Fuzzy subgroups on direct product of groups over a \( t \)-norm

Rasul Rasuli*

Mathematics Department, Faculty of Science Payame Noor University (PNU), Tehran, Iran.

Copyright 2017 © Rasul Rasuli. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, fuzzy subgroups on direct product of groups over a \( t \)-norm has been discussed. By using a \( t \)-norm \( T \), we characterize some basic properties of \( T \)-fuzzy direct product of groups and normal \( T \)-fuzzy direct product of groups. Also we define the concept normal subgroups between \( T \)-fuzzy direct product of groups and prove some basic properties.

Keywords: Group theory, Norms, Fuzzy set theory.

1 Introduction

The notion of a fuzzy subset of a set is due to Lotfi Zadeh ([11]). At present this concept has been applied to many mathematical branches, such as group, functional analysis, probability theory, topology, and so on. The notion of fuzzy subgroup was introduced by A. Rosenfeld et.al [5], [10] in his pioneering paper. Many authors [2], [3], [6], [7], [8] applied the concept of fuzzy sets for studies in fuzzy semigroups, fuzzy groups, fuzzy rings, fuzzy ideals, fuzzy semirings and fuzzy near-rings and so on. In fact many basic properties in group theory are found to be carried over to fuzzy groups. In 1979 Anthony and Sherwood [4] redefined a fuzzy subgroup of a group using the concept of triangular norm \( (t \)-norm, for short). In this paper, we use a \( t \)-norm \( T \) to introduce the notion of \( T \)-fuzzy direct product of groups and normal \( T \)-fuzzy direct product of groups, and investigate some of their properties.

2 \( T \)-fuzzy groups and normal \( T \)-fuzzy subgroups on direct product of groups

Definition 2.1. Let \( G_1, G_2 \) be two arbitrary groups with a multiplicative binary operations and identities \( e_1, e_2 \) respectively. A fuzzy subset of \( G_1 \times G_2 \), we mean a function from \( G_1 \times G_2 \) into \([0,1]\). The set of all fuzzy subsets of \( G_1 \times G_2 \) is called the \([0,1]\)-power set of \( G_1 \times G_2 \) and is denoted \([0,1]^{G_1 \times G_2}\).

Definition 2.2. ([1]) A \( t \)-norm \( T \) is a function \( T : [0,1] \times [0,1] \to [0,1] \) having the following four properties:

\( T \) is a neutral element,

\( T \) is monotonicity,

\( T \) is commutativity,

\( T \) is associativity,

for all \( x, y, z \in [0,1] \).

*Corresponding author. Email address: rasulirasul@yahoo.com
We say that $T$ is idempotent if for all $x \in [0,1], T(x,x) = x$.

**Example 2.1.** The basic $t$-norms are $T_m(x,y) = \min\{x,y\}, T_b(x,y) = \max\{0,x+y-1\}$ and $T_p(x,y) = xy$, which are called standard intersection, bounded sum and algebraic product respectively.

**Definition 2.3.** Let $\mu$ be a fuzzy subset of a group $G_1 \times G_2$. Then $\mu$ is called a fuzzy subgroup of $G_1 \times G_2$ under a $t$-norm $T$ ($T$-fuzzy subgroup) iff for all $(x_1,y_1),(x_2,y_2) \in G_1 \times G_2$ $(TFI)$ $\mu((x_1,y_1)\cdot(x_2,y_2)) \geq T(\mu(x_1,y_1),\mu(x_2,y_2))$

$(TF2)$ $\mu(x_1,y_1)^{-1} \geq \mu(x_1,y_1)$.

Denote by $TF(G_1 \times G_2)$, the set of all $T$-fuzzy subgroups of $G_1 \times G_2$.

**Example 2.2.** Let $Z_2 = \{0,1\}, Z_3 = \{0,1,2\}$ be two additive groups. Then $Z_2 \times Z_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$. Define fuzzy set $\mu$ in $Z_2 \times Z_3$ by $\mu(0,0) = 0.8, \mu(1,0) = 0.7, \mu(0,2) = \mu(0,1) = 0.6$, $\mu(1,1) = \mu(1,2) = 0.5$. If $T(x,y) = T_b(x,y) = \max\{0,x+y-1\}$, then $\mu \in TF(Z_2 \times Z_3)$.

