Complex fuzzy dynamical systems and Stability of the equilibrium Point

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Abstract
In this work, we define a complex fuzzy dynamical systems and we introduce the concept of complex fuzzy equilibrium point. Also we obtain some result regarding the stability. Finally, some examples are given to illustrate the obtained results.

Keywords: Dynamical systems, Equilibrium points, Fuzzy sets.

1 Introduction

A first characterization of dynamical systems in fuzzy metric space is presented in [7]. Let \( \phi_t \) be a flow generated by solutions of autonomous differential equation. M.T. Mizukoshi et al. showed in [9] that the family of applications \( \phi_t \), indexed by \( t \in \mathbb{R} \), obtained by Zadeh’s extension (see [13]) on initial condition of \( \phi_t \), satisfies conditions that characterize \( \phi_t \) as a dynamical system in the metric space \( E^n \). The stability of fuzzy dynamical systems is presented in [2, 8].

Since D. Ramot introduced a new concept in the context of fuzzy sets theory (complex fuzzy set) and there are many examples of application namely solar activity, signal processing, etc (see [12]).

With the same principle of Zadeh’s extension, in this paper we are interested in defining a complex fuzzy dynamical system and we study the stability of complex fuzzy dynamical systems.

2 Preliminaries

Firstly, we review some basic concepts, notations and technical results that are necessary in our study.

Let \( \mathcal{P}_K(\mathbb{R}^n) \) denote the family of all nonempty compact convex subsets of \( \mathbb{R}^n \) and define the addition and scalar multiplication in \( \mathcal{P}_K(\mathbb{R}^n) \) as usual. Let \( A \) and \( B \) be two nonempty bounded subsets of \( \mathbb{R}^n \). The distance between \( A \) and \( B \) is defined by the Hausdorff metric,

\[
d(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}
\]

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where $\| \|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. Then it is clear that $(\mathcal{P}_K(\mathbb{R}^n), d)$ becomes a complete and separable metric space (see [11]).

Denote
\[ E^n = \{ u : \mathbb{R}^n \rightarrow [0,1] \mid u \text{ satisfies (i)-(iv) below} \} , \]
where

(i) $u$ is normal i.e there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,

(ii) $u$ is fuzzy convex,

(iii) $u$ is upper semicontinuous,

(iv) $[u]^0 = cl\{ x \in \mathbb{R}^n \mid u(x) > 0 \}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^{\alpha} = \{ t \in \mathbb{R}^n \mid u(t) \geq \alpha \}$. Then from (i)-(iv), it follows that the $\alpha$-level set $[u]^{\alpha} \in \mathcal{P}_K(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$.

According to Zadeh’s extension principle, we have addition and scalar multiplication in fuzzy number space $E^n$ as follows:

\[ ([u] + [v])^{\alpha} = [u]^{\alpha} + [v]^{\alpha} , \quad [ku]^{\alpha} = k[u]^{\alpha} \]

where $u, v \in E^n$, $k \in \mathbb{R}^n$ and $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \rightarrow \mathbb{R}^+$ by the equation

\[ D(u,v) = \sup_{0 \leq \alpha \leq 1} d\left([u]^{\alpha}, [v]^{\alpha}\right) \]

where $d$ is the Hausdorff metric for non-empty compact sets in $\mathbb{R}^n$. Then it is easy to see that $D$ is a metric in $E^n$.

Using the results in [11], we know that

1. $(E^n, D)$ is a complete metric space;
2. $D(u+w,v+w) = D(u,v)$ for all $u, v, x \in E^n$;
3. $D(ku,kv) = |k| D(u,v)$ for all $u, v \in E^n$ and $k \in \mathbb{R}^n$.

Note that, $(E^n, D)$ is a complete metric space which can be embedded isomorphically as a cone in a Banach space (see [10]).

On $E^n$, we can define the subtraction $\odot$, called the $H$-difference (see [4]) as follows: $u \odot v$ has sense if there exists $w \in E^n$ such that $u = v + w$.

Denote $\mathcal{C}_a = \mathcal{C}([0,a], E^n) = \{ f : [0,a] \rightarrow E^n; f \text{ is continuous on}[0,a]\}$, endowed with the metric

\[ H(u,v) = \sup_{t \in [0,a]} D(u(t), v(t)) \]

Then $(\mathcal{C}_a, H)$ is a complete metric space. We define the following set

\[ \tilde{E}^{2n} = \{ (u,v) \in E^n \times E^n \mid \exists x_0 \in \mathbb{R}^n s.t \ u(x_0) = v(x_0) = 1 \} \]

For $f = (u,v) \in \tilde{E}^{2n}$, define $[f]^{(\alpha,\beta)} = [u]^{\alpha} \cap [v]^{\beta}$ for all $\alpha, \beta \in [0,1]$.

