On Study of Some Intuitionistic Fuzzy Operators for Intuitionistic Fuzzy Algebraic Structures

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Abstract
In 1965, Fuzzy Set Theory was defined by Zadeh as an extension of crisp sets [10]. K.T. Atanassov generalized fuzzy sets in to Intuitionistic Fuzzy Sets in 1983[1]. Intuitionistic Fuzzy Modal Operator was firstly defined in and the other operators were defined by several authors [2, 3, 5]. Some properties of them have studied until now. Intuitionistic fuzzy algebraic structures were studied by many authors[4, 6, 7, 9]. In this study, we examined some intuitionistic fuzzy operators for intuitionistic fuzzy algebraic structures.

Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy modal operators, Intuitionistic fuzzy algebraic structures.

1 Introduction

The original concept of fuzzy sets in Zadeh [10] was introduced as an extension of crisp sets by enlarging the truth value set to the real unit interval [0, 1]. In fuzzy set theory, if the membership degree of an element x is μ(x) then the nonmembership degree is 1 − μ(x) and thus it is fixed.

Intuitionistic fuzzy sets have been introduced by Atanassov in 1983 [1] and form an extension of fuzzy sets by enlarging the truth value set to the lattice [0, 1]². In this study, we examined some intuitionistic fuzzy operators for intuitionistic fuzzy algebraic structures.

Definition 1.1. Let L = [0, 1] then

\[ L^* = \{(x_1, x_2) \in [0, 1]^2 : x_1 + x_2 \leq 1\} \]

is a lattice with \( (x_1, x_2) \leq (y_1, y_2) \) \iff \( x_1 \leq y_1 \text{ and } x_2 \geq y_2 \).

For \( (x_1, y_1), (x_2, y_2) \in L^* \), the operators \( \land \text{ and } \lor \) on \( (L^*, \leq) \) are defined as following;

- \( (x_1, y_1) \land (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2)) \)
- \( (x_1, y_1) \lor (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2)) \)

For each \( J \subseteq L^* \)

\[ \sup J = \{\sup \{x : (x, y \in [0, 1]) : ((x, y) \in J)\} \} \]

\[ \inf J = \{\inf \{x : (x, y \in [0, 1]) : ((x, y) \in J)\} \} \]

and

\[ \inf J = \{\inf \{y : (x, y \in [0, 1]) : ((x, y) \in J)\} \} \]

\[ \sup J = \{\sup \{y : (x, y \in [0, 1]) : ((x, y) \in J)\} \} \]
2 Preliminaries

Definition 2.1. [1] An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form

\[ A = \{ < x, \mu_A(x), \nu_A(x) > : x \in X \} \]

where \( \mu_A(x), (\mu_A : X \to [0,1]) \) is called the “degree of membership of \( x \) in \( A \)”, \( \nu_A(x), (\nu_A : X \to [0,1]) \) is called the “degree of non-membership of \( x \) in \( A \)”, and where \( \mu_A \) and \( \nu_A \) satisfy the following condition:

\[ \mu_A(x) + \nu_A(x) \leq 1, \text{ for all } x \in X. \]

The hesitation degree of \( x \) is defined by \( \pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \)

Definition 2.2. [1] An IFS \( A \) is said to be contained in an IFS \( B \) (notation \( A \sqsubseteq B \)) if and only if, for all \( x \in X : \mu_A(x) \leq \mu_B(x) \) and \( \nu_A(x) \geq \nu_B(x) \).

It is clear that \( A = B \) if and only if \( A \sqsubseteq B \) and \( B \sqsubseteq A \).

Definition 2.3. [1] Let \( A \in \text{IFS} \) and let \( A = \{ < x, \mu_A(x), \nu_A(x) > : x \in X \} \) then the above set is called the complement of \( A \):

\[ A^c = \{ < x, \nu_A(x), \mu_A(x) > : x \in X \} \]

Definition 2.4. [3] Let \( X \) be a set and \( A = \{ < x, \mu_A(x), \nu_A(x) > : x \in X \} \in \text{IFS}(X) \).