**Definition 2.4.** Let $\mu_1, \mu_2 \in TF(G_1 \times G_2)$ and $(x_1,y_1) \in G_1 \times G_2$. We define

(1) $\mu_1 \subseteq \mu_2$ iff $\mu_1(x_1,y_1) \leq \mu_2(x_1,y_1)$,

(2) $\mu_1 \supseteq \mu_2$ iff $\mu_1(x_1,y_1) \geq \mu_2(x_1,y_1)$,

(3) $\mu_1 \cap \mu_2 = (\mu_1 \cap \mu_2)(x_1,y_1)$ and $\mu_1 \cap \mu_2 \cap \mu_3 = (\mu_1 \cap \mu_2) \cap \mu_3 = \mu_1 \cap (\mu_2 \cap \mu_3)$ (property (T3 and T4)).

**Lemma 2.1.** ([1]) Let $T$ be a $t$-norm. Then

$$T(T(x,y),T(w,z)) \geq T(T(x,w),T(y,z)),$$

for all $x,y,w,z \in [0,1]$.

**Proposition 2.1.** Let $\mu_1, \mu_2 \in TF(G_1 \times G_2)$. Then $\mu_1 \cap \mu_2 \in TF(G_1 \times G_2)$.

**Proof.** Let $(x_1,y_1),(x_2,y_2) \in G_1 \times G_2$.

$$(\mu_1 \cap \mu_2)((x_1,y_1)\cdot(x_2,y_2)) = T(\mu_1((x_1,y_1)\cdot(x_2,y_2)),\mu_2((x_1,y_1)\cdot(x_2,y_2))) \geq T(T(\mu_1(x_1,y_1),\mu_2(x_1,y_1)),T(\mu_1(x_2,y_2),\mu_2(x_2,y_2))) = T(\mu_1(x_1,y_1),\mu_2(x_1,y_1)) = (\mu_1 \cap \mu_2)((x_1,y_1),\mu_2(x_1,y_1)).$$

Also $(\mu_1 \cap \mu_2)^{-1} = T(\mu_1(x_1,y_1)^{-1},\mu_2(x_1,y_1)^{-1}) \geq T(\mu_1(x_1,y_1),\mu_2(x_1,y_1)) = (\mu_1 \cap \mu_2)((x_1,y_1)\cdot(x_2,y_2))$.

**Corollary 2.1.** Let $I_n = \{1,2,\ldots,n\}$. If $\{\mu_i \mid i \in I_n\} \subseteq TF(G_1 \times G_2)$, then $\mu = \cap_{i \in I_n} \mu_i \in TF(G_1 \times G_2)$.

**Example 2.3.** Let $Z_3 = \{0,1,2\}$ be an additive group.

Then $Z_3 \times Z_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$. Define fuzzy sets $\mu_1, \mu_2$ in $Z_3 \times Z_3$ by $\mu_1(0,0) = 0.9, \mu_2(0,0) = 0.8, \mu_1(0,1) = \mu_2(0,1) = 0.7, \mu_1(1,0) = \mu_2(1,0) = 0.6, \mu_1(1,1) = \mu_2(1,1) = 0.5$ and $\mu_2(0,2) = 0.7, \mu_2(1,0) = 0.6, \mu_2(1,1) = 0.5, \mu_2(2,0) = 0.4, \mu_2(2,1) = 0.3$ respectively. If $T(x,y) = T_b(x,y) = \max\{0,x+y-1\}$, then $\mu_1 \cap \mu_2 \in TF(Z_3 \times Z_3)$.

**Lemma 2.2.** Let $\mu$ be a fuzzy subset of a finite group $G_1 \times G_2$ and $T$ is idempotent. If $\mu$ satisfies condition (TF1) of Definition 2.3, then $\mu \in TF(G_1 \times G_2)$.

**Proof.** Let $(x,y) \in G_1 \times G_2$ and $(x,y) \neq (e_1,e_2)$. Since $G_1 \times G_2$ is finite, $(x,y)$ has finite order, say $n > 1$.

So $(x,y)^n = (e_1,e_2)$ and $(x,y)^{-1} = (x,y)^{n-1}$. Now by using (TF1) repeatedly, we have that $\mu((x,y)^{-1}) = \mu((x,y)^{n-1}) = \mu((x,y)^{n-2}) \geq T(\mu((x,y)^{n-1}),\mu((x,y)^{n-2})) \geq T(\mu((x,y)^{n-1}),\mu((x,y)^{n-2}),\ldots,\mu(x,y)) = \mu(x,y)$. □
Lemma 2.3. Let $\mu \in TF(G_1 \times G_2)$. If $T$ be idempotent, then for all $(x, y) \in G_1 \times G_2$, and $n \geq 1$, 
(1) $\mu(e_1, e_2) \geq \mu(x, y)$;
(2) $\mu(x, y)^n \geq \mu(x, y)$;
(3) $\mu(x, y) = \mu((x, y)^{-1})$.