For $f = (u_f,v_f), g = (u_g,v_g) \in \tilde{E}^{2n}$ and $c$ is a scalar, let

\[ f + g = (u_f + u_g, v_f + v_g), \quad cf = (cu_f, cv_f). \]

Define $\tilde{D} : \tilde{E}^{2n} \times \tilde{E}^{2n} \rightarrow \mathbb{R}^+$ by the equation

\[ \tilde{D}(f,g) = \tilde{D}\left((u_f,v_f), (u_g,v_g)\right) = \max \{ D(u_f,u_g), D(v_f,v_g) \} . \]
Then \( D \) is a metric in \( E^{2n} \) and \( (\tilde{E}^{2n}, D) \) is a complete metric space (see [6]).

There exists an embedding \( I \) from \( E^{2n} \) into a Banach space (see [1]).

Recall from [10] that on \( E^n \) we define \( \bar{D} \in E^n \) by \( \bar{D}(x) = 1 \) when \( x = 0 \) and \( \bar{D}(x) = 0 \) otherwise. The zero element on \( E^{2n} \) then reads \( \bar{D}_2(x) = (\bar{D}(x), \bar{D}(x)) \in E^{2n} \). We have \( \bar{D}_2(0) = (1, 1) \), verifying that \( \bar{D}_2 \in E^{2n} \).

The polar representation of the membership function as presented in [14]

\[
u(V, z) = r(V)e^{i\theta}(z).
\]

For \( x \in \mathbb{R}^n \), the polar form of \( f \) is defined as follows:

\[
f(x) = r(x)e^{2i\phi(x)},
\]

where \( r, \phi : \mathbb{R}^n \to [0, 1] \), and we denote \( f \) by \((r, \phi)\). The scaling factor is taken to be \( 2\pi \), allowing the range of \( f \) to be the entire unit circle.

Because \( e^{2i\phi} \) is periodic, we take the value of \( \phi \) giving the maximum distance from \( e^0, \phi = 0.5 \), to be the maximum membership value. Now, while \( |r|^n \) can be defined just as \(|u|^n\) above, the corresponding level sets for \( \phi \), denoted \([\phi]^{(\beta)}\), must be defined differently to account for the periodicity:

\[
[\phi]^{(0)} = \{x \in \mathbb{R}^n : 0 < \phi(x) < 1\},
\]

\[
[\phi]^{(\beta)} = [\phi]^{(1-\beta)}, \text{ for all } \beta \in [0, 1].
\]

We can then define the level sets \([f]^{(\alpha, \beta)}\) as \([f]^{(\alpha, \beta)} = [r]^{\alpha} \cap [\phi]^{\beta}\), or by the relations

\[
[f]^{(\alpha, \beta)} = \{x \in \mathbb{R}^n : r(x) \geq \alpha, \phi(x) \in [\beta, 1-\beta]\}
\]

\[
[f]^{(\alpha, 0)} = \{x \in \mathbb{R}^n : r(x) \geq \alpha > 0, 0 < \phi(x) < 1\}
\]

\[
[f]^{(0, \beta)} = \{x \in \mathbb{R}^n : r(x) > 0, \phi(x) \in [\beta, 1-\beta], \beta \in [0, 0.5]\}
\]

\[
[f]^{(0, 0)} = \{x \in \mathbb{R}^n : r(x) > 0, 0 < \phi(x) < 1\}
\]

together with

\[
[f]^{(\alpha, 1-\beta)} = [f]^{(\alpha, 1-\beta)}, \text{ for all } \alpha, \beta \in [0, 1].
\]

Denote

\[
F^n = \{w : \mathbb{R}^n \to [0, 1] \mid w \text{ satisfies (i)-(iv) below}\},
\]

where

(i) There exists an \( x_0 \in \mathbb{R}^n \) such that \( w(x_0) = 0.5\),

(ii) \( w \) is monotone,

(iii) \( w \) is upper semi-continuous on \( K_1 \) and lower semi-continuous on \( K_2 \) where,

\( K_1 = \{x \in \mathbb{R}^n \mid 0 \leq w(x) < 0.5\} \) and \( K_2 = \{x \in \mathbb{R}^n \mid 0.5 < w(x) \leq 1\} \),

(iv) \( K_1 \cup K_2 \) is compact.

