sum_{n_j \geq 1} \frac{|z_j|^{n_j}}{n_j^{r_j+s/d}} = d^{-s} \prod_{j=1}^{d} \text{Li}_{r_j+s/d}(|z_j|).$$

The resulting polylogarithmic functions converge according to the theorem’s assumed parameter space, that is for $d \in \mathbb{N}_2, r_j > 0, s \in \mathbb{C}$ when $|z_j| < 1, j = \frac{1}{\nu}, d$ and $r_j + \Re(s)/d > 1$ when $|z_j| = 1, j = \frac{1}{\nu}, d$.

Further, specifying $\alpha_j(x_j) = x_j^{-r_j}$, $\lambda_j(x_j) = \mathcal{I}(x_j), j = \frac{1}{\nu}, d$ and $\nu \equiv 1$, Theorem 2.1 gives (2.18). Indeed, by quoted specifications we get

$$\tilde{\zeta}_{MT,d}(r; s \mid z) = \int_{\mathbb{R}^d_+} \prod_{j=1}^{d} \frac{1}{\Gamma(r_j)} \cdot \int_{\mathbb{R}_+} x_j^{r_j-1} \sum_{j=1}^{t_j} (e^{-x_j z_j})^{n_j} \, dx_j \cdot \frac{dt}{(t \circ 1)^{s+d}}. \quad (2.18)$$

Now, obvious steps lead to the asserted integral expression. \(\Box\)

Putting $z_j = 1, j = \frac{1}{\nu}, d$ in (2.18), we get the following

Corollary 2.2. Let $d \in \mathbb{N}_2, r_j > 0$ and $s \in \mathbb{C}$, $r_j + \Re(s)/d > 1$. Then we have

$$\zeta_{MT,d}(r; s) = \prod_{j=1}^{d} \frac{1}{\Gamma(r_j)} \cdot \int_{\mathbb{R}^d_+} \prod_{j=1}^{d} x_j^{r_j-1} \left\{ 1 - (e^{-x_j z_j})^{t_j} \right\} \frac{dt \, dx}{(t \circ 1)^{s+d}}.$$

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Finally, the related integral expressions for the Tornheim double sum become:

\[
\begin{align*}
\bar{T}(p, q, r | z) &= \frac{z_1 z_2}{\Gamma(p) \Gamma(q)} \cdot \int_{\mathbb{R}^+} x_1^{p-1} x_2^{q-1} \frac{(1 - (e^{-x_1 z_1})^{[t_1]}) (1 - (e^{-x_2 z_2})^{[t_2]})}{(e^{x_1 - z_1}) (e^{x_2 - z_1}) (t_1 + t_2)^{s+2}} \, dt \, dx, \\
T(p, q, r) &= \frac{1}{\Gamma(p) \Gamma(q)} \cdot \int_{\mathbb{R}^+} x_1^{p-1} x_2^{q-1} \frac{(1 - (e^{-x_1 z_1})^{[t_1]}) (1 - (e^{-x_2 z_2})^{[t_2]})}{(e^{x_1 - 1}) (e^{x_2 - 1}) (t_1 + t_2)^{s+2}} \, dt \, dx.
\end{align*}
\]

3 Mellin–Barnes integral formulae

This section deals with the derivation of Mellin–Barnes integral representation for the function \( \tilde{\mathcal{H}}_{d}(a; s; \alpha; \lambda | z) \) defined in (1.7).

**Theorem 3.1.** Let all parameters and variables of \( \tilde{\mathcal{H}}_{d}(a; s; \alpha; \lambda | z) \) satisfy (2.10). Then there holds true the formula

\[
\tilde{\mathcal{H}}_{d}(a; s; \alpha; \lambda | z) = \frac{1}{(2\pi i)^{d}} \int_{\mathbb{R}^{d}} \prod_{j=1}^{d} \left\{ \Gamma(-\xi_j) \Gamma(1 + \xi_j) (-z_j)^{\xi_j} \right\} \frac{d\xi}{(a + \lambda_1(\xi_1) + \cdots + \lambda_d(\xi_d))^{s}}
\]

where \(|\text{arg}(-z_{j})| < \pi, j = \frac{1}{1,d}\). Here \( \int_{\mathbb{R}^{d}} \) stands for \( d \)-ple line integral on the imaginary axis, that is \( \int_{-i\infty}^{i\infty} \) and \( d\xi \) := \( d\xi_1 \cdots d\xi_d \).

