Some results of approximate fixed points in fuzzy Banach spaces

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Abstract
In this paper, first we introduce some aspects of approximate fixed points, such as approximate fixed point property for subsets of a fuzzy Banach space and approximate eigenvalues of a self mapping on a fuzzy normed space. Then as their consequences, we prove more results for Fixed Point Theory on Fuzzy Banach and Banach Spaces.

Keywords: Approximate eigenvalues, Approximate fixed point property, Fixed points, Reflexive fuzzy Banach spaces.
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1 Introduction

There are plenty of problems in pure and applied mathematics which can be solved by means of fixed point theory. But in some theoretical situations it will be difficult to derive conditions for existence of fixed points for certain types of mappings and the requirement will be only an approximation to the fixed points. In such a case naturally we will use the concept of approximate fixed points ($\varepsilon$-fixed points), see [1]. Also in several situations of practical utility, the mapping under consideration may not have an exact fixed point due to some restrictions on the space or the map. Practice proves that in many real situations, an approximate solution is more than sufficient. So the existence of fixed points is not strictly required (see [3]). Most of these ambitions, persuade us to introduce the concept of approximate fixed points. Approximate fixed points have been viewed from different scopes. A number of authors have considered the notion of approximate fixed points on metric spaces, from distinct views (see [18] and [4]). Some results, have studied the approximate fixed points for multivalued maps (for example see [12]) and approximate fixed point property in product spaces (see [15]). Recently the notion of approximate fixed point, for mappings on fuzzy normed spaces, has been discussed (for example see [10]). We believe that presenting an extended idea, in the realm of fuzzy metric and fuzzy normed spaces, as the extensions of the metric and the normed spaces, will help to find a theoretical and applied solutions for more flexible situations. This is a strong ambition to study the concept of approximate fixed point theory in fuzzy spaces.

In the first part of this paper, we introduce some notions such as approximate fixed point property for subsets of a fuzzy Banach space, as their applications, we prove some fixed point results for the mappings on a fuzzy Banach space, and we study the approximate eigenvalues of a self mapping on a fuzzy normed space. Then we see that these conditions can be transferred to the other structures. On the other hand, we apply the proved results to impose the interesting conditions for fixed point theorems on Banach spaces.

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2 Preliminaries

Definition 2.1. [7] A triangular norm (t-norm) * on \([0,1]\) is defined as an increasing, commutative and associative mapping \(\ast : [0,1]^2 \to [0,1]\) satisfying \(1 \ast x = x, \text{ for all } x \in [0,1]\).

A t-norm can also be defined recursively as an \((n + 1)\)-ary operation \((n \in \mathbb{N})\) by \(\ast^1 = \ast\) and
\[
\ast^n = (x_1, x_2, x_{n+1}) = (\ast^{n-1}(x_1, x_2, x_n), x_{n+1}),
\]
for \(n \geq 2\).

Definition 2.2. [20] A fuzzy normed space (FN-space) is a 3-tuple \((X, N, \ast)\) where \(X\) is a linear space, \(\ast\) is a t-norm and \(N\) is a fuzzy set on \(X \times (0,\infty)\), such that for all \(x, y \in X\) and all \(s, t > 0\) the following conditions hold:
\begin{align*}
(FN-1) \quad & N(x, t) > 0, \\
(FN-2) \quad & N(x, t) = 1 \text{ if and only if } x = 0, \\
(FN-3) \quad & N(\lambda x, t) = N(x, \frac{t}{|\lambda|}), \quad \forall \lambda \neq 0, \\
(FN-4) \quad & N(x, t) \ast N(y, s) \leq N(x + y, t + s), \\
(FN-5) \quad & N(x, \cdot) : (0,\infty) \to [0,1] \text{ is continuous,} \\
(FN-6) \quad & \lim_{t \to \infty} N(x, t) = 1.
\end{align*}

Example 2.1. Let \((X, \|\cdot\|)\) be a normed space, we define \(a \ast b = ab\) or \(a \ast b = \min(a,b)\) and
\[
N_t(x, t) = \frac{t}{t + \|x\|}.
\]
Then \((X, N_t, \ast)\) is a fuzzy normed space, and \(N_t\) is called the standard fuzzy norm, induced by norm \(\|\cdot\|\).