Now, take

\[
\tilde{E}^{2n} = \{(r, \phi) \in E^n \times F^n \mid \exists x_0 \in \mathbb{R}^n \text{ s.t. } r(x_0) = 1 \text{ and } \phi(x_0) = 0.5\}.
\]

\( E^{2n} \) is embeddable into a Banach space (see [6]). Then the following results, apply equally to the space \( E^{2n} \) in the Cartesian case and to the space \( E^{2n} \) in the polar case.

For brevity, we shall let \( E = E^{2n} \) when dealing with the Cartesian complex form, and \( E = \tilde{E}^{2n} \) when dealing with the polar complex form.

We define differentiability as in [5] in terms of the Hukuhara difference. For \( f, g \in E \), if there exists \( h \in E \) such that \( g + h = f \), we write \( f - g = h \) and call \( h \) the difference of \( f \) and \( g \).

Let \( I = [0, a] \subset \mathbb{R} \) be a compact interval.
Definition 2.1. We call a mapping $F : I \rightarrow \mathbb{E}$ differentiable at $t_0 \in I$ if there exists some $F'(t_0) \in \mathbb{E}$ such that the following limits exist and are equal to $F'(t_0) \in \mathbb{E}$:

$$
\lim_{h \rightarrow 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{F(t_0) - F(t_0 - h)}{h}
$$

Let us establish some notation and recall known results about fuzzy dynamical system.

Definition 2.2. We say that a family of continuous maps, defined on the complete metric space $(\bar{X}, \bar{d})$ let

$$
\phi : \mathbb{R}^+ \times X \rightarrow X
$$

is a dynamical system, if $\phi$ satisfies

1. $\phi_0(x_0) = x_0$
2. $\phi_t \phi_s(x_0) = \phi_{t+s}(x_0)$,

for all $t, s \in \mathbb{R}^+$ and $x_0 \in X$.

The set $X$ is called phase space of the dynamical system. In the case of $X \subset \mathbb{R}^2$, $X$ is said to be phase plane.

In dynamical systems context, an orbit of a point $x_0 \in X$ is the subset of the phase space defined by

$$
\theta(x_0) = \bigcup_{t \in \mathbb{R}^+} \phi_t(x_0) = \{\phi_t(x_0), \ t \in \mathbb{R}^+\}.
$$

and for each subset $B \subset X$ we have $\theta(B) = \bigcup_{x_0 \in B} \theta(x_0)$.

The set $\theta(x_0)$ is called periodic orbit if there exists $\tau > 0$ such that $\phi_{t+\tau}(x_0) = \phi_t(x_0)$. The smallest number $\tau > 0$ for which this property is satisfied is called period of the orbit [3].

The $\omega$-limit of a subset $B \subset X$ is defined as

$$
W(B) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \phi_t(B).
$$

Theorem 2.1. (see [8]) Let $\phi : U \rightarrow U$ be a deterministic dynamical system. Then $\hat{\phi}$, defined by Zadeh’s extension applied in $\phi$ has the following properties:

1. $\hat{\phi}_0(x_0) = x_0$, $\forall x_0 \in \mathcal{F}(U)$,
2. $\hat{\phi}_{t+s}(x_0) = \hat{\phi}_t \hat{\phi}_s(x_0)$, $\forall x_0 \in \mathcal{F}(U)$, $t, s \geq 0$.

Thus, $\hat{\phi}$ is a dynamical system in $\mathcal{F}(U)$ and we will call it fuzzy dynamical system.

Theorem 2.2. ([2]) Let $\hat{\phi}$ be Zadeh’s extension of a dynamical system $\phi$. For every $x, y$, the following expression holds

$$
D(\hat{\phi}_t(x), \hat{\phi}_t(y)) \leq e^{kt}D(x, y), \text{ for some constants } k > 0, t \geq 0.
$$

Theorem 2.3. ([8]) Let $\bar{x} \in U$ be an equilibrium point of $\phi$. Then

1. $\bar{x}$ is stable for $\phi$ if and only if $X(\bar{x})$ is stable for $\hat{\phi}$;
2. $\bar{x}$ is asymptotically stable for $\phi$ if and only if $X(\bar{x})$ is asymptotically stable for $\hat{\phi}$.