**Proof.** Assuming that non of the poles of the Gamma functions \( \Gamma(-\xi_j) \) at \( \xi_j = n_j, n_j \in \mathbb{N}_0, j = 1, d \) coincide with any poles of \( \Gamma(1 + \xi_j) \) which ones are situated at \( \xi_j = -1 - n_j, n_j \in \mathbb{N}_0 \) and for zeros of the denominator function \( a + \lambda_1(\xi_1) + \cdots + \lambda_d(\xi_d) \), which cannot be real. Here the integration paths in the complex \( \xi_j \)-planes, which start at the point \(-ix\) and terminate at the point \( ix\) separate the poles of \( \Gamma(-\xi_j), j = \frac{1}{1,d} \) from the poles \( \xi_j = -1 - n_j, n_j \in \mathbb{N}_0 \). Evaluating the residues the functions \( \Gamma(-\xi_j), j = \frac{1}{1,d} \) at \( \xi_j = n_j \), applying the calculus of residues, we obtain (3.19). \( \Box \)

**Corollary 3.1.** Let \( a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; s \in \mathbb{C}, \Re(s) > d \) and \( \nu > 0 \). Then we have

\[
\tilde{\zeta}_{d}(s; a|\nu, z) = \frac{1}{(2\pi i)^{d}} \int_{\mathbb{R}^{d}} \prod_{j=1}^{d} \left\{ \Gamma(-\xi_j) \Gamma(1 + \xi_j) (-z_j)^{\xi_j} \right\} \frac{d\xi}{(a + \nu \circ \xi)^{s}}
\]

where \(|\text{arg}(-z_{j})| < \pi, j = \frac{1}{1,d}\).

**Corollary 3.2.** Let \( a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; s \in \mathbb{C}, \Re(s) > d, \nu > 0 \). Then

\[
\zeta_{d}(s; a|\nu) = \frac{1}{(2\pi i)^{d}} \int_{\mathbb{R}^{d}} \prod_{j=1}^{d} \left\{ \Gamma(-\xi_j) \Gamma(1 + \xi_j) (-1)^{\xi_j} \right\} \frac{d\xi}{(a + \nu \circ \xi)^{s}}
\]

**Remark 3.1.** For \( d = 1, \nu = 1 \) Corollary 3.1 reduces to

\[
\Phi(z, s, a) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\Gamma(-\xi) \Gamma(1 + \xi) (-z)^{\xi}}{(a + \xi)^{s}} \, d\xi,
\]

where \(|\text{arg}(-z)| < \pi, \Re(s) > 1\), see [23].
If, however, we calculate the residues of $\Gamma(1+\xi_j)$ at the simple poles $\xi_j = -1-n_j, n_j \in \mathbb{N}_0, j = 1, d$, we arrive at the following analytic continuation formula.

**Theorem 3.2.** Let all parameters and variables of $\tilde{\eta}_d(a, s; \alpha, \lambda | z)$ satisfy (2.10). Then there holds true the formula

$$
\tilde{\eta}_d(a, s; \alpha, \lambda | z) = \sum_{n \in \mathbb{N}_0^d} \prod_{j=1}^{d} (-z_j)^{-n_j-1} \frac{(a + \lambda_1(1 + n_j) + \cdots + \lambda_d(1 + n_d))^s}{(a + \lambda_1(1 + n_j) + \cdots + \lambda_d(1 + n_d))^s},
$$

where $a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}, \Re(s) > d$.