For each \(t > 0\), \(N(x, t)\) as a function of \(x \in X\), is continuous (see [20]).

In the following, we remind some definitions and results which will be used in the next, to see the details, we refer the readers to [10].

Definition 2.3. Let \((X, N, \ast)\) be a fuzzy normed space and \(A\) be a nonempty subset of \(X\). The fuzzy diameter \(D_A\) of \(A\) is defined by
\[
D_A(x) = \sup_{t < x} \inf_{p \in A} N(p, t) \text{ and } D_A(\infty) = 1.
\]

\(A\) is bounded if and only if
\[
\sup_{x \in A} \inf_{p \in A} N(p, x) = 1, \quad \forall p \in A.
\]

Definition 2.4. Let \((X, N, \ast)\) be a FN-space. A map \(f : X \to X\) is said to be \(\varepsilon\)-continuous if for each \(p \in X\) and \(\varepsilon > 0\),
\[
R_{f(\mathcal{N}(p, \varepsilon))}(\varepsilon) > 1 - \varepsilon,
\]
where \(R_A(\varepsilon) = \sup_{\delta < \varepsilon} \inf_{p \in A} N(p, \delta), \quad \varepsilon > 0\) is called the fuzzy radius of \(A\).

Definition 2.5. Let \((X, N, \ast)\) be a FN-space, \(A \subseteq X\) and \(f : A \to A\) be a single valued self mapping. We say that \(p \in A\) is an \(\varepsilon\)-fixed point of \(f : A \to A\) if for \(\varepsilon > 0\),
\[
\sup_{t < \varepsilon} N(f(p) - p, t) = 1.
\]

Also, we will say that \(f\) has the approximate fixed point property \((a, f, p, p)\) if the function \(f\) posses at least one \(\varepsilon\)-fixed point.

We remind that in a fuzzy normed space \((X, N, \ast)\), a subset \(A \subseteq X\) is called F-bounded if there exists \(t > 0\) and \(0 < r < 1\) such that \(N(x, t) > 1 - r\), for all \(x \in A\) (for more details, see [11]).
Theorem 2.1. Let A be a nonempty F-bounded convex subset of \((X,N,*)\) and \(f: A \rightarrow A\) be an \(\varepsilon\)-continuous continuous function. Then \(f\) has the a.f.p.p.

Definition 2.6. Suppose that \((X,N,*)\) be a fuzzy normed space. A mapping \(f: X \rightarrow X\) is said to be \(\alpha\)-contraction if there exists \(\alpha \in (0,1)\) such that for all \(x, y \in X\) and \(t > 0\)

\[N(f(x) - f(y), \alpha t) \geq N(x - y, t).\]

If \(\alpha = 1\) we say that \(f\) is non expansive.

Corollary 2.1. Suppose that \((X,N,*)\) be a fuzzy normed space and \(f: X \rightarrow X\) be \(\alpha\)-contraction. Then \(f\) has the a.f.p.p.

Proposition 2.1. [17] Every \(\alpha\)-contraction in a complete fuzzy normed space has a unique fixed point.

Theorem 2.3. Let \(X\) be a normed linear space and \(T\) be a self mapping on \(X\). We say that \(T\) is semi-compact if, whenever there exists a sequence \(\{x_n\}\) in \(X\) with \(\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0\), then \(\{x_n\}\) has a convergent subsequence.

Lemma 2.1. Let \(X\) be a normed linear space (on a field \(\mathbb{F}\)) and \(T: X \rightarrow X\) be a self mapping on \(X\). \(T\) is said to be homogeneous if for each \(\lambda \in \mathbb{F}\), \(T(\lambda x) = \lambda Tx\).

