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Research Article

On Algebraic Study of Type-2 Fuzzy Finite State Automata

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Abstract

Theories of fuzzy sets and type-2 fuzzy sets are powerful mathematical tools for modeling various types of uncertainty. In this paper we introduce the concept of type-2 fuzzy finite state automata and discuss the algebraic study of type-2 fuzzy finite state automata, i.e., to introduce the concept of homomorphisms between two type-2 fuzzy finite state automata, to associate a type-2 fuzzy transformation semigroup with a type-2 fuzzy finite state automata. Finally, we discuss several product of type-2 fuzzy finite state automata and shown that these product is a categorical product.

Keywords: Type-2 fuzzy set, type-2 fuzzy finite state automata, type-2 fuzzy transformation semigroup, categorical product.

1 Introduction

The study of fuzzy automata was initiated by [10] and [12] in 1960 after the introduction of fuzzy set theory by [13]. Much later a considerably simpler notion of a fuzzy finite state machine (which is almost identical to a fuzzy automata) was introduced by [6, 8]. Somewhat different notions were introduced subsequently in [3, 4] and [9]. The concept of homomorphism, transformation semigroup, product property play an important role in the study of finite state machine [2]. Much later [7] introduced these ideas for fuzzy finite state machines and explore their algebraic properties (cf.,[6] for more detail).

Type-2 fuzzy sets, firstly proposed by [14] in 1975 as an extension of type-1 fuzzy set in which the membership function falls into a fuzzy set in the interval $[0,1]$. Because type-1 fuzzy sets not able to directly model such uncertainties because their membership functions are totally crisp. Type-2 fuzzy sets are capable to improve such uncertainties because their membership function are already fuzzy. Now, in this paper, we mainly introduced the concept of automata theory in type-2 fuzzy sets, which consists of a state-set, an input-set, a transition map and a primary membership function which have been fuzzified. Further by this concept of type-2 fuzzy finite state automata we discuss the concepts of homomorphism, transformation semigroup and direct product for type-2 fuzzy finite state automata.

This paper is organized as follows: In section 2, we recall some basic definitions and properties of fuzzy automata, type-2 fuzzy sets, type-2 fuzzy relations. In section 3, we recall the concept of type-2 fuzzy finite state automata and with the help of this type-2 fuzzy finite state automata we discuss the concept of homomorphism. In section 4, recall the concept of type-2 fuzzy transformation semigroups by using congruence relation and lastly, we discuss the direct product and the general direct product of type-2 fuzzy finite state automata and shown that this product is a categorical product.

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2 Preliminaries

In this section we recall some basic notions relevant to fuzzy automata, type-2 fuzzy sets, type-2 fuzzy relations and collect some results, which we need in the subsequent sections. Throughout this paper, X is a nonempty set.

Definition 2.1. *[12] A fuzzy automaton is a triple $M = (Q, X, \delta)$, where Q is a nonempty set (the set of states of M), X is a monoid (the input monoid of M), whose identity shall be denoted as e_X, and $\delta : Q \times X \times Q \to [0,1]$ is a map, such that $\forall p, q \in Q, \forall x, y \in X$, we have*

$$\delta(q, e_X, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

and $\delta(q, xy, p) = \vee_{r \in Q}\{\delta(q, x, r) \wedge \delta(r, y, p)\}$

Definition 2.2. *[5]. A type-2 fuzzy set, denoted $\tilde{A}$ is characterized by a type-2 membership function $\mu_{\tilde{A}}(x,u) : X \times J_x \to [0,1], \forall x \in X, \forall u \in J_x \subseteq [0,1]$, i.e., $\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u)) : x \in X, u \in J_x \subseteq [0,1]\}$, In which $0 \leq \mu_{\tilde{A}}(x,u) \leq 1$. $\tilde{A}$ can be expressed as $\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x,u)/(x,u), J_x \subseteq [0,1]$,*
where $\int\int$ denote the union over all admissible x and u.

A class of type-2 fuzzy sets of the universe X is denoted by $\tilde{F}(X)$.

Definition 2.3. *[5]. A vertical slice denoted $\mu_{\tilde{A}}(x)$ of $\tilde{A}$, is the intersection between the two-dimensional plane whose axes are u and $\mu_{\tilde{A}}(x,u)$ and the three-dimensional type-2 membership function $\tilde{A}$, i.e.,*
$\mu_{\tilde{A}}(x) = \mu_{\tilde{A}}(x = x', u) = \int_{u \in J_{x'}} f_{x'}(u)/u, J_{x'} \subseteq [0,1]$
in which $0 \leq f_{x'}(u) \leq 1$. In terms of vertical slice, a type-2 fuzzy set $\tilde{A}$ can also be re-expressed as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$$

or as the following:

$$\tilde{A} = \int_{x \in X} \mu_{\tilde{A}}(x)/x = \int_{x \in X}[\int_{u \in J_x} f_x(u)/u]/x, J_x \subseteq [0,1],$$

where $f_x(u) = \mu_{\tilde{A}}(x,u)$.

The vertical slice $\mu_{\tilde{A}}(x)$ is also called the secondary membership function, and its domain is called the primary membership of x, which is denoted by J_x, where $J_x \subseteq [0,1]$ for any $x \in X$. The amplitude of the secondary membership function is called the secondary grade.

Definition 2.4. *[5]. The footprint of uncertainty denoted $D\tilde{A}$, is the union of all of the primary memberships: i.e., $D\tilde{A} = \cup_{x \in X} J_x$, which represents the uncertainty in the primary memberships of a type-2 fuzzy set $\tilde{A}$.*

Let $D\tilde{A}(x) = J_x, \forall x \in X$. Then a type-2 fuzzy set $\tilde{A}$ can be re-expressed as:

$$\tilde{A} = \int\int_{(x,u) \in D\tilde{A}} \mu_{\tilde{A}}(x,u)/(x,u).$$

Definition 2.5. *[11]. Let X and Y be two nonempty universes. Then type-2 fuzzy set $\tilde{R} \in \tilde{F}(X \times Y)$ is said to be a type-2 fuzzy binary relation from X to Y as:*

$$\tilde{R} = \{(((x,y),u), \mu_{\tilde{R}}((x,y),u)) : (x,y) \in X \times Y, u \in J_{(x,y)} \subseteq [0,1]\},$$

in which $0 \leq \mu_{\tilde{R}}((x,y),u) \leq 1$. $\tilde{R}$ can be expressed as follows:

$$\tilde{R} = \int_{(x,y) \in X \times Y} \int_{u \in J_{(x,y)}} \mu_{\tilde{R}}((x,y),u)/((x,y),u), J_{(x,y)} \subseteq [0,1].$$

The footprint of uncertainty $FOU(\tilde{R})$ is denoted by

(i) $D\tilde{R}(x,y) = J_{(x,y)}$,

(ii) $D\tilde{R} = FOU(\tilde{R}) = \cup_{(x,y) \in X \times Y} D\tilde{R}(x,y)$.

3 Type-2 fuzzy finite state automata

In this section, we recall the concepts related to a type-2 fuzzy finite state automata and introduced the concept of homomorphism between two type-2 fuzzy finite state automata.

Definition 3.1. *A type-2 fuzzy finite state automaton is a four tuple* $\widetilde{M}=(Q,X,\tilde{\delta},J_x)$, *where* Q *is a nonempty finite set (the set of states of* $\widetilde{M}$ *),* X *is a monoid (whose elements are the input symbols),* $\tilde{\delta}$ *is a type-2 fuzzy subset of* $(Q\times X\times Q)\times J_x$ *, i.e., a map* $\tilde{\delta}:(Q\times X\times Q)\times J_x\to[0,1]$, *where* $J_x\subseteq[0,1]$ *is a primary membership of x such that* $\forall p,q\in Q,\forall x,y\in X$, *we have*
$D\tilde{\delta}(q,e,p)=[\mu_{\underline{D}\tilde{\delta}}(q,e,p),\mu_{\overline{D}\tilde{\delta}}(q,e,p)]=\begin{cases}1 & \text{if } q=p\\ 0 & \text{if } q\neq p\end{cases}$

with $\mu_{\underline{D}\tilde{\delta}}(q,xy,p)=\vee_{r\in Q}\{\mu_{\underline{D}\tilde{\delta}}(q,x,r)\wedge\mu_{\underline{D}\tilde{\delta}}(r,y,p)\}$ *and* $\mu_{\overline{D}\tilde{\delta}}(q,xy,p)=\vee_{r\in Q}\{\mu_{\overline{D}\tilde{\delta}}(q,x,r)\wedge\mu_{\overline{D}\tilde{\delta}}(r,y,p)\}$.