1. \( nA = \{ < x, 1 - (1 - \mu_A(x))^n, (\nu_A(x))^n > : x \in X \} \)
2. \( A^n = \{ < x, (\mu_A(x))^n, 1 - (1 - \nu_A(x))^n > : x \in X \} \)
3. \( \square A = \{ < x, \mu_A(x), 1 - \mu_A(x) > : x \in X \} \)
4. \( \diamond A = \{ < x, 1 - \nu_A(x), \nu_A(x) > : x \in X \} \)

In 2007, the author [5] defined a new operator and studied some of its properties. This operator is named \( E_{\alpha,\beta} \) and defined as follows:

Definition 2.5. [5] Let \( X \) be a set and \( A = \{ < x, \mu_A(x), \nu_A(x) > : x \in X \} \in \text{IFS}(X), \alpha, \beta \in [0,1] \). We define the following operator:

\[ E_{\alpha,\beta}(A) = \{ < x, \beta(a \mu_A(x) + 1 - \alpha), \alpha(\beta \nu_A(x) + 1 - \beta) > : x \in X \} \]

In 2007, Atanassov introduced the operator \( \sqcap_{\alpha,\beta,\gamma,\delta} \) which is a natural extension of all these operators in [3].

Definition 2.6. [3] Let \( X \) be a set, \( A \in \text{IFS}(X) \), \( \alpha, \beta, \gamma, \delta \in [0,1] \) such that

\[ \max(\alpha, \beta) + \gamma + \delta \leq 1 \]

then the operator \( \sqcap_{\alpha,\beta,\gamma,\delta} \) defined by

\[ \sqcap_{\alpha,\beta,\gamma,\delta}(A) = \{ < x, \alpha \mu_A(x) + \gamma, \beta \nu_A(x) + \delta > : x \in X \} \]

In 2010, the author [5] defined a new operator which is a generalization of \( E_{\alpha,\beta} \).

Definition 2.7. [5] Let \( X \) be a set and \( A \in \text{IFS}(X) \), \( \alpha, \beta, \omega, \theta \in [0,1] \). We define the following operator:

\[ Z_{\alpha,\beta,\omega,\theta}(A) = \{ < x, \beta(a \mu_A(x) + \omega - \omega, \alpha(\beta \nu_A(x) + \theta - \beta) > : x \in X \} \]
3 Some Intuitionistic Fuzzy Algebraic Structures

Definition 3.1. [6] Let G be a grupoid, \( A \in IFS(G) \). If for all \( x, y \in G \),

\[
A(xy) \geq \max(A(x), A(y))
\]

then \( A \) called an intuitionistic fuzzy ideal over \( G \), shortly \( IFI(G) \).

Definition 3.2. [7] Let \( G \) be a grup and \( A \in IFS(G) \) a grupoid. If for all \( x \in G \),

\[
A(x^{-1}) \geq A(x)
\]

then \( A \) called an intuitionistic fuzzy subgroup over \( G \), shortly \( IFG(G) \).

Definition 3.3. [7] Let \( G \) be a grup and \( A \in IFS(G) \) a intuitionistic fuzzy subgroup. If for all \( x, y \in G \),

\[
A(xy) = A(yx)
\]

then \( A \) called an intuitionistic fuzzy normal subgroup over \( G \), shortly \( IFNG(G) \).

Definition 3.4. [9] Let \( R \) be a ring and \( A \in IFS(R) \). If for all \( x, y \in R \) the following conditions are valid then \( A \) called an intuitionistic fuzzy ring over \( R \), shortly \( IFR(R) \):

1. \( A(x + y) \geq A(x) \land A(y) \)
2. \( A(-x) \geq A(x) \)
3. \( A(xy) \geq A(x) \land A(y) \)

Definition 3.5. [4] Let \( X \) be a set and \( A \in IFS(X) \). If set \( A \) contain two elements, at least, like that \( m_A(x_0) = 1 \) and \( n_A(x_1) = 1 \) then \( A \) called an intuitionistic fuzzy normal(IF normal) over \( X \).

4 Implementation of Some Intuitionistic Fuzzy Operators on Intuitionistic Fuzzy Algebraic Structures

4.1 On Intuitionistic Fuzzy Ideals

Theorem 4.1. Let \( G \) be a grupoid and \( A \in IFS(G) \).