Corollary 2.3. A subset \(C\) of a normed linear space \(X\) is said to be starshaped if there exists \(p \in C\) such that \(rx + (1-r)p \in C\), \(\forall x \in C\) and \(0 \leq r \leq 1\). In this case \(p\) is called the star center of \(C\).

Definition 2.7. Let \((X,\|\cdot\|,d)\) be a normed linear space and \(T\) be a self mapping on \(X\). We say that \(T\) is semi-compact if, whenever there exists a sequence \(\{x_n\}\) in \(X\) with \(\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0\), then \(\{x_n\}\) has a convergent subsequence.

Definition 2.8. Let \(X\) be a normed linear space (on a field \(\mathbb{F}\)) and \(T: X \rightarrow X\) be a self mapping on \(X\). \(T\) is said to be demi-closed if for any sequence \(\{x_n\}\) weakly convergent to an element \(x_0\) with \(\{h(x_n)\}\) norm-convergent to an element \(y_0\) then \(h(x_0) = y_0\).

Definition 2.9. Let \(X\) be a normed linear space and \(T: X \rightarrow X\) be a self mapping on \(X\). \(T\) is said to be semi-compact if, whenever there exists a sequence \(\{x_n\}\) weakly convergent to an element \(x_0\) with \(\{h(x_n)\}\) norm-convergent to an element \(y_0\) then \(h(x_0) = y_0\).

Definition 2.10. Let \(X\) be a normed linear space and \(T: X \rightarrow X\) be a self mapping on \(X\). \(T\) is said to be semi-compact if, whenever there exists a sequence \(\{x_n\}\) weakly convergent to an element \(x_0\) with \(\{h(x_n)\}\) norm-convergent to an element \(y_0\) then \(h(x_0) = y_0\).

Definition 2.11. Let \(S\) be a subset of a normed linear space \(X\). A map \(T: S \rightarrow X\) is said to be semi-compact if for any bounded sequence \(\{x_n\}\) in \(S\) such that \((x_n - Tx_n) \rightarrow z\) in \(S\), as \(n \rightarrow \infty\), there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) and a point \(p \in S\) such that \(x_{n_k} \rightarrow p\), as \(k \rightarrow \infty\), and \((I - T)p = z\).

3 Main Results

Most of results in this section extend the results of approximate fixed points on normed spaces. So we use some of their ideas, especially the consequences proposed in [2].

By definition 2.6 and (FN-6), we easily see that the following lemma holds:

Lemma 3.1. Every \(\alpha\)-contraction is continuous.

Definition 3.1. A subset \(K\) of a fuzzy normed space \((X,N,*)\), is said to have approximate fixed point property (s.a.f.p.p) if, for some \(t > 0\) and every non expansive map \(T: K \rightarrow K\), satisfies the property that

\[\sup_{x \in K} N(x - Tx, t) = 1.\]

Definition 3.2. Let \((X,N,*)\) be a fuzzy normed space and \(T: X \rightarrow X\) be a self mapping on \(X\). Then \(T\) is said to be semi-compact if, for some \(t > 0\), whenever there exists a sequence \(\{x_n\}\) in \(X\) with \(\lim_{n \rightarrow \infty} N(x_n - Tx_n, t) = 1\), then \(\{x_n\}\) has a convergent subsequence.