Example 3.1. *Consider the type-2 fuzzy finite state automata* $(Q,X,\tilde{\delta},J_x)$, *where* $Q=\{p,q,r\}$, $X=\{x,y\}$ *with* $J_x\subseteq[0,1]$ *and* $\tilde{\delta}$ *is a type-2 fuzzy subset of* $(Q\times X\times Q)\times J_x$ *defined as*

$$D\tilde{\delta}(q,e,p)=[\mu_{\underline{D}\tilde{\delta}}(q,e,p),\mu_{\overline{D}\tilde{\delta}}(q,e,p)]=\begin{cases}1 & \text{if } q=p\\ 0 & \text{if } q\neq p\end{cases}\quad \textit{with}$$

$$\mu_{\underline{D}\tilde{\delta}}=\begin{matrix} & p & q & r\\ p & \\ q & \\ r & \end{matrix}\begin{pmatrix}1.0 & 0.5 & 0.6\\ 0.2 & 1.0 & 0.2\\ 0.3 & 0.4 & 1.0\end{pmatrix},\ \mu_{\overline{D}\tilde{\delta}}=\begin{matrix} & p & q & r\\ p & \\ q & \\ r & \end{matrix}\begin{pmatrix}1.0 & 0.5 & 0.6\\ 0.2 & 1.0 & 0.2\\ 0.3 & 0.4 & 1.0\end{pmatrix}$$

also $\mu_{\underline{D}\tilde{\delta}}(q,x,p)=0.2,\mu_{\underline{D}\tilde{\delta}}(q,x,q)=1.0,\mu_{\underline{D}\tilde{\delta}}(q,x,r)=0.2,\mu_{\overline{D}\tilde{\delta}}(q,x,p)=0.6,\mu_{\overline{D}\tilde{\delta}}(q,x,r)=0.8,\mu_{\overline{D}\tilde{\delta}}(q,x,q)=1.0,$ *and*

$$\mu_{\underline{D}\tilde{\delta}}=\begin{matrix} & p & q & r\\ p & \\ q & \\ r & \end{matrix}\begin{pmatrix}0.9 & 0.5 & 0.4\\ 0.5 & 0.8 & 0.6\\ 0.6 & 0.4 & 0.6\end{pmatrix},\ \mu_{\overline{D}\tilde{\delta}}=\begin{matrix} & p & q & r\\ p & \\ q & \\ r & \end{matrix}\begin{pmatrix}1.0 & 0.6 & 0.6\\ 0.8 & 0.9 & 0.8\\ 0.7 & 0.5 & 0.8\end{pmatrix}$$

with $\mu_{\underline{D}\tilde{\delta}}(p,y,p)=0.9,\mu_{\underline{D}\tilde{\delta}}(q,y,p)=0.5,\mu_{\underline{D}\tilde{\delta}}(r,y,p)=0.6,\mu_{\overline{D}\tilde{\delta}}(p,y,p)=1.0,\mu_{\overline{D}\tilde{\delta}}(q,y,p)=0.8,\mu_{\overline{D}\tilde{\delta}}(r,y,p)=0.7.$ *Now* $\mu_{\underline{D}\tilde{\delta}}(q,xy,p)=\vee_{r\in Q}\{\mu_{\underline{D}\tilde{\delta}}(q,x,r)\wedge\mu_{\underline{D}\tilde{\delta}}(r,y,p)\}=\vee\{\mu_{\underline{D}\tilde{\delta}}(q,x,p)\wedge\mu_{\underline{D}\tilde{\delta}}(p,y,p),\mu_{\underline{D}\tilde{\delta}}(q,x,q)\wedge\mu_{\underline{D}\tilde{\delta}}(q,y,p),\mu_{\underline{D}\tilde{\delta}}(q,x,r)\wedge\mu_{\underline{D}\tilde{\delta}}(r,y,p)\}=\vee\{0.2\wedge0.9,1.0\wedge0.5,0.2\wedge0.6\}=\vee\{0.2,0.5,0.2\}=0.5$ *and* $\mu_{\overline{D}\tilde{\delta}}(q,xy,p)=\vee_{r\in Q}\{\mu_{\overline{D}\tilde{\delta}}(q,x,r)\wedge\mu_{\overline{D}\tilde{\delta}}(r,y,p)\}=\vee\{\mu_{\overline{D}\tilde{\delta}}(q,x,p)\wedge\mu_{\overline{D}\tilde{\delta}}(p,y,p),\mu_{\overline{D}\tilde{\delta}}(q,x,q)\wedge\mu_{\overline{D}\tilde{\delta}}(q,y,p),\mu_{\overline{D}\tilde{\delta}}(q,x,r)\wedge\mu_{\overline{D}\tilde{\delta}}(r,y,p)\}=\vee\{0.6\wedge1.0,1.0\wedge0.8,0.8\wedge0.7\}=\vee\{0.6,0.8,0.7\}=0.8.$ *Thus* $D\tilde{\delta}(q,xy,p)=[\mu_{\underline{D}\tilde{\delta}}(q,xy,p),\mu_{\overline{D}\tilde{\delta}}(q,xy,p)]=[0.5,0.8]$.

Definition 3.2. *Let* $\widetilde{M_1}=(Q_1,X_1,\tilde{\delta}_1,J_{x_1})$ *and* $\widetilde{M_2}=(Q_2,X_2,\tilde{\delta}_2,J_{x_2})$ *be type-2 fuzzy finite state automata. A* **homomorphism** *from* $\widetilde{M_1}$ *to* $\widetilde{M_2}$ *is a pair of maps* (f,g), *where* $f:Q_1\to Q_2$ *and* $g:X_1\to X_2$ *are functions such that* $D\tilde{\delta}_1(q,x,p)\leq D\tilde{\delta}_2(f(q),g(x),f(p)),\forall p,q\in Q_1,\forall x\in X_1$.

A homomorphism $(f,g):\widetilde{M_1}\to\widetilde{M_2}$ is called an isomorphism if f and g are both one-one and onto.

Definition 3.3. *Let* $\widetilde{M_1}=(Q_1,X_1,\tilde{\delta}_1,J_{x_1})$ *and* $\widetilde{M_2}=(Q_2,X_2,\tilde{\delta}_2,J_{x_2})$ *be type-2 fuzzy finite state automata and* $(f,g):\widetilde{M_1}\to\widetilde{M_2}$ *be a homomorphism. Let* $g^*:X_1^*\to X_2^*$ *be a map such that* $g^*(e)=e$ *and* $g^*(wa)=g^*(w)g(a),\forall w\in X_1^*,a\in X_1$.

The class of all type-2 fuzzy finite state automata and there homomorphisms obviously forms a category (under obvious composition of maps). We shall denote it by **T2FFSA**

Lemma 3.1. *Let $\widetilde{M_1} = (Q_1, X_1, \tilde{\delta}_1, J_{x_1})$ and $\widetilde{M_2} = (Q_2, X_2, \tilde{\delta}_2, J_{x_2})$ be type-2 fuzzy finite state automata and $(f,g) : \widetilde{M_1} \to \widetilde{M_2}$ be a homomorphism. Then $g^*(xy) = g^*(x)g^*(y), \forall x, y \in X_1^*$.*

Proof. Let $x, y \in X_1^*$. We prove the result by induction on $|y| = n$. If $n = 0$, then $y = e$ and so $xy = xe = x$. Thus $g^*(xy) = g^*(x) = g^*(x)e = g^*(x)g^*(e) = g^*(x)g^*(y)$, whereby, the result is true for $n = 0$. Also. let the result be true $\forall z \in X_1^*$ such that $|z| = n-1, n > 0$ and $y = za$, where $a \in X_1$. Then $g^*(xy) = g^*(xza) = g^*(xz)g(a) = g^*(x)g^*(z)g(a) = g^*(x)g^*(za) = g^*(x)g^*(y)$. Hence the result is true for $|y| = n$. □

Proposition 3.1. *Let $\widetilde{M_1} = (Q_1, X_1, \tilde{\delta}_1, J_{x_1})$ and $\widetilde{M_2} = (Q_2, X_2, \tilde{\delta}_2, J_{x_2})$ be type-2 fuzzy finite state automata and $(f,g) : \widetilde{M_1} \to \widetilde{M_2}$ be a homomorphism. Then $D\tilde{\delta}_1^*(q,x,p) \leq D\tilde{\delta}_2^*(f(q), g*(x), f(p)), i.e.,$
$\mu_{\underline{D}\tilde{\delta}_1^*}(q,x,p) \leq \mu_{\underline{D}\tilde{\delta}_2^*}(f(q), g^*(x), f(p))$ and $\mu_{\overline{D}\tilde{\delta}_1^*}(q,x,p) \leq \mu_{\overline{D}\tilde{\delta}_2^*}(f(q), g^*(x), f(p)), \forall p, q \in Q_1, \forall x \in X_1^*$.*