**Corollary 3.3.** Let $a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}, \Re(s) > d$ and $\nu > 0$. Then, for $|\arg(-z_j)| < \pi, j = 1, d$:

$$
\tilde{\zeta}_d(s; a|\nu, z) = \sum_{n \in \mathbb{N}_0^d} \prod_{j=1}^{d} (-z_j)^{-n_j-1} \frac{(a + 1 + \nu + n \circ \nu)^s}{(a + 1 + \nu + n \circ \nu)^s}.
$$

**Corollary 3.4.** Suppose that $a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}, \Re(s) > d, \nu > 0$. Then

$$
\zeta_d(s; a|\nu) = (-1)^d \sum_{n \in \mathbb{N}_0^d} \prod_{j=1}^{d} (-1)^{-n_j} \frac{(a + 1 + \nu + n \circ \nu)^s}{(a + 1 + \nu + n \circ \nu)^s}.
$$

**Remark 3.2.** For $d = 1, \nu = 1$ Corollary 3.3 reduces to

$$
\Phi(z, s, a) = -\frac{1}{z} \sum_{n \in \mathbb{N}_0} \frac{(-z)^{-n}}{(a + 1 + n)^s},
$$

where $|\arg(-z)| < \pi, \Re(s) > 1$.

Relations (1.5), (1.8) and (2.11) connect Hilbert’s $d$–linear $d$–variate sum $\tilde{\eta}_d$ and the related Hurwitz–Euler Eta functions $\eta_d, \tilde{\eta}_d$. By similar considerations as for $\zeta_d, \tilde{\zeta}_d$, we deduce the following results (avoiding the proofs which only modestly differ of here exposed ones).

**Corollary 3.5.** Let $a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}, \Re(s) > d$. Also assume $\nu > 0$. Then we have

$$
\tilde{\eta}_d(s; a|\nu, z) = \frac{1}{(2\pi i)^d} \int_{i\mathbb{R}^d} \prod_{j=1}^{d} \frac{\{\Gamma(-\xi_j)\Gamma(1 + \xi_j)z_j^{\xi_j}\}}{(a + \nu \circ \xi)^s} \, d\xi,
$$

where $|\arg(-z_j)| < \pi, j = 1, d$.

**Corollary 3.6.** Letting $a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}, \Re(s) > d$ and $\nu > 0$, it follows

$$
\eta_d(s; a|\nu) = \frac{1}{(2\pi i)^d} \int_{i\mathbb{R}^d} \prod_{j=1}^{d} \frac{\{\Gamma(-\xi_j)\Gamma(1 + \xi_j)\}}{(a + \nu \circ \xi)^s} \, d\xi.
$$
4 Mellin transforms

By the application of the multidimensional Mellin inversion formula, the following results readily follow from (3.20) and (3.21). Because it simplicity, we omit their proofs.

Theorem 4.1. Let $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$, $s \in \mathbb{C}$, $\Re(s) > d$ and $\nu > 0$. Then we have

$$\int_{\mathbb{R}_+^d} z_1^{s_1-1} \cdots z_d^{s_d-1} \widetilde{\zeta}_d(s; a|\nu, z) \, dz = \frac{\prod_{j=1}^d \Gamma(\xi_j) \Gamma(1-\xi_j)}{(a-\nu \circ \xi)^s},$$

where $|\arg(-z_j)| < \pi$, $j = 1, \ldots, d$.

Corollary 4.1. Let $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$, $s \in \mathbb{C}$, $\Re(s) > d$ and $\nu > 0$. Then we have

$$\int_{\mathbb{R}_+^d} z_1^{s_1-1} \cdots z_d^{s_d-1} \widetilde{\eta}_d(s; a|\nu, z) \, dz = (-1)^{\xi_0-1-d} \frac{\prod_{j=1}^d \Gamma(\xi_j) \Gamma(1-\xi_j)}{(a-\nu \circ \xi)^s},$$

where $|\arg(-z_j)| < \pi$, $j = 1, \ldots, d$.

Acknowledgements

The authors would like to express their sincere thanks to an unknown Reviewer who provided constructive helpful advices and suggestions to improve the article.

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http://dx.doi.org/10.1098/rsta.1901.0006