Theorem 3.1. F-bounded closed convex subsets of a fuzzy Banach space, always have the s.a.f.p.p. A subset \(K\) of a fuzzy normed space \((X,N,*)\) satisfies the s.a.f.p.p. if for any non expansive map \(T: K \rightarrow K\), there exists a sequence \(\{x_n\} \subseteq K\) such that, for some \(t > 0\),

\[\lim_{n \rightarrow \infty} N(x_n - Tx_n, t) = 1.\]
Proof. Let $K$ be a $F$-bounded closed convex subset a fuzzy Banach space $(X,N,\ast)$ and $T : K \rightarrow K$ be any non expansive map on $K$. By Lemma 3.1 $T$ is continuous and also for each $\varepsilon > 0$, $T$ is $\varepsilon$-continuous. Now from Theorem 2.1, $T$ has the a.f.p.p. and so by definition 2.5, we see that $\sup_{x \in K}N(x - Tx, t) = 1$, for some $t > 0$. Hence the first part of theorem has been proved. Suppose that $K$ satisfies the s.a.f.p.p. and $T : K \rightarrow K$ be any non expansive map on $K$. By definition 3.1, for each $n \in \mathbb{N}$, there exists $x_n \in K$ such that

$$N(x_n - Tx_n, t) > 1 - \frac{1}{n}.$$ 

As $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} N(x_n - Tx_n, t) = 1,$$

and hence the sequence $\{x_n\}$ is as desired. Conversely if there exists $x_n \in K$ such that $N(x_n - Tx_n, t) > 1 - \frac{1}{n}$, we see that $\sup_{x \in K}N(x - Tx, t) = 1$.

If $(X, || . ||)$ is a normed space and we consider $(X,N_x,\min)$ as its corresponding standard fuzzy topological space, then they have the same topology and hence, as an application, we deduce that:

**Corollary 3.1.** (see [2]) Bounded closed convex subsets of a Banach space $(X,|| . ||)$, always have the s.a.f.p.p. A subset $K$ of a Banach space $(X,|| . ||)$ satisfies the s.a.f.p.p. iff for any non expansive map $T : K \rightarrow K$ there exists a sequence $\{x_n\} \subseteq K$ such that

$$\lim_{n \rightarrow \infty} ||x_n - Tx_n|| = 0.$$

In the following theorem we recognize a large number of subsets possessing the s.a.f.p.p.

**Theorem 3.2.** If $A$ is a dense subset of a fuzzy Banach space $(X,N,\ast)$ having the s.a.f.p.p. then $A$ has the s.a.f.p.p.

**Proof.** For a non expansive map $T : X \rightarrow X$, first we observe that, for $t > 0$,

$$\sup \{N(x - Tx, t), x \in A\} = \sup \{N(x - Tx, t), x \in X\}.$$

It is clear that

$$\sup \{N(x - Tx, t), x \in A\} \leq \sup \{N(x - Tx, t), x \in X\}.$$ 

Suppose that $y \in X$. Then there exists a sequence $\{y_n\} \subseteq A$ such that $y_n \rightarrow y$. Also we know that

$$\sup \{N(x - Tx, t), x \in A\} \geq \sup \{N(y_n - Ty_n, t), \forall n\}.$$ 

By taking limsup, we have

$$\limsup \{N(x - Tx, t), x \in A\} \geq N(y - Ty, t).$$

This is true for all $y \in X$ and hence

$$\sup \{N(x - Tx, t), x \in A\} \geq \sup \{N(y - Ty, t), y \in X\}.$$ 

Now for any non expansive map $T : A \rightarrow A$, given any $x \in X$, there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \rightarrow x$, as $n \rightarrow \infty$. Then, as $T : A \rightarrow A$ is continuous, it can be extended to $X$ by $Tx = \lim_{n \rightarrow \infty} Tx_n$. Hence we can see that $T$ is a non expansive map on $X$, and hence, by our observation

$$\sup \{N(x - Tx, t), x \in A\} = \sup \{N(x - Tx, t), x \in X\}.$$ 

Since $X$ has the s.a.f.p.p., for each non expansive map $T$ on $A$, we have

$$\sup \{N(x - Tx, t), x \in A\} = 1,$$

and so $A$ has the s.a.f.p.p.
Theorem 3.3. Each F-bounded, closed and starshaped subset of a fuzzy Banach space $\langle X, N, \ast \rangle$ has the s.a.f.p.p.