Proof. Let $(x, y) \in G_1 \times G_2$ and $n \geq 1$.

(1) $\mu(e_1, e_2) = \mu((x, y)(x, y)^{-1}) \geq T(\mu(x, y), \mu((x, y)^{-1})) = T(\mu(x, y), \mu(x, y)) = \mu(x, y)$.

(2) $\mu(x, y)^n = \mu((x, y)(x, y)^{-1} \cdots (x, y)(x, y)^{-1}) \geq T(\mu(x, y), \mu(x, y), \cdots, \mu(x, y)) = \mu(x, y)$.

(3) $\mu(x, y) = \mu((x, y)^{-1})$.

Proposition 2.2. Let $\mu \in TF(G_1 \times G_2)$ and $(x_1, y_1) \in G_1 \times G_2$. If $T$ be idempotent, then $\mu((x_1, y_1)(x_2, y_2)) = \mu(x_1, y_1) \forall (x_2, y_2) \in G_1 \times G_2$ if and only if $\mu((x_1, y_1) = \mu(e_1, e_2)$.

Proof. Suppose that $\mu((x_1, y_1)(x_2, y_2)) = \mu(x_1, y_1) \forall (x_2, y_2) \in G_1 \times G_2$. Then by letting $(x_2, y_2) = (e_1, e_2)$, we get that $\mu((x_1, y_1) = \mu(e_1, e_2)$.

Conversely, suppose that $\mu((x_1, y_1) = \mu(e_1, e_2)$. By Lemma 2.3 we have $\mu((x_1, y_1)(x_2, y_2)) = \mu(x_1, y_1) \forall (x_2, y_2) \in G_1 \times G_2$. Now $\mu((x_1, y_1)(x_2, y_2)) = T(\mu(x_1, y_1), \mu(x_2, y_2)) \geq T(\mu(x_1, y_1), \mu(x_1, y_1)(x_2, y_2)) = \mu(x_1, y_1)(x_2, y_2)$.

Also $\mu(x_1, y_1)(x_2, y_2) = \mu((x_1, y_1)(x_2, y_2)) \geq T(\mu((x_1, y_1)(x_2, y_2)), \mu((x_1, y_1)(x_2, y_2)))$.

Example 2.4. Let $Z_2 = \{0, 1\}$ and $\mu$ be a fuzzy set in $Z_2 \times Z_2$ as $\mu(0, 0) = 0.5, \mu(0, 1) = 0.3, \mu(1, 0) = 0.4, \mu(1, 1) = 0.2$. If $f(x, y) = T_{\mu}(x, y) = \min\{x, y\} \forall (x, y) \in Z_2 \times Z_2$, then $\mu((x_1, y_1)(x_2, y_2)) = \mu(x_1, y_1) \forall (x_2, y_2) \in Z_2 \times Z_2$ if and only if $\mu((x_1, y_1) = \mu(0, 0)$.

Definition 2.5. Let $f$ be a mapping from $G_1 \times G_2$ into $H_1 \times H_2$, $\mu \in [0, 1]^{G_1 \times G_2}$ and $\nu \in [0, 1]^{H_1 \times H_2}$. Following [9] $f(\mu) \in [0, 1]^{H_1 \times H_2}$ and $f^{-1}(\nu) \in [0, 1]^{G_1 \times G_2}$, defined by $\forall (x_2, y_2) \in H_1 \times H_2, f(\mu)(x_2, y_2) = \sup\{\mu((x_1, y_1) \mid (x_1, y_1) \in G_1 \times G_2, f(x_1, y_1) = (x_2, y_2)) \}$. If $f^{-1}(x_2, y_2) \neq \emptyset$ and $f(\mu)(x_2, y_2) = \emptyset$ if $f^{-1}(x_2, y_2) = \emptyset$. Also $\forall (x_1, y_1) \in G_1 \times G_2, f^{-1}(\nu)(x_1, y_1) = \nu(f(x_1, y_1))$.

Lemma 2.4. Let $\mu \in TF(G_1 \times G_2)$ and $H_1 \times H_2$ be a group. Suppose that $f$ is a homomorphism of $G_1 \times G_2$ into $H_1 \times H_2$. Then $\mu(e_1, e_2)$.