3 Main Results

In this section we give a definition of dynamical system in $\mathbb{E}$. 

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3.1 Cartesian complex fuzzy dynamical system

Let $\varphi_i$ be a dynamical system defined in a subset $A \subseteq \mathbb{R}^n$, and $\psi_i$ be a dynamical system defined in a subset $B \subseteq \mathbb{R}^n$, we define the family of applications $\tilde{\varphi}_t$ on $\mathcal{P}(A) \times \mathcal{P}(B)$, defined by
\[
\tilde{\varphi}_t(x, y) = (\tilde{\varphi}_t(x), \tilde{\psi}_t(y)), \quad (x, y) \in \mathcal{P}(A) \times \mathcal{P}(B)
\]
where $\tilde{\varphi}_t$ is defined by Zadeh’s extension applied in $\varphi_i$, and $\tilde{\psi}_t$ is defined by Zadeh’s extension applied in $\psi_i$.
For $(x, y) \in \mathcal{P}(A) \times \mathcal{P}(B)$, we have
\[
\tilde{\varphi}_0(x, y) = (\tilde{\varphi}_0(x), \tilde{\psi}_0(y)) = (x, y).
\]
For $(x, y) \in \mathcal{P}(A) \times \mathcal{P}(B)$, and $t, s \geq 0$, we have
\[
\tilde{\varphi}_{t+s}(x, y) = (\tilde{\varphi}_{t+s}(x), \tilde{\psi}_{t+s}(y)) = (\tilde{\varphi}_t \tilde{\varphi}_s)(x, y).
\]
Let $(u, v) \in \tilde{E}^{2n}$, then there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = v(x_0) = 1$, suppose that $\varphi_i(x_0) = \psi_i(x_0)$, we have
\[
\tilde{\varphi}_t(u)(\varphi_i(x_0)) = \tilde{\psi}_t(v)(\psi_i(x_0)) = \psi_i(v)(\varphi_i(x_0)) = 1.
\]
Thus, $\tilde{\varphi}_t(u, v) = (\tilde{\varphi}_t(u), \tilde{\psi}_t(v))$ is well defined on $\tilde{E}^{2n}$.
For $(x, y) \in \tilde{E}^{2n}$, $t, s \geq 0$, we have
\[
\tilde{D}((\tilde{\varphi}_t(x), \tilde{\psi}_t(y)), (\tilde{\varphi}_s(x), \tilde{\psi}_s(y))) = \max\{D(\tilde{\varphi}_t(x), \tilde{\varphi}_s(x)), D(\tilde{\psi}_t(y), \tilde{\psi}_s(y))\}
\]
So, $t \mapsto \tilde{\varphi}_t(x, y)$ is continuous, since $t \mapsto \tilde{\varphi}_t(x)$ and $t \mapsto \tilde{\psi}_t(y)$ are continuous. Then, $\tilde{\varphi}_t$ is a dynamical system in $\tilde{E}^{2n}$ and we will call it Cartesian complex fuzzy dynamical system.
Particularly, if $\varphi_i$ be a dynamical system defined in a subset $A \subseteq \mathbb{R}^n$, then $\tilde{\varphi}_t = (\tilde{\varphi}_t, \tilde{\psi}_t)$ is a Cartesian complex fuzzy dynamical system.

3.2 Polar complex fuzzy dynamical system

Let $\varphi_i$ be a dynamical system defined in a subset $A \subseteq \mathbb{R}^n$ and $\psi_i$ be a dynamical system defined in a subset $B \subseteq \mathbb{R}^n$, we define the family of applications $\hat{\varphi}_t$ on $\mathcal{P}(A) \times \mathcal{P}(B)$, defined by
\[
\hat{\varphi}_t(x, y) = (\hat{\varphi}_t(x), \hat{\psi}_t(y)), \quad (x, y) \in \mathcal{P}(A) \times \mathcal{P}(B)
\]
where $\hat{\varphi}_t$ is defined by Zadeh’s extension applied in $\varphi_i$,
\[
\hat{\varphi}_t(y) = \begin{cases} 
\frac{1}{2} \hat{\psi}_t(2y), & \text{if } y \in K_1 \\
\frac{1}{2} (2 - \hat{\psi}_t(2 - 2y)), & \text{if } y \in K_2
\end{cases}
\]
and $\hat{\psi}_t$ is defined by Zadeh’s extension applied in $\psi_i$.
Note that $\hat{\varphi}_t$ is well defined on $F^n$. Indeed
For $y' \in E^n$, we can write
\[
y = \begin{cases} 
\frac{1}{2} y', & \text{if } y \in K_1 \\
\frac{1}{2} (2 - y'), & \text{if } y \in K_2
\end{cases}
\]
We then have $y \in F^n, y' = 2y$ on $K_1$ and $y' = 2 - 2y$ on $K_2$. Thus $\hat{\psi}_t$ is well defined on $E^n$. Therefore $\hat{\phi}_t$ is well defined on $F^n$.