**Proof.** Suppose that $C$ be a F-bounded, closed and starshaped subset of a fuzzy normed space $X$ with center $p$. For any non expansive map $T : C \rightarrow C$ and $n = 1, 2, ..., \infty$, define $T_n : C \rightarrow C$, by

$$T_n x = \frac{n}{n+1} T x + \frac{1}{n+1} p, \forall x \in C.$$ 

Then $T_n$ is a contraction on $C$. So by proposition 2.1, it has a unique fixed point $x_n \in C$. Now for each $n \in \mathbb{N}$ and $t > 0$, consider

$$N(T x_n - x_n, t) = N(T x_n - T_n x_n, t) = N(T x_n - \frac{n}{n+1} T x_n - \frac{1}{n+1} p, t) = N(T x_n - p, t(n+1)).$$

Since $C$ is F-bounded, as $n \rightarrow \infty$, by (FN-6), we see that $N(T x_n - x_n, t) \rightarrow 1$ and hence $X$ has the s.a.f.p.p. $\square$

As applications of the above theorem, we have two interesting results about the s.a.f.p.p. on normed spaces and extended fixed point theory.

**Corollary 3.2.** (see [2]) Every closed bounded starshaped subset of a Banach space $X$ has the s.a.f.p.p.

**Corollary 3.3.** For any F-bounded, closed and starshaped subset $C$ of a fuzzy Banach space $\langle X, N, \ast \rangle$, suppose that $T : C \rightarrow C$ be a non expansive and semi-compact map. Then $T$ has a fixed point in $C$.

**Proof.** Take $p$ in the star center of $C$. Then, according to Theorem 3.3, we have a contraction $T_n : C \rightarrow C$ and a sequence $\{x_n\} \in C$ such that $N(T x_n - x_n, t) \rightarrow 1$, as $n \rightarrow \infty$. Since $T$ is semi-compact, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to $x^* \in C$. By continuity of $T$, $T x_{n_k} \rightarrow T x^*$. Finally, consider

$$T_{n_k} x_{n_k} = x_{n_k} = \frac{n_k}{n_k+1} T x_{n_k} + \frac{1}{n_k+1} p.$$ 

As $k \rightarrow \infty$, we have $T x^* = x^*$ and hence $x^*$ will be a fixed point of $T$ in $C$. $\square$

**Corollary 3.4.** For any F-bounded, closed and starshaped subset $C$ of a reflexive fuzzy Banach space $\langle X, N, \ast \rangle$, suppose that $T : X \rightarrow X$ be a non expansive mapping such that be invariant under $C$ and $I - T$ is demi-closed. Then $T$ has a fixed point.

**Proof.** For F-bounded, closed and starshaped subset $C$ of a reflexive fuzzy Banach space $\langle X, N, \ast \rangle$, let $T : X \rightarrow X$ be a non expansive mapping such that $T(C) \subset C$ and $I - T$ is demi-closed. For $n = 1, 2, ..., \infty$, define $T_n : C \rightarrow C$ by

$$T_n x = \frac{n}{n+1} T x + \frac{1}{n+1} p, \forall x \in C,$$

then by theorem 3.3, we have an F-bounded sequence $\{x_n\} \in C$ such that for some $t > 0$, $N(T x_n - x_n, t) \rightarrow 1$, as $n \rightarrow \infty$, and since $X$ is reflexive, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which convergent weakly to $x^*$. But $I - T$ is demi-closed. Then $(I - T)x^* = 0$ and hence $T x^* = x^*$. $\square$

Also if we consider the min (or product) $t$-norm and use the equivalency between the standard fuzzy normed topology and the ordinary normed topology, we have the following results:

**Corollary 3.5.** (see [2]) Let $C$ be a closed bounded starshaped subset of a Banach space $X$. Suppose that $T : C \rightarrow C$ be a non expansive and semi-compact map on $C$. Then $T$ has a fixed point in $C$.