Proof. Let $(u_1, u_2), (v_1, v_2) \in H_1 \times H_2$ and $(x_1, y_1), (x_2, y_2) \in G_1 \times G_2$. If $(u_1, u_2) \notin f(G_1 \times G_2)$ or $(v_1, v_2) \notin f(G_1 \times G_2)$, then $f(\mu)(u_1, u_2) = f(\mu)(v_1, v_2) = 0 \leq \mu((u_1, u_2)(v_1, v_2))$. Suppose $(u_1, u_2) = f(x_1, y_1)$ and $(v_1, v_2) = f(x_2, y_2)$ then $f(\mu)((u_1, u_2)(v_1, v_2)) = \sup\{\mu((x_1, y_1)(x_2, y_2)) \mid (u_1, u_2) = f(x_1, y_1), (v_1, v_2) = f(x_2, y_2)\}$.

Also since $\mu \in TF(G_1 \times G_2)$ we have $f(\mu)((u_1, u_2)^{-1}) = f(\mu)(u_1, u_2)$.

Lemma 2.5. Let $H_1 \times H_2$ be a group and $\nu \in TF(H_1 \times H_2)$. If $f$ be a homomorphism of $G_1 \times G_2$ into $H_1 \times H_2$, then $f^{-1}(\nu) \in TF(G_1 \times G_2)$.

Proof. Let $(x_1, y_1), (x_2, y_2) \in G_1 \times G_2$. Then $f^{-1}(\nu)((x_1, y_1)(x_2, y_2)) = \nu(f((x_1, y_1)(x_2, y_2))) = \nu(f(x_1, y_1)f(x_2, y_2)) = T(\nu(f(x_1, y_1)), \nu(f(x_2, y_2))) = T(f^{-1}(\nu)(x_1, y_1), f^{-1}(\nu)(x_2, y_2))$.

Definition 2.6. We say $\mu \in TF(G_1 \times G_2)$ is a normal $T$-fuzzy subgroup of $G_1 \times G_2$ if for all $(x_1, y_1), (x_2, y_2) \in G_1 \times G_2, \mu((x_1, y_1)(x_2, y_2)) = \mu(x_2, y_2)$. Also we denote by $NTF(G_1 \times G_2)$ the set of all normal $T$-fuzzy subgroups of $G_1 \times G_2$.
Proposition 2.3. Let $\mu \in NTF(G_1 \times G_2)$ and $H_1 \times H_2$ be a group. Suppose that $f$ is an epimorphism of $G_1 \times G_2$ onto $H_1 \times H_2$. Then $f(\mu) \in NTF(H_1 \times H_2)$.

Proof. From Lemma 2.4 we have $f(\mu) \in TF(H_1 \times H_2)$. Let $(x_1,y_1),(x_2,y_2) \in H_1 \times H_2$. Since $f$ is a surjection, $f(u_1,u_2) = (x_1,y_1)$ for some $(u_1,u_2) \in G_1 \times G_2$. Then

$$f(\mu)((x_1,y_1),(x_2,y_2)(x_1,y_1)^{-1}) = \sup\{\mu(w_1,w_2) \mid (w_1,w_2) \in G_1 \times G_2, f((x_1,y_1)(x_2,y_2)(x_1,y_1)^{-1})\}
$$

$$= \sup\{\mu((u_1,u_2)^{-1}(w_1,w_2)(u_1,u_2)) \mid (w_1,w_2) \in G_1 \times G_2, f((u_1,u_2)^{-1}(w_1,w_2)(u_1,u_2)) = (x_2,y_2)\}
$$

$$= \sup\{\mu(w_1,w_2) \mid (w_1,w_2) \in G_1 \times G_2, f(w_1,w_2) = (x_2,y_2)\} = f(\mu)(x_2,y_2).$$

□

Proposition 2.4. Let $H_1 \times H_2$ be a group and $v \in NTF(G_1 \times G_2)$ . Suppose that $f$ is a homomorphism of $G_1 \times G_2$ into $H_1 \times H_2$. Then $f^{-1}(v) \in NTF(G_1 \times G_2)$.