Proof. Let $p, q \in Q_1$ and $x \in X_1^*$. we prove the result by induction on $|x| = n$. If $n = 0$, then $x = e$ and $g^*(x) = g^*(e) = e$. Now

$$\mu_{\underline{D}\tilde{\delta}_1^*}(q,e,p) = \mu_{\overline{D}\tilde{\delta}_1^*}(q,e,p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases} =$$

$\mu_{\underline{D}\tilde{\delta}_2^*}(f(q), e, f(p)) = \mu_{\overline{D}\tilde{\delta}_2^*}(f(q), e, f(p))$.
Let the result be true for all $y \in X^*$ such that $|y| = n-1, n > 0$ and $x = ya$, where $a \in X_1, y \in X_1^*$ with $|y| = n-1$. Then $\mu_{\underline{D}\tilde{\delta}_1^*}(q,x,p) = \mu_{\underline{D}\tilde{\delta}_1^*}(q,ya,p) = \vee_{r \in Q_1}\{\mu_{\underline{D}\tilde{\delta}_1^*}(q,y,r) \wedge \mu_{\underline{D}\tilde{\delta}_1^*}(r,a,p)\} \leq \vee_{r \in Q_1}\{\mu_{\underline{D}\tilde{\delta}_2^*}(f(q), g^*(y), f(r)) \wedge \mu_{\underline{D}\tilde{\delta}_2^*}(f(r), g(a), f(p))\} \leq \vee_{r' \in Q_2}\{\mu_{\underline{D}\tilde{\delta}_2^*}(f(q), g^*(y), r') \wedge \mu_{\underline{D}\tilde{\delta}_2^*}(r', g^*(a), f(p))\} = \mu_{\underline{D}\tilde{\delta}_2^*}(f(q), g^*(y)g(a), f(p)) = \mu_{\underline{D}\tilde{\delta}_2^*}(f(q), g^*(ya), f(p)) = \mu_{\underline{D}\tilde{\delta}_2^*}(f(q), g^*(x), f(p))$. Similarly, we have $\mu_{\overline{D}\tilde{\delta}_1^*}(q,x,p) \leq \mu_{\overline{D}\tilde{\delta}_2^*}(f(q), g^*(x), f(p))$. □

4 Type-2 fuzzy transformation semigroup

In this section, we recall some basic concepts related to a transformation semigroups both in finite state automata and in type-2 fuzzy finite state automata (cf. [2], [6]). Also, we discuss the direct and the general direct product and shown that this product is a categorical product in type-2 fuzzy finite state automata.
Recall from [6] that an equivalence relation $\sim$ on a semigroup $(X, *)$ is called a **congruence relation** on X if $\forall a, b \in X, a \sim b \Rightarrow \tilde{\delta}(q,a,p) = \tilde{\delta}(q,b,p), \forall p, q \in Q$.

Let $(Q, X, \tilde{\delta}, J_x)$ be a type-2 fuzzy finite state automaton. Define a relation $\simeq$ on X^* by $x \simeq y \Leftrightarrow \mu_{\underline{D}\tilde{\delta}}(q,x,p) = \mu_{\underline{D}\tilde{\delta}}(q,y,p)$ and $\mu_{\overline{D}\tilde{\delta}}(q,x,p) = \mu_{\overline{D}\tilde{\delta}}(q,y,p), \forall p, q \in Q$ and $\forall x, y \in X^*$. Then we have the following.

Proposition 4.1. *Let $(Q, X, \tilde{\delta}, J_x)$ be a type-2 fuzzy finite state automaton. Then the relation $\simeq$ is a congruence relation on X^*.*

Proof. It is obvious that the relation $\simeq$ is an equivalence relation on X^*. Let $x, y \in X^*$ such that $x \simeq y$ and $z \in X^*$. Then $\forall p, q \in Q, \mu_{\underline{D}\tilde{\delta}}(q,xz,p) = \vee_{r \in Q}\{\mu_{\underline{D}\tilde{\delta}}(q,x,r) \wedge \mu_{\underline{D}\tilde{\delta}}(r,z,p)\} = \vee_{r \in Q}\{\mu_{\underline{D}\tilde{\delta}}(q,y,r) \wedge \mu_{\underline{D}\tilde{\delta}}(r,z,p)\} = \mu_{\underline{D}\tilde{\delta}}(q,yz,p)$. Similarly, we have $\mu_{\overline{D}\tilde{\delta}}(q,xz,p) = \mu_{\overline{D}\tilde{\delta}}(q,yz,p)$. Thus $xz \simeq yz$. Similarly $zx \simeq zy$. Hence $\simeq$ is a congruence relation on X^*. □

For a given type-2 fuzzy finite state automaton $\widetilde{M} = (Q, X, \tilde{\delta}, J_x)$, let $[x] = \{y \in X^* : x \simeq y\}$ and $E(\widetilde{M}) = \{[x] : x \in X^*\}$. Define a binary operation $*$ on $E(\widetilde{M})$ by $[x] * [y] = [xy], \forall [x], [y] \in E(\widetilde{M})$. Then we have the following.

Proposition 4.2. *Let $\widetilde{M} = (Q, X, \tilde{\delta}, J_x)$ be a type-2 fuzzy finite state automaton. Then $(E(\widetilde{M}), *)$ is a finite semigroup with identity.*

Proof. It is obvious that $*$ is associative and well defined. For $[x] \in E(\widetilde{M})$, we have $[x] * [e] = [xe] = [x] = [ex] = [e] * [x]$, where $[e]$ is the identity of $(E(\widetilde{M}), *)$. Thus $(E(\widetilde{M}), *)$ is a semigroup with identity. Let $x \in X^*$ and let $x = x_1 x_2 .....x_n$, where $x_1, x_2, ..., x_n \in X$. Then for all $p, q \in Q$,
$\mu_{\underline{D}\tilde{\delta}^*}(q,x,p) = \vee_{q_1, q_2, ..., q_{n-1} \in Q}\{\mu_{\underline{D}\tilde{\delta}}(q, x_1, q_1) \wedge \mu_{\underline{D}\tilde{\delta}}(q_1, x_2, q_2) \wedge ..... \wedge \mu_{\underline{D}\tilde{\delta}}(q_{n-1}, x_n, p)\}$ and $\mu_{\overline{D}\tilde{\delta}^*}(q,x,p) = \vee_{q_1, q_2, ..., q_{n-1} \in Q}\{\mu_{\overline{D}\tilde{\delta}}(q, x_1, q_1) \wedge \mu_{\overline{D}\tilde{\delta}}(q_1, x_2, q_2) \wedge ..... \wedge \mu_{\overline{D}\tilde{\delta}}(q_{n-1}, x_n, p)\}$. Thus as Q is finite, then $\mu_{\underline{D}\tilde{\delta}^*}(q,x,p)$ and $\mu_{\overline{D}\tilde{\delta}^*}(q,x,p)$ is finite. Hence $(E(\widetilde{M}), *)$ is a finite semigroup with identity. □

Now, we define another type of congruence relation on X^*. Let $\widetilde{M} = (Q, X, \tilde{\delta}, J_x)$ be a type-2 fuzzy finite state automaton. Define a relation $\simeq$ on X^* by $x \simeq y$ as $\mu_{\underline{D}\tilde{\delta}}(q,x,p) > 0 \Leftrightarrow \mu_{\underline{D}\tilde{\delta}}(q,y,p) > 0$ and $\mu_{\overline{D}\tilde{\delta}}(q,x,p) > 0 \Leftrightarrow \mu_{\overline{D}\tilde{\delta}}(q,y,p) > 0, \forall p,q \in Q$ and $\forall x,y \in X^*$. Then we have the following.