For $y \in F^n$, we have
\[
\hat{\phi}_0(y) = \begin{cases} 
\frac{1}{2} \hat{\psi}_0(2y) = y, & \text{if } y \in K_1 \\
\frac{1}{2} (2 - \hat{\psi}_0(2 - 2y)) = y, & \text{if } y \in K_2 
\end{cases}
\]

For $y \in F^n, t, s \geq 0$, we have
\[
\hat{\phi}_{t+s}(y) = \begin{cases} 
\frac{1}{2} \hat{\psi}_{t+s}(2y) = \frac{1}{2} \hat{\psi}_t \hat{\psi}_s(2y), & \text{if } y \in K_1 \\
\frac{1}{2} (2 - \hat{\psi}_{t+s}(2 - 2y)) = \frac{1}{2} (2 - \hat{\psi}_t \hat{\psi}_s(2 - 2y)), & \text{if } y \in K_2 
\end{cases}
\]

and
\[
\hat{\phi}_t \hat{\phi}_s(y) = \begin{cases} 
\frac{1}{2} \hat{\psi}_t(2\hat{\phi}_s(y)) = \frac{1}{2} \hat{\psi}_t \hat{\psi}_s(2y), & \text{if } y \in K_1 \\
\frac{1}{2} (2 - \hat{\psi}_t(2 - 2\hat{\phi}_s(y))) = \frac{1}{2} (2 - \hat{\psi}_t \hat{\psi}_s(2 - 2y)), & \text{if } y \in K_2 
\end{cases}
\]

Then $\hat{\phi}_{t+s}(y) = \hat{\phi}_t \hat{\phi}_s(y)$.

In addition, we have $t \mapsto \hat{\phi}_t(y)$ is continuous for all $y \in F^n$ with continuity of $t \mapsto \hat{\psi}_t(z), z \in E^n$.

It follows that
1. $\overline{\hat{\phi}}_0(x, y) = (x, y)$;
2. $\overline{\hat{\phi}}_{t+s}(x, y) = \overline{\hat{\phi}}_t \overline{\hat{\phi}}_s(x, y)$;
3. $t \mapsto \overline{\hat{\phi}}_t(x, y)$ is continuous for all $(x, y) \in E^n \times F^n$.

Let $(u, v) \in \hat{E}^{2n}_s$, $w = 2v$ and $w' = 2 - 2v$, then there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$ and $v(x_0) = \frac{1}{2}$, suppose that, $\varphi_1(x_0) = \psi_1(x_0)$, then we have

$w(x_0) = 1, \quad \text{and} \quad w'(x_0) = 1$.

and
\[
\hat{\phi}_t(v)(\varphi_1(x_0)) = \hat{\phi}_t(v)(\psi_1(x_0)) = \begin{cases} 
\frac{1}{2} \psi_0(w)(\psi_1(x_0)) = \frac{1}{2}, & \text{if } v \in K_1 \\
\frac{1}{2} (2 - \hat{\psi}_t(w')(\psi_1(x_0))) = \frac{1}{2}, & \text{if } v \in K_2 
\end{cases}
\]

Therefore,
\[
\hat{\phi}_t(u)(\varphi_1(x_0)) = 1 \quad \text{and} \quad \hat{\phi}_t(v)(\varphi_1(x_0)) = \frac{1}{2}.
\]

Consequently, $\overline{\hat{\phi}}_t(u, v) = \left( \hat{\phi}_t(u), \hat{\phi}_t(v) \right)$ is well defined on $\hat{E}^{2n}_s$ and $\overline{\hat{\phi}}_t$ is a dynamical system in $\hat{E}^{2n}_s$ and we will call it Polar complex fuzzy dynamical system.

we shall let $\hat{E} = \hat{E}^{2n}$ or $\hat{E}^{2n}_s$.