Proof. By Lemma 2.5, $f^{-1}(v) \in TF(G_1 \times G_2)$. Now for any $(x_1,y_1),(x_2,y_2) \in G_1 \times G_2$, we have

$$f^{-1}(v)((x_1,y_1),(x_2,y_2)(x_1,y_1)^{-1}) = v(f((x_1,y_1)(x_2,y_2)(x_1,y_1)^{-1}))
$$

$$= v(f(x_1,y_1)f(x_2,y_2)f((x_1,y_1)^{-1})) = v(f(x_1,y_1)f(x_2,y_2)f^{-1}(x_1,y_1))
$$

$$= v(f(x_2,y_2)) = f^{-1}(v)(x_2,y_2).$$

Hence $f^{-1}(v) \in NTF(G_1 \times G_2)$. □

Example 2.5. Let $f$ be an epimorphism of $Z_2 \times Z_2$ onto $Z_2 \times Z_2$ such that $f(x,y) = (x,y)$ for all $(x,y) \in Z_2 \times Z_2$. Let $T(x,y) = T_p(x,y) = xy \forall (x,y) \in Z_2 \times Z_2$ and define $\mu \in NTF(Z_2 \times Z_2)$ as $\mu(0,0) = \mu(0,1) = \mu(1,0) = \mu(1,1) = 0.4$. Then $f(\mu), f^{-1}(\mu) \in NTF(Z_2 \times Z_2)$.

Proposition 2.5. Let $\mu_1, \mu_2 \in NTF(G_1 \times G_2)$. Then $\mu_1 \cap \mu_2 \in NTF(G_1 \times G_2)$.

Proof. Since $\mu_1, \mu_2 \in NTF(G_1 \times G_2)$ then from definition of normal $T$-fuzzy subgroups of $G_1 \times G_2$ for all $(x_1,y_1), (x_2,y_2) \in G_1 \times G_2$ we have that

$$(\mu_1 \cap \mu_2)((x_1,y_1)(x_2,y_2)(x_1,y_1)^{-1}) = T(\mu_1((x_1,y_1)(x_2,y_2)(x_1,y_1)^{-1}), \mu_2((x_1,y_1)(x_2,y_2)(x_1,y_1)^{-1}))
$$

$$= T(\mu_1(x_2,y_2), \mu_2(x_2,y_2))
$$

$$= (\mu_1 \cap \mu_2)(x_2,y_2).$$

□

Corollary 2.2. Let $I_n = \{1, 2, ..., n\}$. If $\{\mu_i \mid i \in I_n\} \subseteq NTF(G_1 \times G_2)$, Then $\mu = \cap_{i \in I_n} \mu_i \in NTF(G_1 \times G_2)$.

Example 2.6. Let $\mu_1(0,0) = \mu_1(0,1) = \mu_1(1,0) = \mu_1(1,1) = 0.1$, and $\mu_2(0,0) = \mu_2(0,1) = \mu_2(1,0) = \mu_2(1,1) = 0.1$. Let $T(x,y) = T_p(x,y) = xy \forall (x,y) \in Z_2 \times Z_3$. Then $\mu_1, \mu_2, \mu_1 \cap \mu_2 \in NTF(Z_2 \times Z_3)$.

Definition 2.7. Let $\mu, \nu \in TF(G_1 \times G_2)$ and $\mu \subseteq \nu$. Then $\mu$ is called a normal subgroup of the subgroup $\nu$, written $\mu \leq \nu$, if for all $(x_1,y_1), (x_2,y_2) \in G_1 \mu((x_1,y_1)(x_2,y_2)(x_1,y_1)^{-1}) \geq T(\mu(x_2,y_2), \nu(x_1,y_1))$.

Proposition 2.6. (1) If $T$ be idempotent, then every $T$-fuzzy subgroup is a normal fuzzy subgroup of itself.

(2) $\mu \in NTF(G_1 \times G_2)$ if and only if $\mu \leq 1_{G_1 \times G_2}$.

Proof. (1) Let $\mu \in TF(G_1 \times G_2)$ and $(x_1,y_1), (x_2,y_2) \in G_1 \times G_2$. It follows that

$$\mu((x_1,y_1)(x_2,y_2)(x_1,y_1)^{-1}) \geq T(\mu((x_1,y_1)(x_2,y_2)), \mu((x_1,y_1)^{-1}))$$

International Scientific Publications and Consulting Services
Conversely, since $m \in NTF(G_1 \times G_2)$ and $(x_1, y_1), (x_2, y_2) \in G_1 \times G_2$. Then $m((x_1, y_1)(x_2, y_2)(x_1, y_1)^{-1}) = m(x_2, y_2) = T(m(x_2, y_2), 1)$. Hence $m \in NTF(G_1 \times G_2)$. Conversely, since $m((x_1, y_1)(x_2, y_2)(x_1, y_1)^{-1}) \geq T(m(x_2, y_2), 1) = T(m(x_2, y_2), 1)$, we have $m \in NTF(G_1 \times G_2)$.