1. If \( A \in IFI(G) \) then \( \square A \in IFI(G) \).
2. If \( A \in IFI(G) \) then \( \bigdiamond A \in IFI(G) \).

Proof. (1)For \( x, y \in G \),

\[
\mu_{\square A}(xy) = \mu_A(xy) \geq \mu_A(x) \lor \mu_A(y)
\]

and

\[
v_{\square A}(xy) = 1 - \mu_A(xy) \leq (1 - \mu_A(x)) \land (1 - \mu_A(y)) = v_{\square A}(x) \land v_{\square A}(y)
\]

So,

\[
\square A(xy) \geq \square A(x) \lor \square A(y)
\]

Theorem 4.2. Let \( G \) be a grupoid and \( A \in IFS(G) \) an ideal.

1. \( nA \in IFS(G) \) is an ideal.
2. \( A^n \in IFS(G) \) is an ideal.
Proof. (1) For $x, y \in G$,
\[ \mu_{nA}(xy) = 1 - (1 - \mu_A(xy))^n \geq 1 - (1 - \mu_A(x))^n \lor 1 - (1 - \mu_A(y))^n = \mu_{nA}(x) \lor \mu_{nA}(y) \]
and
\[ \nu_{nA}(xy) = (\nu_A(xy))^n \leq (\nu_A(x))^n \land (\nu_A(y))^n = \nu_{nA}(x) \land \nu_{nA}(y) \]
So,
\[ nA(xy) \geq nA(x) \lor nA(y) \]
(2) It can be proved similarly. \hfill \square

**Theorem 4.3.** Let $G$ be a groupoid and $A \in IFS(G)$ an ideal then $E_{\alpha,\beta}(A) \in IFS(G)$ is an ideal.

**Proof.** For $x, y \in G$,
\[ \mu_{E_{\alpha,\beta}(A)}(xy) = \beta(\alpha \mu_A(xy) + 1 - \alpha) \geq \beta(\alpha \mu_A(x) + 1 - \alpha) \lor \beta(\alpha \mu_A(y) + 1 - \alpha) \]
and
\[ \nu_{E_{\alpha,\beta}(A)}(xy) = \alpha(\beta \nu_A(xy) + 1 - \beta) \leq \alpha(\beta \nu_A(x) + 1 - \beta) \land \alpha(\beta \nu_A(y) + 1 - \beta) \]
So,
\[ E_{\alpha,\beta}(A)(xy) \geq E_{\alpha,\beta}(A)(x) \lor E_{\alpha,\beta}(A)(y) \]
\hfill \square

**Theorem 4.4.** Let $G$ be a groupoid and $A \in IFS(G)$ an ideal then $\Box_{\alpha,\beta,\gamma,\delta}(A) \in IFS(G)$ is an ideal.

**Proof.** For $x, y \in G$,
\[ \mu_{\Box_{\alpha,\beta,\gamma,\delta}(A)}(xy) = \alpha \mu_A(xy) + \gamma \geq (\alpha \mu_A(x) + \gamma) \lor (\alpha \mu_A(y) + \gamma) \]
and
\[ \nu_{\Box_{\alpha,\beta,\gamma,\delta}(A)}(xy) = \beta \nu_A(xy) + \delta \leq (\beta \nu_A(x) + \delta) \land (\beta \nu_A(y) + \delta) \]
So,
\[ \Box_{\alpha,\beta,\gamma,\delta}(A)(xy) \geq \Box_{\alpha,\beta,\gamma,\delta}(A)(x) \lor \Box_{\alpha,\beta,\gamma,\delta}(A)(y) \]
\hfill \square

**Theorem 4.5.** Let $G$ be a groupoid and $A \in IFS(G)$ an ideal then $Z^{\omega,\theta}_{\alpha,\beta}(A) \in IFS(G)$ is an ideal.