Proposition 4.3. *Let $(Q,X,\tilde{\delta},J_x)$ be a type-2 fuzzy finite state automaton. Then the relation $\simeq$ is a congruence relation on X^*.*

Proof. It is obvious that the relation $\simeq$ is an equivalence relation on X^*. Let $x,y \in X^*$ such that $x \simeq y$ and $z \in X^*$. Then $\forall p,q \in Q$,
$\mu_{\underline{D}\tilde{\delta}^*}(q,zx,p) = \vee_{r\in Q}\{\mu_{\underline{D}\tilde{\delta}^*}(q,z,r) \wedge \mu_{\underline{D}\tilde{\delta}^*}(r,x,p)\} > 0 \Leftrightarrow \exists r \in Q$ such that $\mu_{\underline{D}\tilde{\delta}^*}(q,z,r) \wedge \mu_{\underline{D}\tilde{\delta}^*}(r,x,p) > 0 \Leftrightarrow \exists r \in Q$ such that $\mu_{\underline{D}\tilde{\delta}^*}(q,z,r) \wedge \mu_{\underline{D}\tilde{\delta}^*}(r,y,p) > 0 \Leftrightarrow \mu_{\underline{D}\tilde{\delta}^*}(q,zy,p) = \vee_{r\in Q}\{\mu_{\underline{D}\tilde{\delta}^*}(q,z,r) \wedge \mu_{\underline{D}\tilde{\delta}^*}(r,y,p)\} > 0$. Similarly, we have $\mu_{\overline{D}\tilde{\delta}^*}(q,zx,p) > 0 \Leftrightarrow \mu_{\overline{D}\tilde{\delta}^*}(q,zy,p) > 0$. Thus $zx \simeq zy$. Similarly $xz \simeq yz$. Hence $\simeq$ is a congruence relation on X^*. □

For given a type-2 fuzzy finite state automaton $\widetilde{M} = (Q, X, \tilde{\delta}, J_x)$. Let $x \in X^*$ and let $[[x]] = \{y \in X^* : x \simeq y\}$ and $\widetilde{E(\widetilde{M})} = \{[[x]] : x \in X^*\}$. Define a binary operation $\widetilde{*}$ on $\widetilde{E(\widetilde{M})}$ by $[[x]]\widetilde{*}[[y]] = [[xy]], \forall [[x]],[[y]] \in \widetilde{E(\widetilde{M})}$. Then we have the following.

Proposition 4.4. *Let $\widetilde{M} = (Q, X, \tilde{\delta}, J_x)$ be a type-2 fuzzy finite state automaton. Then $(\widetilde{E(\widetilde{M})}, \widetilde{*})$ is a finite semigroup with identity and $[x] \to [[x]]$ is a homomorphism of $E(\widetilde{M})$ onto $\widetilde{E(\widetilde{M})}$.*

Proof. Associativity of the $\widetilde{*}$ is trivial. For $[[x]] \in \widetilde{E(\widetilde{M})}$, we have $[[x]]\widetilde{*}[[e]] = [[xe]] = [[x]] = [[ex]] = [[e]]\widetilde{*}[[x]]$, whereby $[[e]]$ is the identity of $(\widetilde{E(\widetilde{M})}, \widetilde{*})$. Thus $(\widetilde{E(\widetilde{M})}, \widetilde{*})$ is a semigroup with identity. Now, define $f : E(\widetilde{M}) \to \widetilde{E(\widetilde{M})}$ by $f([x]) = [[x]], \forall [x] \in E(\widetilde{M})$. Let $x,y \in X^*$ and $[x] \simeq [y]$. Then $\forall p,q \in Q, \mu_{\underline{D}\tilde{\delta}^*}(q,x,p) = \mu_{\underline{D}\tilde{\delta}^*}(q,y,p)$ and $\mu_{\overline{D}\tilde{\delta}^*}(q,x,p) = \mu_{\overline{D}\tilde{\delta}^*}(q,y,p)$. Thus $\forall p,q \in Q, \mu_{\underline{D}\tilde{\delta}^*}(q,x,p) > 0 \Leftrightarrow \mu_{\underline{D}\tilde{\delta}^*}(q,y,p) > 0$ and $\mu_{\overline{D}\tilde{\delta}^*}(q,x,p) > 0 \Leftrightarrow \mu_{\overline{D}\tilde{\delta}^*}(q,y,p) > 0$. Thus $x \simeq y$ or $[[x]] = [[y]]$. Hence f is well defined. Thus obviously f is an onto homomorphism. Also, as $E(\widetilde{M})$ is finite $\widetilde{E(\widetilde{M})}$ is finite. □

Definition 4.1. *Let $\widetilde{M} = (Q, X, \tilde{\delta}, J_x)$ be a type-2 fuzzy finite state automaton. Then the type-2 fuzzy subset $x^{\widetilde{M}}$ of $Q \times Q$ by $x^{\widetilde{M}}(q,p) = D\tilde{\delta}^*(q,x,p) = [\mu_{\underline{D}\tilde{\delta}^*}(q,x,p), \mu_{\overline{D}\tilde{\delta}^*}(q,x,p)], \forall p,q \in Q$ and $\forall x \in X^*$.*

Proposition 4.5. *Let $\widetilde{M} = (Q, X, \tilde{\delta}, J_x)$ be a type-2 fuzzy finite state automaton. Let $S_{\widetilde{M}} = \{x^{\widetilde{M}} : x \in X^*\}$. Then*
(1) $x^{\widetilde{M}} o y^{\widetilde{M}} = (xy)^{\widetilde{M}}, \forall x,y \in X^*$
(2) $(S_{\widetilde{M}}, o)$ is a finite semigroup with identity, where o is defined as $(x^{\widetilde{M}} o y^{\widetilde{M}})(q,p) = \vee_{r\in Q}\{x^{\widetilde{M}}(q,r) \wedge y^{\widetilde{M}}(r,p)\}$.

Proof. (1) Let $p,q \in Q$. Then $(xy)^{\widetilde{M}}(q,p) = D\tilde{\delta}^*(q,x,p) = [\mu_{\underline{D}\tilde{\delta}^*}(q,x,p), \mu_{\overline{D}\tilde{\delta}^*}(q,x,p)]$. Where $\mu_{\underline{D}\tilde{\delta}^*}(q,xy,p) = \vee_{r\in Q}\{\mu_{\underline{D}\tilde{\delta}^*}(q,x,r) \wedge \mu_{\underline{D}\tilde{\delta}^*}(r,y,p)\}$ and $\mu_{\overline{D}\tilde{\delta}^*}(q,xy,p) = \vee_{r\in Q}\{\mu_{\overline{D}\tilde{\delta}^*}(q,x,r) \wedge \mu_{\overline{D}\tilde{\delta}^*}(r,y,p)\}$. Hence $(xy)^{\widetilde{M}}(q,p) = D\tilde{\delta}^*(q,xy,p) = \vee_{r\in Q}\{D\tilde{\delta}^*(q,x,r) \wedge D\tilde{\delta}^*(r,y,p)\}$. Thus $(xy)^{\widetilde{M}} = x^{\widetilde{M}} o y^{\widetilde{M}}$.
(2) $S_{\widetilde{M}}$ is finite since Q and $D\tilde{\delta}^*$ is finite and $e^{\widetilde{M}}$ is the identity elements. Thus $(S_{\widetilde{M}}, o)$ is a finite semigroup with identity. □

Definition 4.2. ***A type-2 fuzzy transformation semigroup*** *(T2FTS, in short) is a* 4*-tuple $\tilde{A} = (Q, S, \tilde{\lambda}, J_v)$, where Q is a nonempty finite set (the set of states of $\tilde{A}$), S is a nonempty finite semigroup, $\tilde{\lambda}$ is a type-2 fuzzy subset of $(Q \times S \times Q) \times J_v$, i.e., $\tilde{\lambda} : (Q \times S \times Q) \times J_v \to [0,1]$, where $J_v \subseteq [0,1]$ is a primary membership of v such that*

(1) if S contains the identity e, then

$$D\tilde{\lambda}(q,e,p) = [\mu_{\underline{D}\tilde{\lambda}}(q,e,p), \mu_{\overline{D}\tilde{\lambda}}(q,e,p)] = \begin{cases} 1 & if\, q = p \\ 0 & if\, q \neq p \end{cases}$$

(2) $\mu_{\underline{D}\tilde{\lambda}}(q,vw,p) = \vee_{r\in Q}\{\mu_{\underline{D}\tilde{\lambda}}(q,v,r)\wedge\mu_{\underline{D}\tilde{\lambda}}(r,w,p)\}$ *and*
$\mu_{\overline{D}\tilde{\lambda}}(q,vw,p) = \vee_{r\in Q}\{\mu_{\overline{D}\tilde{\lambda}}(q,v,r)\wedge\mu_{\overline{D}\tilde{\lambda}}(r,w,p)\}, \forall p,q\in Q$ *and* $\forall v,w\in S$.

If in addition, $\forall p,q\in Q$ and $\forall v,w\in S, \mu_{\underline{D}\tilde{\lambda}}(q,v,p) = \mu_{\underline{D}\tilde{\lambda}}(q,w,p)$ and $\mu_{\overline{D}\tilde{\lambda}}(q,v,p) = \mu_{\overline{D}\tilde{\lambda}}(q,w,p) \Rightarrow v = w$ holds. Then $(Q,S,\tilde{\lambda},J_v)$ is called **faithful T2FTS**.