**Theorem 3.1.** Let $\overline{\hat{\phi}}_t$ be a dynamical system in $E$. Then for each $(u, v), (u', v') \in E$, we have
\[
\hat{D} \left( \overline{\hat{\phi}}_t(u, v), \overline{\hat{\phi}}_t(u', v') \right) \leq e^{at} \hat{D}((u, v), (u', v')),
\]

For some constants $a > 0$ and $t \geq 0$. 
The equilibrium points of $\varphi_b$ point for the complex fuzzy dynamical system $A$

Let

Note that, $\varphi$, whose solution is given by

Example 3.1.

Theorem 3.2.

Definition 3.1.

Stability of complex fuzzy equilibrium Point

In this section, we establish a result relating the stability of the equilibrium points.

Definition 3.1. We say that $(u, v)$ is a complex fuzzy equilibrium point for $\Phi_t$ when

$$\Phi_t (u, v) = (u, v),$$

for each $t \geq 0$.

Theorem 3.2. $x$ is a common equilibrium point for $\varphi_b$ and $\psi_b$ if and only if, $(x_{[1]}, x_{[2]})$ is a complex fuzzy equilibrium point for the complex fuzzy dynamical system $\Phi_t$ defined by 3.1.

Proof. We have $\varphi_b (x) = \psi_b (x) = x$, if and only if

$$\Phi_t (x_{[1]}, x_{[2]}) = (\Phi_t (x_{[1]}), \psi_b (x_{[1]})) = (x_{[\Phi_b(x_{[1])]}, x_{[\psi_b(x_{[1])}]}) = (x_{[1]}, x_{[2]}).$$

Example 3.1. Consider the deterministic problem

$$\begin{cases}
    x' = x(1-x) \\
    x(0) = x_0,
\end{cases}
\tag{3.3}$$

whose solution is given by

$$\varphi_b (x_0) = \frac{x_0}{x_0 + (1-x_0)e^{-t}}. \tag{3.4}$$

Note that, $\varphi_b$ is a dynamical system on $\mathbb{R}^+$. The equilibrium points of $\varphi_b$ are 0 and 1. From the previous theorem $(x_{[0]}, x_{[0]})$ and $(x_{[1]}, x_{[1]})$ are complex fuzzy equilibrium Points of $\Phi_t = (\Phi_b, \Phi_c)$ the complex fuzzy dynamical system obtained by Zadehs extension applied to $\varphi_b$.

Let $A, B \subset \mathbb{R}^n$ and consider $\hat{C}_{AB} \subset \hat{E}^{2n}$ defined by

$$\hat{C}_{AB} = \left\{ (u, v) \in \hat{E}^{2n} : [u]^A \subset A, [v]^B \subset B \right\}$$
Theorem 3.3. If $A$ is invariant by $\varphi$, and $B$ is invariant by $\psi$, then $\hat{C}_{AB}$ is invariant by $\hat{\pi}$.

Proof. Suppose $(u, v) \in \hat{C}_{AB}$, we have

$$\hat{\pi}(u, v) = (\hat{\varphi}(u), \hat{\psi}(v)),$$

and

$$[\hat{\varphi}(u)]^\alpha = \varphi([u]^{\alpha}), \quad [\hat{\psi}(v)]^\beta = \psi([v]^{\beta}).$$

Since, $A$ is invariant by $\varphi$, $B$ is invariant by $\psi$ and $(u, v) \in \hat{C}_{AB}$, then

$$[u]^\alpha \subset A, \quad [v]^\beta \subset B,$$

and

$$\varphi([u]^\alpha) \subset A, \quad \psi([v]^\beta) \subset B.$$ 

It follows that $\hat{\pi}(u, v) \in \hat{C}_{AB}$, i.e $\hat{C}_{AB}$ is invariant by $\hat{\pi}$. 

Corollary 3.1. A is invariant by $\varphi$, and $\psi$, if and only if $\hat{C}_{AA}$ is invariant by $\hat{\pi}$.

Proof. ($\Rightarrow$) It follows immediately from the previous theorem. 