**Proof.** For $x, y \in G$,
\[ \mu_{Z^{\omega,\theta}_{\alpha,\beta}(A)}(xy) = \beta(\alpha \mu_A(xy) + \omega - \omega \alpha) \geq \beta(\alpha \mu_A(x) + \omega - \omega \alpha) \lor \beta(\alpha \mu_A(y) + \omega - \omega \alpha) \]
and
\[ \nu_{Z^{\omega,\theta}_{\alpha,\beta}(A)}(xy) = \alpha(\beta \nu_A(xy) + \theta - \theta \beta) \leq \alpha(\beta \nu_A(x) + \theta - \theta \beta) \land \alpha(\beta \nu_A(y) + \theta - \theta \beta) \]
Therefore, we obtain
\[ Z^{\omega,\theta}_{\alpha,\beta}(A)(xy) \geq Z^{\omega,\theta}_{\alpha,\beta}(A)(x) \lor Z^{\omega,\theta}_{\alpha,\beta}(A)(y). \]
4.2 On Intuitionistic Fuzzy Subgroups

Theorem 4.6. Let $G$ be a group and $A \in IFS(G)$.

1. If $A \in IFG(G)$ then $\square A \in IFG(G)$.
2. If $A \in IFG(G)$ then $\Diamond A \in IFG(G)$.

Proof. It is clear that, if $A \in IFG(G)$ then it means $A \in IFI(G)$ and for all $x \in G$, $A(x^{-1}) \geq A(x)$.
So, it will be enough to prove the correctness of the second condition.

(2) For $x \in G$

$$\mu_{\square A}(x^{-1}) = 1 - v_A(x^{-1}) \geq 1 - v_A(x) = \mu_{\Diamond A}(x)$$

and

$$v_{\square A}(x^{-1}) = v_A(x^{-1}) \leq v_A(x) = v_{\Diamond A}(x)$$

The other property can be proved same way.

Theorem 4.7. Let $G$ be a group and $A \in IFS(G)$.

1. If $A \in IFG(G)$ then $nA \in IFG(G)$.
2. If $A \in IFG(G)$ then $A^n \in IFG(G)$.

Proof. (2) For $x \in G$,

$$\mu_{A^n}(x^{-1}) = (\mu_A(x^{-1}))^n \geq (\mu_A(x))^n = \mu_{A^n}(x)$$

and

$$v_{A^n}(x^{-1}) = 1 - (1 - (v_A(x^{-1}))^n \leq 1 - (1 - (v_A(x))^n = v_{A^n}(x)$$

On the other hand, we know that $A^n(xy) \geq A^n(x) \land A^n(y), x, y \in G$.
It indicates that $A^n \in IFG(G)$.

Theorem 4.8. Let $G$ be a group and $A \in IFS(G)$. If $A$ is an intuitionistic fuzzy subgroup on $G$ then $E_{\alpha, \beta}(A) \in IFG(G)$.

Proof. It is clear that for $x, y \in G$, $E_{\alpha, \beta}(A)(xy) \geq E_{\alpha, \beta}(A)(x) \lor E_{\alpha, \beta}(A)(y)$.

$$\mu_{E_{\alpha, \beta}(A)}(x^{-1}) = \beta(\alpha \mu_A(x^{-1}) + 1 - \alpha) \geq \beta(\alpha \mu_A(x) + 1 - \alpha) = \mu_{E_{\alpha, \beta}(A)}(x)$$

and

$$v_{E_{\alpha, \beta}(A)}(x^{-1}) = \alpha(\beta v_A(x^{-1}) + 1 - \beta) \leq \alpha(\beta v_A(x) + 1 - \beta) = v_{E_{\alpha, \beta}(A)}(x)$$

So, $E_{\alpha, \beta}(A) \in IFG(G)$.

Theorem 4.9. Let $G$ be a group and $A \in IFS(G)$ an intuitionistic fuzzy group then $\square_{\alpha, \beta, \gamma, \delta}(A) \in IFS(G)$ is an intuitionistic fuzzy subgroup.