Let $\tilde{A} = (Q,S,\tilde{\lambda},J_v)$ be a T2FTS which is not faithful. Define a relation $\sim$ on S by $\forall v,w\in S$ and $p,q\in Q, v\sim w \Leftrightarrow \mu_{\underline{D}\tilde{\delta}}(q,v,p) = \mu_{\underline{D}\tilde{\delta}}(q,w,p)$ and $\mu_{\overline{D}\tilde{\delta}}(q,v,p) = \mu_{\overline{D}\tilde{\delta}}(q,w,p)$. Then it can be easily seen that $\sim$ is an equivalence relation on S. Also, let $v,w,t\in S$ and $v\sim w$. Then $\mu_{\underline{D}\tilde{\lambda}}(q,vt,p) = \vee_{r\in Q}\{\mu_{\underline{D}\tilde{\lambda}}(q,v,r)\wedge\mu_{\underline{D}\tilde{\lambda}}(r,t,p)\} = \vee_{r\in Q}\{\mu_{\underline{D}\tilde{\lambda}}(q,w,r)\wedge \mu_{\underline{D}\tilde{\lambda}}(r,t,p)\} = \mu_{\underline{D}\tilde{\lambda}}(q,wt,p)$. Similarly, we have $\mu_{\overline{D}\tilde{\lambda}}(q,vt,p) = \mu_{\overline{D}\tilde{\lambda}}(q,wt,p)$. Thus $vt\sim wt$. Similarly $tv\sim tw$. Hence $\sim$ is a congruence relation on S.

Let $[v]$ be the equivalence class of v induced by the relation $\sim$ and $S/\sim = \{[v] : v\in S\}$. Define $\tilde{\lambda}' : (Q\times S/\sim \times Q)\times J_x \to [0,1]$ by $\mu_{\underline{D}\tilde{\lambda}'}(q,[x],p) = \mu_{\underline{D}\tilde{\lambda}}(q,x,p)$ and $\mu_{\overline{D}\tilde{\lambda}'}(q,[x],p) = \mu_{\overline{D}\tilde{\lambda}}(q,x,p), \forall p,q\in Q$ and $\forall [x]\in S/\sim$. Now
$\mu_{\underline{D}\tilde{\lambda}'}(q,[e],p) = \mu_{\overline{D}\tilde{\lambda}'}(q,[e],p) = \begin{cases} 1 & \text{if } q=p \\ 0 & \text{if } q\neq p\end{cases}$
Also, $\mu_{\underline{D}\tilde{\lambda}'}(q,[x][y],p) = \mu_{\underline{D}\tilde{\lambda}'}(q,[xy],p) = \mu_{\underline{D}\tilde{\lambda}}(q,xy,p) = \vee_{r\in Q}\{\mu_{\underline{D}\tilde{\lambda}}(q,x,r)\wedge\mu_{\underline{D}\tilde{\lambda}}(r,y,p)\} = \vee_{r\in Q}\{\mu_{\underline{D}\tilde{\lambda}'}(q,[x],r)\wedge \mu_{\underline{D}\tilde{\lambda}'}(r,[y],p)\}$. Similarly, we have $\mu_{\overline{D}\tilde{\lambda}'}(q,[x][y],p) = \mu_{\overline{D}\tilde{\lambda}'}(q,[xy],p) = \vee_{r\in Q}\{\mu_{\overline{D}\tilde{\lambda}'}(q,[x],r)\wedge\mu_{\overline{D}\tilde{\lambda}'}(r,[y],p)\}, \forall [x],[y] \in S/\sim$. Again, let $\mu_{\underline{D}\tilde{\lambda}'}(q,[x],p) = \mu_{\underline{D}\tilde{\lambda}'}(q,[y],p)$ and $\mu_{\overline{D}\tilde{\lambda}'}(q,[x],p) = \mu_{\overline{D}\tilde{\lambda}'}(q,[y],p), \forall p,q\in Q$. Then $\mu_{\underline{D}\tilde{\lambda}}(q,x,p) = \mu_{\underline{D}\tilde{\lambda}}(q,y,p)$ and $\mu_{\overline{D}\tilde{\lambda}}(q,x,p) = \mu_{\overline{D}\tilde{\lambda}}(q,y,p), \forall p,q\in Q$. Thus $x\sim y$, whereby $[x]=[y]$ showing that $(Q,S/\sim,\tilde{\lambda}',J_x)$ is a faithful T2FTS.

Proposition 4.6. *Let $\widetilde{M} = (Q,X,\tilde{\delta},J_x)$ be a type-2 fuzzy finite state automaton. Then $(Q,E(\widetilde{M}),\tilde{\lambda},J_{[v]})$ is a faithful T2FTS, where $\mu_{\underline{D}\tilde{\lambda}}(q,[x],p) = \mu_{\underline{D}\tilde{\delta}^*}(q,x,p)$ and $\mu_{\overline{D}\tilde{\lambda}}(q,[x],p) = \mu_{\overline{D}\tilde{\delta}^*}(q,x,p), \forall p,q\in Q$ and $\forall x\in X^*$.*

Proof. $E(\widetilde{M})$ is a finite semigroup with identity $[e]$ from Proposition 4.2. Obviously,

$$\mu_{\underline{D}\tilde{\lambda}}(q,[e],p) = \mu_{\overline{D}\tilde{\lambda}}(q,[e],p) = \begin{cases} \mu_{\underline{D}\tilde{\delta}^*}(q,e,p) = \mu_{\overline{D}\tilde{\delta}^*}(q,e,p) = 1 & \text{if } q=p \\ \mu_{\underline{D}\tilde{\delta}^*}(q,e,p) = \mu_{\overline{D}\tilde{\delta}^*}(q,e,p) = 0 & \text{if } q\neq p\end{cases}$$

Let $q\in Q$ and $[x],[y]\in E(\widetilde{M})$. Then $\mu_{\underline{D}\tilde{\lambda}}(q,[x]*[y],p) = \mu_{\underline{D}\tilde{\lambda}}(q,[xy],p) = \mu_{\underline{D}\tilde{\delta}^*}(q,xy,p) = \vee_{r\in Q}\{\mu_{\underline{D}\tilde{\delta}^*}(q,x,r)\wedge \mu_{\underline{D}\tilde{\delta}^*}(r,y,p)\} = \vee_{r\in Q}\{\mu_{\underline{D}\tilde{\lambda}}(q,[x],r)\wedge\mu_{\underline{D}\tilde{\lambda}}(r,[y],p)\}$. Similarly, we have $\mu_{\overline{D}\tilde{\lambda}}(q,[x]*[y],p) = \mu_{\overline{D}\tilde{\lambda}}(q,[xy],p) = \vee_{r\in Q}\{ \mu_{\overline{D}\tilde{\lambda}}(q,[x],r)\wedge\mu_{\overline{D}\tilde{\lambda}}(r,[y],p)\}$. Also, let $\mu_{\underline{D}\tilde{\lambda}}(q,[x],p) = \mu_{\underline{D}\tilde{\lambda}}(q,[y],p)$ and $\mu_{\overline{D}\tilde{\lambda}}(q,[x],p) = \mu_{\overline{D}\tilde{\lambda}}(q,[y],p)$, i.e., $\mu_{\underline{D}\tilde{\delta}^*}(q,x ,p) = \mu_{\underline{D}\tilde{\delta}^*}(q,y,p)$ and $\mu_{\overline{D}\tilde{\delta}^*}(q,x,p) = \mu_{\overline{D}\tilde{\delta}^*}(q,y,p), \forall q\in Q$. Thus $x\sim y$ or $[x]=[y]$. Hence $(Q,E(\widetilde{M}),\tilde{\lambda},J_{[x]})$ is a faithful T2FTS. □

For type-2 fuzzy finite state automaton $\widetilde{M} = (Q,X,\tilde{\delta},J_x)$, we shall denote by T2FTS $(\widetilde{M})$, the T2FTS $(Q,E(\widetilde{M}),\tilde{\lambda},J_{[v]})$, and call it the T2FTS associated with $\widetilde{M}$.

Definition 4.3. *A homomorphism from a T2FTS $\tilde{A}_1 = (Q_1,S_1,\tilde{\lambda}_1,J_{v_1})$ to T2FTS $\tilde{A}_2 = (Q_2,S_2,\tilde{\lambda}_2,J_{v_2})$ is a pair of maps (α,β), where $\alpha : Q_1\to Q_2$ and $\beta : S_1\to S_2$ are functions such that*
(1) $\beta(uv) = \beta(u)\beta(v), \forall u,v\in S_1$,
(2) If S_1 and S_2 contain the identity e_1 and e_2 respectively, then $\beta(e_1) = e_2$, and
(3) $\mu_{\underline{D}\tilde{\lambda}_1}(q,v,p) \le \mu_{\underline{D}\tilde{\lambda}_2}(\alpha(q),\beta(v),\alpha(p))$. and
$\mu_{\overline{D}\tilde{\lambda}_1}(q,v,p) \le \mu_{\overline{D}\tilde{\lambda}_2}(\alpha(q),\beta(v),\alpha(p)), \forall p,q\in Q_1, \forall v\in S_1$.