**Example 2.7.** Let $G = \{1, -1\}$ be a productive group and $u(1, 1) = (1, -1) = 0.6, m(1, -1) = (1, 1) = 0.5$. If $T(x, y) = T_n(x, y) = \min\{x, y\} \forall(x, y) \in G \times G$, then $m \geq m$. Also $m \in NTF(G \times G)$ if and only if $m \geq 1_{G \times G}$.

**Lemma 2.6.** Let $T$ be idempotent. If $m \in NTF(G_1 \times G_2)$ and $v \in TF(G_1 \times G_2)$, then $m \cap v \geq v$.

**Proof.** From Proposition 2.1 we have $(m \cap v)(x_1, y_1) = T(m(x_1, y_1), v(x_1, y_1)) = T(m(x_2, y_2), v(x_1, y_1)) = T(m(x_2, y_2), v(x_1, y_1))$.

Hence $m \cap v \geq v$.

**Proposition 2.7.** Let $T$ be idempotent and $m_1, m_2, \xi \in TF(G_1 \times G_2)$. If $m_1 \geq \xi$, then $m_1 \cap m_2 \geq \xi$.

**Proof.** Clearly, $m_1 \cap m_2 \in TF(G_1 \times G_2)$ and $m_1 \cap m_2 \leq \xi$. If $(x_1, y_1), (x_2, y_2) \in G_1 \times G_2$, then $(m_1 \cap m_2)((x_1, y_1)(x_2, y_2)(x_1, y_1)^{-1}) = T(m_1((x_1, y_1)(x_2, y_2)(x_1, y_1)^{-1}), m_2((x_1, y_1)(x_2, y_2)(x_1, y_1)^{-1}))$

$\geq T(m_1(x_2, y_2), m_2(x_2, y_2), \xi(x_1, y_1))$

$= T(m_1(x_2, y_2), m_2(x_2, y_2), \xi(x_1, y_1))$

$= T(m_1(x_2, y_2), m_2(x_2, y_2), \xi(x_1, y_1))$

$= T(T(m_1(x_2, y_2), m_2(x_2, y_2), \xi(x_1, y_1)))$

Therefore, $m_1 \cap m_2 \geq \xi$.
Proposition 2.9. Let $H_1 \times H_2$ be a group. Let $\mu, \nu \in TF(H_1 \times H_2)$ and $\mu \triangleright= \nu$. If $f$ be a homomorphism from $G_1 \times G_2$ into $H_1 \times H_2$, then $f^{-1}(\mu) \triangleright= f^{-1}(\nu)$.

Proof. From Lemma 2.5 we have $f^{-1}(\mu), f^{-1}(\nu) \in TF(G_1 \times G_2)$. Let $(x_1, y_1), (x_2, y_2) \in G_1 \times G_2$. Now

$$f^{-1}(\mu)((x_1, y_1)(x_2, y_2)(x_1, y_1)^{-1}) = \mu(f((x_1, y_1)(x_2, y_2)(x_1, y_1)^{-1}))$$

$$= \mu(f(x_1, y_1)f(x_2, y_2)f^{-1}(x_1, y_1)) \geq T(\mu(f(y)), \nu(f(x_1, y_1)))$$

$$= T(f^{-1}(\mu)(x_2, y_2), f^{-1}(\nu)(x_1, y_1)),$$

hence $f^{-1}(\mu) \triangleright= f^{-1}(\nu)$.

Example 2.9. Let $f$ be a homomorphism from $Z_2 \times Z_2$ into $Z_2 \times Z_2$ such that $f(x, y) = (x, y)^{-1}$ for all $(x, y) \in Z_2 \times Z_2$. Let $T(x, y) = T_0(x, y) = xy \forall (x, y) \in Z_2 \times Z_2$. Define $\mu, \nu \in TF(Z_2 \times Z_2)$ as $\mu(0, 0) = 0.6, \mu(0, 1) = 0.5, \mu(1, 0) = 0.4, \mu(1, 1) = 0.3, \nu(0, 0) = \nu(0, 1) = \nu(1, 0) = \nu(1, 1) = 0.5$. Now we have that $\mu \triangleright= \nu, f(\mu) \triangleright= f(\nu), f^{-1}(\mu) \triangleright= f^{-1}(\nu)$

Acknowledgements

The Author would like to thank the reviewers for carefully reading the manuscript and making several helpful comments to increase the quality of the paper.

References