Proof. For $x \in G$,

$$\mu_{\square_{\alpha, \beta, \gamma, \delta}(A)}(x^{-1}) = \alpha \mu_A(x^{-1}) + \gamma \geq \alpha \mu_A(x) + \gamma$$

$$= \mu_{\square_{\alpha, \beta, \gamma, \delta}(A)}(x)$$

and

$$v_{\square_{\alpha, \beta, \gamma, \delta}(A)}(x^{-1}) = \beta v_A(x^{-1}) + \delta \leq \beta v_A(x) + \delta$$

$$= v_{\square_{\alpha, \beta, \gamma, \delta}(A)}(x)$$

Therefore $\square_{\alpha, \beta, \gamma, \delta}(A) \in IFG(G)$.

Theorem 4.10. Let $G$ be a group and $A \in IFS(G)$. If $A$ is an intuitionistic fuzzy subgroup on $G$ then $Z_{\alpha, \beta}(A) \in IFG(G)$.

Proof. It is clear.
4.3 On Intuitionistic Fuzzy Normal Subgroups

Theorem 4.11. Let $G$ be a group and $A \in IFS(G)$.

1. If $A \in IFNG(G)$ then $\Box A \in IFNG(G)$.
2. If $A \in IFNG(G)$ then $\Diamond A \in IFNG(G)$.

Proof. (1) If $A \in IFNG(G)$ then $\Box A \in IFG(G)$.
For $x, y \in G$

$$\mu_{\Box A}(xy) = \mu_A(xy) = \mu_A(yx) = \mu_{\Box A}(yx)$$

and

$$\nu_{\Box A}(xy) = 1 - \mu_A(xy) = 1 - \mu_A(yx) = \nu_{\Box A}(yx)$$

So, $\Box A \in IFNG(G)$ and we can prove the other property similarly.

Theorem 4.12. Let $G$ be a group and $A \in IFS(G)$.

1. If $A \in IFNG(G)$ then $nA \in IFNG(G)$.
2. If $A \in IFNG(G)$ then $A^n \in IFNG(G)$.

Proof. (1) For $x, y \in G$,

$$\mu_{nA}(xy) = 1 - (1 - \mu_A(xy))^n = 1 - (1 - \mu_A(yx))^n = \mu_{nA}(yx)$$

and

$$\nu_{nA}(xy) = (\nu_A(xy))^n = (\nu_A(yx))^n = \nu_{nA}(yx)$$

We obtain that $nA \in IFNG(G)$.
(2) It is clear.

Theorem 4.13. Let $G$ be a group and $A \in IFS(G)$. If $A$ is an intuitionistic fuzzy normal subgroup on $G$ then $E_{\alpha, \beta}(A) \in IFNG(G)$.

Proof. It is clear that $E_{\alpha, \beta}(A) \in IFG(G)$. For $x, y \in G$,

$$\mu_{E_{\alpha, \beta}(A)}(xy) = \beta(\alpha \mu_A(xy)) + 1 - \alpha = \beta(\alpha \mu_A(yx)) + 1 - \alpha = \mu_{E_{\alpha, \beta}(A)}(yx)$$

and

$$\nu_{E_{\alpha, \beta}(A)}(xy) = \alpha(\beta \nu_A(xy)) + 1 - \beta = \alpha(\beta \nu_A(yx)) + 1 - \beta = \nu_{E_{\alpha, \beta}(A)}(yx)$$

So, $E_{\alpha, \beta}(A) \in IFNG(G)$.

Theorem 4.14. Let $G$ be a group and $A \in IFS(G)$ an intuitionistic fuzzy normal subgroup then $\Box_{\alpha, \beta, \gamma, \delta}(A) \in IFS(G)$ is an intuitionistic fuzzy normal subgroup.

Theorem 4.15. Let $G$ be a group and $A \in IFS(G)$. If $A$ is an intuitionistic fuzzy subgroup on $G$ then $Z_{\alpha, \beta}^{\omega, \theta}(A) \in IFNG(G)$.