A homomorphism $(\alpha,\beta) : \tilde{A}_1\to\tilde{A}_2$ is called an isomorphism if α and β are both one-one and onto.
Let S be a semigroup with identity e and $(Q,S,\tilde{\lambda},J_v)$ be a faithful T2FTS. Define type-2 fuzzy finite state automaton $\widetilde{M} = (Q,X,\tilde{\delta},J_x)$ by taking $\tilde{\delta} = \tilde{\lambda}$. Consider type-2 fuzzy finite state automaton$(\widetilde{M}) = (Q,E(\widetilde{M}),\rho,J_{[v]})$, where $E(\widetilde{M}) = S^*/\sim$ with $\mu_{\underline{D}\rho}(q,[v],p) = \mu_{\underline{D}\tilde{\delta}^*}(q,v,p)$ and $\mu_{\overline{D}\rho}(q,[v],p) = \mu_{\overline{D}\tilde{\delta}^*}(q,v,p)$. Now, for all $p,q\in Q$,

$\mu_{\underline{D}\rho}(q,[e],p) = \mu_{\overline{D}\rho}(q,[e],p)$

$$= \begin{cases} \mu_{\underline{D}\tilde{\delta}^*}(q,e,p) = \mu_{\overline{D}\tilde{\delta}^*}(q,e,p) = 1 & \text{if } q = p \\ \mu_{\underline{D}\tilde{\delta}^*}(q,e,p) = \mu_{\overline{D}\tilde{\delta}^*}(q,e,p) = 0 & \text{if } q \neq p \end{cases}$$

Hence $\mu_{\underline{D}\rho}(q,[e],p) = \mu_{\underline{D}\rho}(q,[\tilde{\lambda}],p)$ and $\mu_{\overline{D}\rho}(q,[e],p) = \mu_{\overline{D}\rho}(q,[\tilde{\lambda}],p)$, where $\tilde{\lambda}$ is the empty word in S^*. Thus $[e] = [\tilde{\lambda}]$.

Proposition 4.7. *Let $\widetilde{M} = (Q,X,\tilde{\delta},J_x)$ be a type-2 fuzzy finite state automata and S be a semigroup with identity e. Then T2FTS $(\widetilde{M})$ is isomorphic to a faithful T2FTS $\tilde{A} = (Q,S,\tilde{\lambda},J_v)$.*

Proof. Let $\alpha : Q \to Q$ and $\beta : S \to E(\widetilde{M})$ be maps such that $\alpha(q) = q$ and $\beta(v) = [v], \forall [v] \in S$ and $\forall q \in Q$. Let $\bullet$ be the binary operation of S and for $a,b \in S, a\bullet b \in S, ab \in S^*$. Then $\mu_{\underline{D}\tilde{\delta}^*}(q,a\bullet b,p) = \mu_{\underline{D}\tilde{\delta}}(q,a\bullet b,p) = \mu_{\underline{D}\tilde{\lambda}}(q,a\bullet b,p) = \vee_{r\in Q}\{\mu_{\underline{D}\tilde{\lambda}}(q,a,r)\wedge\mu_{\underline{D}\tilde{\lambda}}(r,b,p)\} = \vee_{r\in Q}\{\mu_{\underline{D}\tilde{\delta}}(q,a,r)\wedge\mu_{\underline{D}\tilde{\delta}}(r,b,p)\} = \mu_{\underline{D}\tilde{\delta}^*}(q,ab,p)$. Similarly, we have $\mu_{\overline{D}\tilde{\delta}^*}(q,a\bullet b,p) = \mu_{\overline{D}\tilde{\delta}^*}(q,ab,p), \forall p,q \in Q$. Thus $[a\bullet b] = [ab]$, showing that $\beta(a\bullet b) = [a\bullet b] = [ab] = [a][b] = \beta(a)\beta(b)$. Also, $\mu_{\underline{D}\rho}(\alpha(q),\beta(v),\alpha(p)) = \mu_{\underline{D}\rho}(q,[v],p) = \mu_{\underline{D}\tilde{\delta}^*}(q,v,p) = \mu_{\underline{D}\tilde{\delta}}(q,v,p) = \mu_{\underline{D}\tilde{\lambda}}(q,v,p)$. Similarly, we have $\mu_{\overline{D}\rho}(\alpha(q),\beta(v),\alpha(p)) = \mu_{\overline{D}\tilde{\lambda}}(q,v,p)$. Now it remains to show that β is one-one and onto. Let $v,w \in S$ be such that $\beta(v) = \beta(w)$. Then $[v] = [w]$. Thus $\mu_{\underline{D}\tilde{\delta}^*}(q,v,p) = \mu_{\underline{D}\tilde{\delta}^*}(q,w,p)$, and $\mu_{\overline{D}\tilde{\delta}^*}(q,v,p) = \mu_{\overline{D}\tilde{\delta}^*}(q,w,p)$ or that $\mu_{\underline{D}\tilde{\delta}}(q,v,p) = \mu_{\underline{D}\tilde{\delta}}(q,w,p) \Rightarrow \mu_{\underline{D}\tilde{\lambda}}(q,v,p) = \mu_{\underline{D}\tilde{\lambda}}(q,w,p)$ and $\mu_{\overline{D}\tilde{\delta}}(q,v,p) = \mu_{\overline{D}\tilde{\delta}}(q,w,p) \Rightarrow \mu_{\overline{D}\tilde{\lambda}}(q,v,p) = \mu_{\overline{D}\tilde{\lambda}}(q,w,p)$, or that $v = w$. As $\tilde{A}$ is faithful. Thus β is one-one. Also, it can be easily seen that if $c_i \in S, i \in [1,n]$, then by induction $[c_1\bullet c_2\bullet .....\bullet c_n] = [c_1c_2.....c_n]$. Finally, let $[x] \in E(M)$. If $x = \tilde{\lambda}$ then $[\tilde{\lambda}] = [e]$ and $\beta(e) = [\tilde{\lambda}]$. Let $x = a_1a_2.....a_n, a_i \in S, i \in [1,n]$. Then $\beta(a_1\bullet a_2\bullet .....\bullet a_n) = [a_1\bullet a_2\bullet .....\bullet a_n] = [a_1a_2.....a_n] = [x]$. Thus β is onto. ☐

Definition 4.4. *Let $\widetilde{M_1} = (Q_1,X_1,\tilde{\delta}_1,J_{x_1})$ and $\widetilde{M_2} = (Q_2,X_2,\tilde{\delta}_2,J_{x_2})$ be a type-2 fuzzy finite state automata. Then the type-2 fuzzy finite state automata $\widetilde{M_1}\times\widetilde{M_2} = (Q_1\times Q_2, X_1\times X_2, \tilde{\delta}_1\times\tilde{\delta}_2, J_{(x_1,x_2)})$ is called* **(full) direct product** *of $\widetilde{M_1}$ and $\widetilde{M_2}$, where $\tilde{\delta}_1\times\tilde{\delta}_2 : ((Q_1\times Q_2)\times(X_1\times X_2)\times(Q_1\times Q_2))\times J_{(x_1,x_2)} \to [0,1]$, is defined as follows:*
$\mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(x_1,x_2),(p_1,p_2)) = \mu_{\underline{D}\tilde{\delta}_1}(q_1,x_1,p_1)\wedge\mu_{\underline{D}\tilde{\delta}_2}(q_2,x_2,p_2)$, and $\mu_{\overline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(x_1,x_2),(p_1,p_2)) = \mu_{\overline{D}\tilde{\delta}_1}(q_1,x_1,p_1)\wedge\mu_{\overline{D}\tilde{\delta}_2}(q_2,x_2,p_2), \forall (q_1,q_2),(p_1,p_2) \in (Q_1\times Q_2)$ and $\forall (x_1,x_2) \in X_1\times X_2$.