**Definition 3.3.** Suppose that $K$ be a subset of a fuzzy normed space $\langle X, N, \ast \rangle$. A map $T : K \rightarrow X$ is said to be demi-compact if for any F-bounded sequence $\{x_n\}$ in $K$, such that $(x_n - T x_n) \rightarrow z \in K$, as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ and a point $p \in K$ such that $x_{n_k} \rightarrow p$, as $k \rightarrow \infty$, and $(I - T)p = z$. 

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Theorem 3.4. Suppose that C be an F-bounded, closed and starshaped subset of a fuzzy normed space (X, N, +) with center p and let 0 ∈ C. Then, each demi-compact map \( T : C \rightarrow C \) has a unique fixed point.

Proof. For a non expansive demi-compact map \( T : C \rightarrow C \) and \( n = 1, 2, \ldots \), define

\[
T_n x = \left[ \frac{n}{n+1} \right] T x + \left[ \frac{1}{n+1} \right] p, \forall x \in C.
\]

Then \( T_n : C \rightarrow C \) is a contraction and hence by proposition 2.1, for each \( \varepsilon > 0 \), it has a \( \varepsilon \)-fixed point \( x_\varepsilon \in C \), in particular for \( n = 1, 2, \ldots \) and for some \( t > 0 \), there exists \( x_n \in C \) such that \( N(T x_n - x_n, t) > 1 - \frac{1}{n} \). Now for some \( t_0 > 0 \), observe that

\[
N(x_n - T x_n, t) \geq N(x_n - T x_n, t - t_0) + N(T x_n - T x_n, t_0)
\]

\[
> (1 - \frac{1}{n}) N(\frac{1}{n+1} T x + \frac{1}{n+1} p - T x_n, t)
\]

\[
= (1 - \frac{1}{n}) N(p - T x_n, (n+1) t_0).
\]

Since \( N(x, t) \) is continuous with respect to \( t \) and also \( * \) is a continuous map, as \( n \rightarrow \infty \) we have \( (I - T)x_n \rightarrow 0 \). Now the bounded sequence \( \{x_n\} \subseteq C \) has a convergent subsequence \( \{x_{n_k}\} \) which converges to a point \( x_0 \in C \) and \( T : C \rightarrow C \) is demi-compact. So, as \( k \rightarrow \infty \), we have \( (I - T)x_0 = 0 \).

In the last part of this section, we try to consider the notion of an approximate eigenvalue for a self mapping on a given space and see some of its aspects.

Definition 3.4. For a fuzzy normed space \( (X, N, +) \), let \( T : X \rightarrow X \) be a self mapping on X. An element \( k \in K \) is said to be an approximate eigenvalue of \( T \) if there exists a sequence \( \{x_n\} \) in \( X \) and a constant \( 0 < \beta < 1 \) such that for each \( n \), \( N(x_n, t_n) = \beta \), for some \( t_n \in \mathbb{R} \), depending on \( x_n \), and moreover \( N(T x_n - k x_n, t) \rightarrow 1 \), as \( n \rightarrow \infty \), for some \( t > 0 \).

Since \( N(x, \cdot) \) is a continuous non decreasing function and \( \lim_{t \rightarrow \infty} N(x, t) = 1 \), by mean value theorem such \( \beta \) always exists.

Definition 3.5. Let \( (X, N, +) \) be a fuzzy normed space. A subset \( K \) of \( X \) is said to have \( F \)-bounded approximate fixed point property (f.b.a.f.p.p.) if for each non expansive map \( T : K \rightarrow K \), there exists an \( F \)-bounded sequence \( \{x_n\} \) in \( K \) such that for some \( t > 0 \)

\[
\lim_{n \rightarrow \infty} N(x_n - T x_n, t) = 1.
\]

Theorem 3.5. Let \( (X, N, +) \) be a fuzzy normed space, \( A \) be a subset of \( X \) such that \( \alpha A \subseteq A \), \( \forall \alpha > 0 \), and \( 0 \notin A \). Also suppose that \( A \) has the f.b.a.f.p.p. Then for each \( \lambda \geq 1 \), \( \lambda \) is an approximate eigenvalue of any non expansive homogeneous map on \( A \).