Proof. $Z_{\alpha, \beta}^{\omega, \theta}(A) \in IFG(G)$ and for $x, y \in G$,

$$\mu_{Z_{\alpha, \beta}^{\omega, \theta}(A)}(xy) = \beta(\alpha \mu_A(xy) + \omega - \omega \alpha) = \beta(\alpha \mu_A(yx) + \omega - \omega \alpha) = \mu_{Z_{\alpha, \beta}^{\omega, \theta}(A)}(yx)$$

and

$$\nu_{Z_{\alpha, \beta}^{\omega, \theta}(A)}(xy) = \alpha(\beta \nu_A(xy) + \theta - \theta \beta) = \alpha(\beta \nu_A(yx) + \theta - \theta \beta) = \nu_{Z_{\alpha, \beta}^{\omega, \theta}(A)}(yx)$$

Therefore, $Z_{\alpha, \beta}^{\omega, \theta}(A) \in IFNG(G)$.  

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4.4 On Intuitionistic Fuzzy Rings

Theorem 4.16. Let R be a ring and A ∈ IFS(R).

1. If A ∈ IFR(R) then □A ∈ IFR(R).
2. If A ∈ IFR(R) then ⊙A ∈ IFR(R).

Proof. (2) For x, y ∈ R,

i. \[ \mu_{\odot A}(x + y) = 1 - \nu_{A}(x + y) \geq 1 - (\nu_{A}(x) \lor \nu_{A}(y)) \]

\[ = (1 - \nu_{A}(x)) \land (1 - \nu_{A}(y)) \]

\[ = \mu_{\odot A}(x) \land \mu_{\odot A}(y) \]

\[ \nu_{\odot A}(x + y) = \nu_{A}(x + y) \leq \nu_{A}(x) \lor \nu_{A}(y) \]

\[ = \nu_{\odot A}(x) \lor \nu_{\odot A}(y) \]

ii. \[ \mu_{\odot A}(-x) = 1 - \nu_{A}(-x) \geq 1 - \nu_{A}(x) = \mu_{\odot A}(x) \]

\[ \nu_{\odot A}(-x) = \nu_{A}(-x) \leq \nu_{A}(x) = \nu_{\odot A}(x) \]

iii. \[ \mu_{\odot A}(xy) = 1 - \nu_{A}(xy) \geq 1 - (\nu_{A}(x) \lor \nu_{A}(y)) \]

\[ = (1 - \nu_{A}(x)) \land (1 - \nu_{A}(y)) \]

\[ = \mu_{\odot A}(x) \land \mu_{\odot A}(y) \]

\[ \nu_{\odot A}(xy) = \nu_{A}(xy) \leq \nu_{A}(x) \lor \nu_{A}(y) \]

\[ = \nu_{\odot A}(x) \lor \nu_{\odot A}(y) \]

Thus, we showed that □A ∈ IFR(R) and we can prove the ⊙A ∈ IFR(R) similarly.

Theorem 4.17. Let R be a ring and A ∈ IFS(R).

1. If A ∈ IFR(R) then nA ∈ IFR(R).
2. If A ∈ IFR(R) then A^n ∈ IFR(R).

Proof. (1) For x, y ∈ R,

i. \[ \mu_{nA}(x + y) = 1 - (1 - \mu_{A}(x + y))^{n} \geq 1 - (1 - \mu_{A}(x))^{n} \lor 1 - (1 - \mu_{A}(y))^{n} \]

\[ = \mu_{nA}(x) \land \mu_{nA}(y) \]

\[ \nu_{nA}(x + y) = (\nu_{A}(x + y))^{n} \leq (\nu_{A}(x))^{n} \lor (\nu_{A}(y))^{n} = \nu_{nA}(x) \lor \nu_{nA}(y) \]

ii. \[ \mu_{nA}(-x) = 1 - (1 - \mu_{A}(-x))^{n} \geq 1 - (1 - \mu_{A}(-x))^{n} = \mu_{nA}(x) \]

\[ \nu_{nA}(-x) = (\nu_{A}(-x))^{n} \leq (\nu_{A}(x))^{n} = \nu_{nA}(x) \]

iii. \[ \mu_{nA}(xy) = 1 - (1 - \mu_{A}(xy))^{n} \geq 1 - (1 - \mu_{A}(x))^{n} \lor 1 - (1 - \mu_{A}(y))^{n} \]

\[ = \mu_{nA}(x) \land \mu_{nA}(y) \]

\[ \nu_{nA}(xy) = (\nu_{A}(xy))^{n} \leq (\nu_{A}(x))^{n} \lor (\nu_{A}(y))^{n} = \nu_{nA}(x) \lor \nu_{nA}(y) \]

It indicates that nA ∈ IFR(R).
Theorem 4.18. Let $R$ be a ring and $A \in \text{IFS}(R)$ an intuitionistic fuzzy ring then $\square_{\alpha, \beta, \gamma, \delta}(A) \in \text{IFS}(R)$ is an intuitionistic fuzzy ring.