An examination of the 'categorical product' in the category **T2FFSA** leads to a concept of 'direct product' of type-2 fuzzy finite state automata, which we illustrate here for two type-2 fuzzy finite state automata $\widetilde{M_1} = (Q_1,X_1,\tilde{\delta}_1,J_{x_1})$ and $\widetilde{M_2} = (Q_2,X_2,\tilde{\delta}_2,J_{x_2})$ as follows.
($X_1\times X_2$, appearing below is the 'direct product' of the monoids X_1 and X_2. Thus it is the cartesian product of X_1 and X_2, considered as a monoid, whose binary operation is defined as $(x_1,x_2)(x_1',x_2') = (x_1x_1',x_2x_2')$, for $(x_1,x_2),(x_1',x_2') \in X_1\times X_2$, and whose identity element is (e_{X_1},e_{X_2}), where e_{X_1} and e_{X_2} are the identities of X_1 and X_2 respectively.)
Define Let $\widetilde{M_1} = (Q_1,X_1,\tilde{\delta}_1,J_{x_1})$ and $\widetilde{M_2} = (Q_2,X_2,\tilde{\delta}_2,J_{x_2})$ be a type-2 fuzzy finite state automata. Then the direct product of $\widetilde{M_1}$ and $\widetilde{M_2}$ is $\widetilde{M_1}\times\widetilde{M_2} = (Q_1\times Q_2, X_1\times X_2, \tilde{\delta}_1\times\tilde{\delta}_2, J_{(x_1,x_2)})$. Then

$$\tilde{\delta}_1\times\tilde{\delta}_2 : ((Q_1\times Q_2)\times(X_1\times X_2)\times(Q_1\times Q_2))\times J_{(x_1,x_2)} \to [0,1]$$

as

$$\mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(x_1,x_2),(p_1,p_2)) = \mu_{\underline{D}\tilde{\delta}_1}(q_1,x_1,p_1)\wedge\mu_{\underline{D}\tilde{\delta}_2}(q_2,x_2,p_2), \text{ and}$$
$$\mu_{\overline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(x_1,x_2),(p_1,p_2)) = \mu_{\overline{D}\tilde{\delta}_1}(q_1,x_1,p_1)\wedge\mu_{\overline{D}\tilde{\delta}_2}(q_2,x_2,p_2)$$

$\forall (q_1,q_2),(p_1,p_2) \in (Q_1\times Q_2)$ and $\forall (x_1,x_2) \in X_1\times X_2$.
Then
$\mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(e_{x_1},e_{x_2}),(p_1,p_2)) = \mu_{\overline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(e_{x_1},e_{x_2}),(p_1,p_2)) = \begin{cases} 1 & \text{if } (q_1,q_2) = (p_1,p_2) \\ 0 & \text{if } (q_1,q_2) \neq (p_1,p_2). \end{cases}$
Next, for $(x_1',x_2') \in X_1\times X_2$,
$\mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(x_1,x_2)(x_1',x_2'),(p_1,p_2)) = \mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(x_1x_1',x_2x_2'),$
$(p_1,p_2)) = \mu_{\underline{D}\tilde{\delta}_1}(q_1,x_1x_1',p_1)\wedge\mu_{\underline{D}\tilde{\delta}_2}(q_2,x_2x_2',p_2) = [\vee\{\mu_{\underline{D}\tilde{\delta}_1}(q_1,x_1,s)\wedge\mu_{\underline{D}\tilde{\delta}_1}(s,x_1',p_1) : s \in Q_1\}]\wedge[\vee\{\mu_{\underline{D}\tilde{\delta}_2}(q_2,x_2,t)$

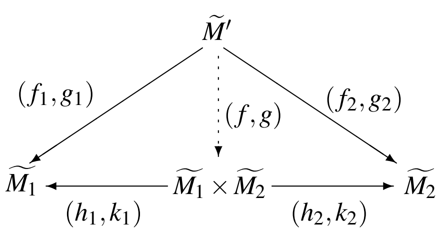
$\wedge\, \mu_{\underline{D}\tilde{\delta}_2}(t,x_2',p_2) : t \in Q_2\}] = \vee\{(\mu_{\underline{D}\tilde{\delta}_1}(q_1,x_1,s) \wedge \mu_{\underline{D}\tilde{\delta}_1}(s,x_1',p_1)) \wedge (\mu_{\underline{D}\tilde{\delta}_2}(q_2,x_2,t) \wedge \mu_{\underline{D}\tilde{\delta}_2}(t,x_2',p_2)) : s \in Q_1, t \in Q_2\}$.
On the other hand,
$\vee\{\mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(x_1,x_2),(s,t)) \wedge \mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((s,t),(x_1',x_2'),(p_1,p_2)) : s \in Q_1, t \in Q_2\} = \vee\{(\mu_{\underline{D}\tilde{\delta}_1}(q_1,x_1,s) \wedge \mu_{\underline{D}\tilde{\delta}_2}(q_2,x_2,t)) \wedge (\mu_{\underline{D}\tilde{\delta}_1}(s,x_1',p_1) \wedge \mu_{\underline{D}\tilde{\delta}_1}(t,x_2',p_2)) : s \in Q_1, t \in Q_2\} = \vee\{(\mu_{\underline{D}\tilde{\delta}_1}(q_1,x_1,s) \wedge \mu_{\underline{D}\tilde{\delta}_1}(s,x_1',p_1)) \wedge (\mu_{\underline{D}\tilde{\delta}_2}(q_2,x_2,t) \wedge \mu_{\underline{D}\tilde{\delta}_1}(t,x_2',p_2)) : s \in Q_1, t \in Q_2\}$. Thus, $\mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(x_1,x_2)(x_1',x_2'),(p_1,p_2)) = \vee\{\mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(x_1,x_2),(s,t)) \wedge \mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((s,t),(x_1',x_2'),(p_1,p_2)) : s \in Q_1, t \in Q_2\}$. Similarly, we can derive $\mu_{\overline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(x_1,x_2)(x_1',x_2'),(p_1,p_2)) = \vee\{\mu_{\overline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(x_1,x_2),(s,t)) \wedge \mu_{\overline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((s,t),(x_1',x_2'),(p_1,p_2)) : s \in Q_1, t \in Q_2\}$

This shows that $(Q_1 \times Q_2, X_1 \times X_2, \tilde{\delta}_1 \times \tilde{\delta}_2, J_{(x_1,x_2)})$ is a type-2 fuzzy finite state automata, which we shall refer to as the *direct product* of the type-2 fuzzy finite state automata $\widetilde{M_1} = (Q_1,X_1,\tilde{\delta}_1,J_{x_1})$ and $\widetilde{M_2} = (Q_2,X_2,\tilde{\delta}_2,J_{x_2})$ and will denote it as $\widetilde{M_1} \times \widetilde{M_2}$.

Remark 4.1. *This direct product may be interpreted as the 'parallel composition' of $\widetilde{M_1}$ and $\widetilde{M_2}$ cf., e.g., Dörfler* [1].

Proposition 4.8. *The direct product $\widetilde{M_1} \times \widetilde{M_2}$ of $\widetilde{M_1}, \widetilde{M_2} \in$* **T2FFSA** *is the categorical direct product in* **T2FFSA**.

Proof. We first need to identify the two 'projection morphisms' from $\widetilde{M_1} \times \widetilde{M_2}$ to $\widetilde{M_1}$ and $\widetilde{M_2}$ in **T2FFSA**. Let $\widetilde{M_1} = (Q_1,X_1,\tilde{\delta}_1,J_{x_1})$ and $\widetilde{M_2} = (Q_2,X_2,\tilde{\delta}_2,J_{x_2})$. Let $h_1 : Q_1 \times Q_2 \to Q_1, h_2 : Q_1 \times Q_2 \to Q_2, k_1 : X_1 \times X_2 \to X_1$ and $k_2 : X_1 \times X_2 \to X_2$ be the projection maps associated with the cartesian products $Q_1 \times Q_2$ and $X_1 \times X_2$. We show that $(h_1,k_1) : \widetilde{M_1} \times \widetilde{M_2} \to \widetilde{M_1}$ and $(h_2,k_2) : \widetilde{M_1} \times \widetilde{M_2} \to \widetilde{M_2}$ are **T2FFSA**-morphisms. Let $((q_1,q_2),(x_1,x_2),(p_1,p_2),u) \in ((Q_1 \times Q_2) \times (X_1 \times X_2) \times (Q_1 \times Q_2) \times J_{(x_1,x_2)}$. Then $\mu_{\underline{D}\tilde{\delta}_1}(h_1(q_1,q_2),k_1(x_1,x_2),h_1(p_1,p_2)) = \mu_{\underline{D}\tilde{\delta}_1}(q_1,x_1,p_1) \geq \mu_{\underline{D}\tilde{\delta}_1}(q_1,x_1,p_1) \wedge \mu_{\underline{D}\tilde{\delta}_2}(q_2,x_2,p_2) = \mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(x_1,x_2),(p_1,p_2))$. Similarly, we have derived $\mu_{\overline{D}\tilde{\delta}_1}(h_1(q_1,q_2),k_1(x_1,x_2),h_1(p_1,p_2)) \geq \mu_{\overline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((q_1,q_2),(x_1,x_2),(p_1,p_2))$. Thus, $\tilde{\delta}_1(h_1(q_1,q_2),k_1(x_1,x_2),h_1(p_1,p_2)) \geq (\tilde{\delta}_1 \times \tilde{\delta}_2)((q_1,q_2),(x_1,x_2),(p_1,p_2)), \forall((q_1,q_2),(x_1,x_2),(p_1,p_2)) \in (Q_1 \times Q_2) \times (X_1 \times X_2) \times (Q_1 \times Q_2)$, whereby $(h_1,k_1) : \widetilde{M_1} \times \widetilde{M_2} \to \widetilde{M_1}$ is an **T2FFSA**-morphism. Similarly, $(h_2,k_2) : \widetilde{M_1} \times \widetilde{M_2} \to \widetilde{M_2}$ can be seen to be an **T2FFSA**-morphism. Next, let $\widetilde{M'} = (Q',X',\tilde{\delta}',J_{x'}) \in$ **T2FFSA** and two **T2FFSA**-morphisms $(f_1,g_1) : \widetilde{M'} \to \widetilde{M_1}$ and $(f_2,g_2) : \widetilde{M'} \to \widetilde{M_2}$ be given. We show that there exists a unique **T2FFSA**-morphism $(f,g) : \widetilde{M'} \to \widetilde{M_1} \times \widetilde{M_2}$ such that the following diagram commutes.