Proof. Let \( A \subseteq X \) and \( \alpha A \subseteq A \), \( \forall \alpha > 0 \), and \( 0 \notin A \). Moreover suppose that \( A \) has the f.b.a.f.p.p., and \( T : A \rightarrow A \) be any non expansive homogeneous map. For any given \( \lambda \geq 1 \) and for each \( x \in A \), define \( T_\lambda x = \frac{1}{\lambda} T x \). Then \( T_\lambda \) on \( A \) is a homogeneous non expansive map. Since \( A \) has the f.b.a.f.p.p., there exists an \( F \)-bounded sequence \( \{x_n\} \) in \( A \) such that \( \lim_{n \rightarrow \infty} N(x_n - T_\lambda x_n, t_0) \rightarrow 1 \), for some \( t_0 > 0 \). Since \( \{x_n\} \) is \( F \)-bounded, by definition 2.3 and mean value theorem, for each \( n \in \mathbb{N} \), there exist \( 0 < \beta_n < 1 \) and \( t_n \in \mathbb{R} \) such that \( N(x_n, t_n) = \frac{\beta_n}{2} \). Take \( \beta = \sup_{n \in \mathbb{N}} \frac{\beta_n}{2} \). Again by mean value theorem and nondecreasing property of \( N(\cdot, t) \) with respect to \( t \), we can choose the sequence \( \{x_n\} \) in \( \mathbb{R} \) such that \( N(x_n, t_0) = \beta \). On the other hand we have

\[
\lim_{n \rightarrow \infty} N(T x_n - \lambda x_n, \lambda t_0) = \lim_{n \rightarrow \infty} N(T_\lambda x_n - x_n, t_0) = 1.
\]

So by definition 3.4, \( \lambda \) is an approximate eigenvalue of \( T \) and hence the proof is complete.

Corollary 3.6. Suppose that \( (X, N, +) \) be a fuzzy normed space, and \( A \) be a subset of \( X \) such that \( \alpha A \subseteq A \), \( \forall \alpha > 0 \), and \( 0 \notin A \). If \( \lambda \) has the f.b.a.f.p.p., then for each homogeneous Lipschitzian map \( T : A \rightarrow A \), the Lipschitzian constant \( L \) is an approximate eigenvalue of \( T \).

Proof. Define \( T_\lambda x = \frac{1}{\lambda} T x \), and use theorem 3.5. 

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Definition 3.6. [9] Let $X$ be a normed linear space. A subset $K$ of $X$ is said to have bounded approximate fixed point property b.a.f.p.p. if for every non-expansive map $T: K \rightarrow K$, there exists a bounded sequence $\{x_n\}$ in $K$ such that

$$\lim_{n \to \infty} \|x_n - Tx_n\| = 0.$$ 

Also by considering the standard fuzzy normed space (with production or min t-norm), as a direct result of theorem 3.5, we have:

Corollary 3.7. (see [9]) Let $X$ be a normed linear space. $A$ be a subset of $X$ such that $\alpha A \subseteq A$, $\forall \alpha > 0$, and $0 \notin A$. Also suppose that $A$ has the b.a.f.p.p. Then for any Lipschitzian map $T: A \rightarrow A$, the Lipschitzian constant $L$ is an approximate eigenvalue of $T$.

4 Conclusion

By introducing some notions such as approximate fixed point property for subsets of a fuzzy Banach space and approximate eigenvalues of a self mapping, we proved some interesting results, and also many important theorems in Fixed Point Theory on the Banach Spaces, arised as their consequences.

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