($\Leftarrow$) Suppose $x \in A$, we have $(\mathcal{X}_{t}, \mathcal{Y}) \in \hat{C}_{AA}$ and $\hat{\pi}(\mathcal{X}_{t}, \mathcal{Y}) \in \hat{C}_{AA}$. Which shows that

$$[\hat{\varphi}(\mathcal{X}_{t})]^\alpha = \varphi(\{x\}) \subset A, \quad [\hat{\psi}(\mathcal{Y})]^\beta = \psi(\{x\}) \subset A.$$

Example 3.2. Consider the dynamical system defined by 3.4. The set $A = [0, 1]$ is invariant by $\varphi$. According to Corollary 3.1,

$$\hat{C}_{AA} = \left\{ (u, v) \in \hat{E}^2 : [u]^{\alpha}, [v]^{\beta} \subset [0, 1], \right\}$$

is invariant by the complex fuzzy dynamical system $\hat{\pi} = (\hat{\varphi}, \hat{\psi})$.

Definition 3.2. Let $A, B \subset \mathbb{R}^n$ and $\theta(A), \theta(B)$ be the orbit determined by $\varphi$, and $\psi$, respectively. The set $\hat{\theta}_{AB}$ defined by

$$\hat{\theta}_{AB} = \left\{ (u, v) \in \hat{E}^{2n} : [u]^{\alpha} \subset \theta(A), [v]^{\beta} \subset \theta(B) \right\},$$

is called a complex fuzzy orbit.

Definition 3.3. Let $A, B \subset \mathbb{R}^n$ and $\omega(A), \omega(B)$ be the $\omega$–limit determined by $\varphi$, and $\psi$, respectively. The set $\hat{\omega}_{AB}$ defined by

$$\hat{\omega}_{AB} = \left\{ (u, v) \in \hat{E}^{2n} : [u]^{\alpha} \subset \omega(A), [v]^{\beta} \subset \omega(B) \right\},$$

is called a complex fuzzy $\omega$–limit.

Corollary 3.2. $\hat{\theta}_{AB}$ and $\hat{\omega}_{AA}$ are invariant by $\hat{\pi}$.

Proof. Immediate consequence of theorem 3.3, since the orbits and $\omega$–limit are invariant sets.
Definition 3.4. Let \((u, v)\) be a complex fuzzy equilibrium point for 
\(\tilde{f} = (\tilde{\phi}, \tilde{\psi})\):

1. \((u, v)\) is stable if and only if, \(\forall \varepsilon > 0, \exists \eta > 0\), such that, for \((u', v')\) \(\in E\)
\[D\left(\left(u', v'\right), (u, v)\right) < \eta \Rightarrow D\left(\tilde{f}(u', v'), (u, v)\right) < \varepsilon, \ \forall t \geq 0.\]

2. \((u, v)\) is asymptotically stable if it is stable and, there exists \(\alpha > 0\) such that
\[\lim_{t \to +\infty} D\left(\tilde{f}(u', v'), (u, v)\right) = 0, \ \forall (u', v') \in E, s.t. D\left(\left(u', v'\right), (u, v)\right) < \alpha.\]

Theorem 3.4. Let \(x\) be a common equilibrium point for \(\phi_t\) and \(\psi_t\), then:

1. \(x\) is stable for \(\phi_t\) and \(\psi_t\), if, and only if, \((x(t), x(s))\) is stable for \(\tilde{f}\).

2. \(x\) is asymptotically stable for \(\phi_t\) and \(\psi_t\), if, and only if, \((x(t), x(s))\) is asymptotically stable for \(\tilde{f}\).

Proof.

1. \((\Rightarrow)\) From theorem 2.3, we have, \(x\) is stable for \(\phi_t\) and \(\psi_t\), if, and only if, \(x(t)\) is stable for \(\tilde{\phi}\) and \(\tilde{\psi}\).
Which shows that, \(\forall \varepsilon > 0\), there exists \(\eta_1, \eta_2 > 0\) such that for \((u, v) \in E \subset E^n \times E^n\)
\[D\left(u, x\left(t\right)\right) < \eta_1, D\left(v, x\left(t\right)\right) < \eta_2 \Rightarrow D\left(\tilde{\phi}(u), x\left(t\right)\right) < \varepsilon, D\left(\tilde{\psi}(v), x\left(t\right)\right) < \varepsilon \ \forall t \geq 0.\]
For \(\eta = \max\{\eta_1, \eta_2\}\), we have
\[\tilde{D}\left(\left(u, v\right), \left(x\left(t\right), x\left(s\right)\right)\right) = \max\left\{D\left(u, x\left(t\right)\right), D\left(v, x\left(s\right)\right)\right\} < \eta,\]
and
\[\tilde{D}\left(\left(\tilde{f}(u, v)\right), \left(x\left(t\right), x\left(s\right)\right)\right) = \tilde{D}\left(\left(\tilde{\phi}(u), \tilde{\psi}(v)\right), \left(x\left(t\right), x\left(s\right)\right)\right) = \max\left\{D\left(\tilde{\phi}(u), x\left(t\right)\right), D\left(\tilde{\psi}(v), x\left(s\right)\right)\right\} < \eta.\]