Here, $(h_1,k_1) : \widetilde{M_1} \times \widetilde{M_2} \to \widetilde{M_1}$ and $(h_2,k_2) : \widetilde{M_1} \times \widetilde{M_2} \to \widetilde{M_2}$ are the projection maps. We choose the f and g in following way.
Let $f : Q' \to Q_1 \times Q_2$ and $g : X' \to X_1 \times X_2$ be the maps given by $f(q') = (f_1(q'), f_2(q'))$ and $g(x') = (g_1(x'), g_2(x'))$, $\forall(q',x') \in Q' \times X'$. Let $((q',x',p'),u') \in ((Q' \times X' \times Q') \times J_{x'})$. As both (f_1,g_1) and (f_2,g_2) are **T2FFSA**-morphisms, $\mu_{\underline{D}\tilde{\delta}_1}(f_1(q'),g_1(x'),f_1(p')) \geq \mu_{\underline{D}\tilde{\delta}'}(q',x',p')$ and $\mu_{\overline{D}\tilde{\delta}_1}(f_1(q'),g_1(x'),f_1(p')) \geq \mu_{\overline{D}\tilde{\delta}'}(q',x',p')$ also $\mu_{\underline{D}\tilde{\delta}_2}(f_2(q'),g_2(x'),f_2(p')) \geq \mu_{\underline{D}\tilde{\delta}'}(q',x',p')$ and $\mu_{\overline{D}\tilde{\delta}_2}(f_2(q'),g_2(x'),f_2(p')) \geq \mu_{\overline{D}\tilde{\delta}'}(q',x',p')$, whereby $\mu_{\underline{D}\tilde{\delta}_1}(f_1(q'),g_1(x'),f_1(p')) \wedge \mu_{\underline{D}\tilde{\delta}_2}(f_2(q'),g_2(x'),f_2(p')) \geq \mu_{\underline{D}\tilde{\delta}'}(q',x',p')$ and $\mu_{\overline{D}\tilde{\delta}_1}(f_1(q'),g_1(x'),f_1(p')) \wedge \mu_{\overline{D}\tilde{\delta}_2}(f_2(q'),g_2(x'),f_2(p')) \geq \mu_{\overline{D}\tilde{\delta}'}(q',x',p')$.
Thus, $\mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}(f(q'),g(x'),f(p')) = \mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}((f_1(q'),f_2(q')),(g_1(x'),g_2(x')),(f_1(p'),f_2(p'))) = \mu_{\underline{D}\tilde{\delta}_1}(f_1(q'),g_1(x'),f_1(p')) \wedge \mu_{\underline{D}\tilde{\delta}_2}(f_2(q'),g_2(x'),f_2(p')) \geq \mu_{\underline{D}\tilde{\delta}'}(q',x',p')$. Similarly, we can derive $\mu_{\underline{D}(\tilde{\delta}_1\times\tilde{\delta}_2)}(f(q'),g(x'),f(p')) \geq \mu_{\underline{D}\tilde{\delta}'}(q',x',p')$. Hence (f,g) is an **T2FFSA**-morphism. Also, the definitions of f and g are such that we obviously have $(h_1,k_1) \circ (f,g) = (f_1,g_1)$ and $(h_2,k_2) \circ (f,g) = (f_2,g_2)$.
To prove the uniqueness of (f,g), let there exist another **T2FFSA**-morphism $(f',g') : \widetilde{M'} \to \widetilde{M_1} \times \widetilde{M_2}$ such that

$(h_1,k_1)\circ(f',g') = (f_1,g_1)$ and $(h_2,k_2)\circ(f',g') = (f_2,g_2)$, i.e., $h_1\circ f' = f_1, k_1\circ g' = g_1, h_2\circ f' = f_2$, and $k_2\circ g' = g_2$. We then have $h_1\circ f' = h_1\circ f, k_1\circ g' = k_1\circ g, h_2\circ f' = h_2\circ f$, and $k_2\circ g' = k_2\circ g$, whereby $f = f'$ and $g = g'$. Thus $(f',g') = (f,g)$, proving the uniqueness of (f,g). Hence the direct product $\widetilde{M_1}\times\widetilde{M_2}$ is the categorical direct product. □

Let $\widetilde{M_1} = (Q_1,X_1,\tilde{\delta}_1,J_{x_1})$ and $\widetilde{M_2} = (Q_2,X_2,\tilde{\delta}_2,J_{x_2})$ be a type-2 fuzzy finite state automata. Let $\overline{X}$ be a finite set and $f:\overline{X}\to X_1\times X_2$ be a map. Also, let $\prod_1$ and $\prod_2$ be the projection mapping of $X_1\times X_2$ onto X_1 and X_2 respectively, i.e., $\prod_1 : X_1\times X_2\to X_1$ and $\prod_2 : X_1\times X_2\to X_2$. Then the concept of generalized direct product of type-2 fuzzy finite state automata are given below.

Definition 4.5. *Let $\widetilde{M_1} = (Q_1,\overline{X},\tilde{\delta}_1,J_x)$ and $\widetilde{M_2} = (Q_2,\overline{X},\tilde{\delta}_2,J_x)$ be a type-2 fuzzy finite state automata. Then the type-2 fuzzy finite state automata $\widetilde{M_1}*\widetilde{M_2} = (Q_1\times Q_2,\overline{X},\tilde{\delta}_1*\tilde{\delta}_2,J_x)$ is called* **general direct product** *of $\widetilde{M_1}$ and $\widetilde{M_2}$, where $\tilde{\delta}_1*\tilde{\delta}_2 : ((Q_1\times Q_2)\times\overline{X}\times(Q_1\times Q_2))\times J_x\to[0,1]$, is defined as follows:*
$\mu_{\underline{D}(\tilde{\delta}_1*\tilde{\delta}_2)}((q_1,q_2),x,(p_1,p_2)) = \mu_{\underline{D}(\tilde{\delta}_1*\tilde{\delta}_2)}((q_1,q_2),(\prod_1(f(x)),\prod_2(f(x))),(p_1,p_2)) = \mu_{\underline{D}\tilde{\delta}_1}(q_1,\prod_1(f(x)),p_1)\wedge\mu_{\underline{D}\tilde{\delta}_2}(q_2,\prod_2(f(x)),p_2)$*, and*
$\mu_{\overline{D}(\tilde{\delta}_1*\tilde{\delta}_2)}((q_1,q_2),x,(p_1,p_2)) = \mu_{\overline{D}(\tilde{\delta}_1*\tilde{\delta}_2)}((q_1,q_2),(\prod_1(f(x)),\prod_2(f(x))),(p_1,p_2)) = \mu_{\overline{D}\tilde{\delta}_1}(q_1,\prod_1(f(x)),p_1)\wedge\mu_{\overline{D}\tilde{\delta}_2}(q_2,\prod_2(f(x)),p_2),\forall(q_1,q_2),(p_1,p_2)\in(Q_1\times Q_2),\forall x\in\overline{X}$.

Remark 4.2. *(1). If $\overline{X} = X_1\times X_2$ and f is the identity map, then the general direct product $\widetilde{M_1}\times\widetilde{M_2}$ reduces to the (full) direct product.*

(2). As, from above Proposition 4.8. If $\overline{X} = X_1\times X_2$ and f is the identity map, then the general direct product $\widetilde{M_1}\times\widetilde{M_2}$ is also, the categorical general direct product.

5 Conclusion

Chiefly inspired from [6] and [7]. We have introduced and studied here the concept of type-2 fuzzy finite state automata, homomorphism between two type-2 fuzzy finite state automata and transformation semigroup associated with a type-2 fuzzy finite state automata. Finally, we have introduced the concept of product of type-2 fuzzy finite state automata. We hope that, like fuzzy finite state machine, rough finite state machine, type-2 fuzzy finite state automata which is another dimension of applications of type-2 fuzzy set theory, will attract the researchers began to work on type-2 fuzzy finite state automata and finding more successful applications of type-2 fuzzy finite state automata.

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