Proof. Let $x, y \in R$,

\begin{align*}
\mu_{\square_{\alpha, \beta, \gamma, \delta}(A)}(x + y) &= \alpha \mu_{\text{a}}(x + y) + \gamma \geq (\alpha \mu_{\text{a}}(x) + \gamma) \wedge (\alpha \mu_{\text{a}}(y) + \gamma) \\
&= \mu_{\square_{\alpha, \beta, \gamma, \delta}(A)}(x) \wedge \mu_{\square_{\alpha, \beta, \gamma, \delta}(A)}(y)
\end{align*}

\begin{align*}
\nu_{\square_{\alpha, \beta, \gamma, \delta}(A)}(x + y) &= \beta \nu_{\text{a}}(x + y) + \delta \leq (\beta \nu_{\text{a}}(x) + \delta) \vee (\beta \nu_{\text{a}}(y) + \delta) \\
&= \nu_{\square_{\alpha, \beta, \gamma, \delta}(A)}(x) \vee \nu_{\square_{\alpha, \beta, \gamma, \delta}(A)}(y)
\end{align*}

Therefore $\square_{\alpha, \beta, \gamma, \delta}(A) \in \text{IFG}(G)$.

Theorem 4.19. Let $R$ be a group and $A \in \text{IFS}(R)$. If $A$ is an intuitionistic fuzzy ring on $R$ then $E_{\alpha, \beta}(A)$ and $Z_{\alpha, \beta}(A)$ are intuitionistic fuzzy rings.

Proof. It is clear.

4.5 On Intuitionistic Fuzzy Normal Sets

Theorem 4.20. Let $X$ be a set and $A \in \text{IFS}(X)$.

1. If $A$ an $\text{IF}_\text{normal}$ set then $\square A$ an $\text{IF}_\text{normal}$ set.

2. If $A$ an $\text{IF}_\text{normal}$ set then $\Diamond A$ an $\text{IF}_\text{normal}$ set.

Proof. (1) We know that, $A$ contain two elements, at least, like that $\mu_{\square}(x_0) = 1$ and $\nu_{\square}(x_1) = 1$. So,

\begin{align*}
\mu_{\square}(x_0) &= \mu_{\square}(x_0) = 1 \text{ and } \nu_{\square}(x_0) = 1 - \mu_{\square}(x_0) = 0 \\
\text{and} \\
\nu_{\square}(x_1) &= 1 - \mu_{\square}(x_1) = 1 \text{ and } \mu_{\square}(x_1) = 0
\end{align*}

Thus, we prove that $\square A$ an $\text{IF}_\text{normal}$ set. Similarly, $\Diamond A$ an $\text{IF}_\text{normal}$ set, too.

Theorem 4.21. Let $X$ be a set and $A \in \text{IFS}(X)$.

1. If $A$ an $\text{IF}_\text{normal}$ set then $nA$ an $\text{IF}_\text{normal}$ set.

2. If $A$ an $\text{IF}_\text{normal}$ set then $A^n$ an $\text{IF}_\text{normal}$ set.

Proof. It can be seen easily.

5 Conclusion

In this paper, we studied the impact on algebraic structures of some intuitionistic fuzzy modal operators. The intuitionistic fuzzy algebraic structures intuitionistic fuzzy ideal, intuitionistic fuzzy subgroup, intuitionistic fuzzy normal subgroup and intuitionistic fuzzy ring are protected by the studying operators. On the other hand, normality only protected by $\square A$, $\Diamond A$, $nA$ and $A^n$ operators. In accordance with this information, we can examine the algebraic structures on intuitionistic fuzzy operators set.
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