\((\Leftarrow)\) If \((x(t), x(s))\) is stable for \(\tilde{f}\), then, there exists \(\eta > 0\), such that, for \((u, v) \in E^{2n}\)
\[\tilde{D}\left(\left(u, v\right), \left(x\left(t\right), x\left(s\right)\right)\right) < \eta \Rightarrow \tilde{D}\left(\left(\tilde{\phi}(u), \tilde{\psi}(v)\right), \left(x\left(t\right), x\left(s\right)\right)\right) < \eta,\]
which imply that,
\[D\left(u, x\left(t\right)\right) < \eta, D\left(v, x\left(s\right)\right) < \eta \Rightarrow D\left(\tilde{\phi}(u), x\left(t\right)\right) < \varepsilon, D\left(\tilde{\psi}(v), x\left(s\right)\right) < \varepsilon.\]
Hence, \(x(t)\) is stable for \(\tilde{\phi}\) and \(\tilde{\psi}\), then by theorem 2.3, \(x\) is stable for \(\phi_t\) and \(\psi_t\).

2. If \((x(t), x(s))\) is asymptotically stable for \(\tilde{f}\), then there exists \(\alpha > 0\) such that, for every \((u, v)\) satisfying
\[\tilde{D}\left(\left(u, v\right), \left(x\left(t\right), x\left(s\right)\right)\right) < \alpha,\]
we have
\[\lim_{t \to +\infty} \tilde{D}\left(\left(\tilde{\phi}(u), \tilde{\psi}(v)\right), \left(x\left(t\right), x\left(s\right)\right)\right) = 0.\]
This implies, for every \((u, v)\) satisfying \(D\left(u, x\left(t\right)\right) < \alpha\) and \(D\left(v, x\left(s\right)\right) < \alpha\), we have
\[\lim_{t \to +\infty} D\left(\tilde{\phi}(u), x\left(t\right)\right) = \lim_{t \to +\infty} D\left(\tilde{\psi}(v), x\left(s\right)\right) = 0.\]
Therefore, \(x(t)\) is asymptotically stable for \(\tilde{\phi}\) and \(\tilde{\psi}\). Also it is easy to see the other implication. According to theorem 2.3, 2 is proved.
Example 3.3. Consider dynamical system \( \Phi_t, \Psi_t : \mathbb{R} \rightarrow \mathbb{R} \) defined by

\[
\Phi_t(x) = e^{k_1 t} x, \quad \Psi_t(y) = e^{k_2 t} y, \quad x, y \in \mathbb{R}, \quad k_1, k_2 > 0, \quad t \geq 0.
\]

Note that 0 is an equilibrium point asymptotically stable for \( \Phi_t \) and \( \Psi_t \).

By theorem 3.2, we have that \( (X_{(0)}, Y_{(0)}) \) is an equilibrium point for \( \overline{\Phi}_t \) in \( \mathbb{R} \).

Moreover, according to theorem 3.4, \( (X_{(0)}, Y_{(0)}) \) is asymptotically stable for \( \overline{\Phi}_t \) defined by (3.1).

4 Conclusion

In this article we consider complex fuzzy dynamical systems obtained by Zadeh’s extension of two dynamical systems defined on two subsets of \( \mathbb{R}^n \) respectively. In the first section of this paper we establish the definition of a cartesian complex fuzzy dynamical system on the space defined by the cartesian representation of the membership function and in the second section we establish the definition of a polar complex fuzzy dynamical system on the space defined by the polar representation of the membership function. Such we have studied the stability of the complex fuzzy equilibrium Point, the stability concept is easily introduced as an extension of the classic theory for differential equations. As we have seen from the example presented, the results obtained in this work are fundamental to the comprehension of the behavior of the solution of complex fuzzy differential equations.

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References


http://dx.doi.org/10.1016/j.ins.2007.02.039

http://dx.doi.org/10.1016/S0165-0114(97)00094-8

http://dx.doi.org/10.1016/0022-247X(86)90093-4

http://dx.doi.org/10.1109/91.995119

http://dx.doi.org/10.1016/S0165-0114(98)00408-4

http://dx.doi.org/10.1002/int.20454