Graded fuzzy ring and control picture by coloring fuzzy location

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Abstract

In this article, the definition of graded fuzzy ring by group and some of related theorems has been investigated. Among it is the relation between a graded fuzzy ring with crisp graded ring by level sets and gain the set of cosets a graded ring on a fuzzy ideal with a special condition is graded ring. After that be graded fuzzy ring on a group ring and its result on Laurent polynomial is showed. Finally we prove a graded fuzzy ring for a specific group ring that is fuzzy on group elements and then we control a system of image processing by setting of standard colors.

Keywords: Fuzzy ring, Graded fuzzy ring, Laurent polynomial, Fuzzy group rings, Standard colors system.

1 Introduction


Graded algebras are much used in commutative algebra and algebraic geometry, homological algebra and algebraic topology. One example is the close relationship between homogeneous polynomials and projective varieties. Eslami explained a concept of graded fuzzy rings on natural numbers by direct sum of fuzzy subgroup and then found a way for fuzzification polynomial rings [1].

This article introduce a graded fuzzy ring by an arbitrary group. We prove that how a fuzzy ring is graded whenever all of its level sets be graded. We effort open a way toward introduction a fuzzy group ring by graded fuzzy ring and induce the results to Laurent polynomial that we extend Eslami work. We show a group ring with fuzzy elements of group can be a graded fuzzy ring. Finally, by definition of binary operation on standard colors system, we display a pixel color by a fuzzy element of group ring and control the resolution of pixel with a control function. Afterwards the picture will be prossece by a graded fuzzy ring.

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2 Preliminaries

In the all parts of this article $R$ is a commutative ring and unital. Let $\mu$ be a fuzzy subset in $R$ that $\mu(0)=1$. If for every $x, y \in R$, we have $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$, then $\mu$ is a fuzzy subgroup of $R$ and we call fuzzy subgroup $\mu$ a fuzzy ring (fuzzy ideal) of $R$ if $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$.

Suppose that $I$ is a nonempty set. For $x, y \in R$ with $i \in I$, we mean from $x = \sum_{i \in I} x_i$ that $x$ can be written as element summation of $x_i$ which $x_i$'s are zero except finite numbers.

Definition 2.1. [7] Let $\{\mu_i \mid i \in I\}$ be a family of fuzzy subsets on $R$. Fuzzy subset $\sum_{i \in I} \mu_i$ on $R$ is defined as below,

$$ (\sum_{i \in I} \mu_i)(x) = \sup \{ \inf \{ \mu_i(x_i) \mid i \in I, x = \sum_{i \in I} x_i \} \} \forall x \in R. $$

Remark 2.1. Let $\{\mu_i \mid i \in I\}$ be a family of fuzzy subgroup, fuzzy ring or fuzzy ideal on $R$, then easily it can be shown that $\sum_{i \in I} \mu_i$ is fuzzy group, fuzzy ring or fuzzy ideal on $R$, respectively. If for every $x \in R$ we set $x_j = x$ if $i = j$ and $x_j = 0$ if $j \neq i$, then by the definition of $\sum_{i \in I} \mu_i$, we have,

$$ \mu_i \subseteq \sum_{j \in I} \mu_j, \forall j \in I. $$

Definition 2.2. [1] Let $\mu_i$ and $\mu_j$ with $i \in I$ be fuzzy subsets on $R$. Then $R$ is called weak direct sum of $\{\mu_i \mid i \in I\}$ if $\mu = \sum_{i \in I} \mu_i$ and $\mu_j \cap \sum_{i \neq j} \mu_i = 0$ such that,

$$ I_0(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} $$

In this case $\mu = \bigoplus_{i \in I} \mu_i$ is written.

Definition 2.3. [7] Let $\mu$ be a fuzzy subset of $R$. We define,

$$ \mu^* = \{ x \in R \mid \mu(x) > 0 \}, \mu_* = \{ x \in R \mid \mu(x) = \mu(0) \}, $$$$ \mu^{\alpha} = \{ x \in R \mid \mu(x) > \alpha \}. $$

the fuzzy subset $\mu$ is called a finite value fuzzy subset on $R$, if $Im(\mu)$ is finite set.

Proposition 2.1. [7] Let $\mu$ be a fuzzy ring (fuzzy ideal), then $\mu^*$ and $\mu_*$ are subrings (ideals) of $R$.

Definition 2.4. [7] Let $\mu$ and $\nu$ be two fuzzy subsets on $R$. Fuzzy subset $\mu \vee \nu$ on $R$ is defined as following,

$$ \mu \vee \nu = \sup \left\{ \inf \{ \min \{ \mu(y_i), \nu(z_i) \} \mid i = 1, \ldots, n \} \mid x = \sum_{i=1}^n y_iz_i, n \in \mathbb{N} \right\}. $$

Proposition 2.2. [7] Let $\mu$ and $\nu$ be fuzzy rings (fuzzy ideals) on $R$. Then $\mu \vee \nu$ is fuzzy ring (fuzzy ideal) on $R$.

Definition 2.5. [4] Let $\mu$ be a fuzzy ideal of $R$ and let $x \in R$. Then the fuzzy subset $x + \mu$ of $R$ defined by

$$ (x + \mu)(r) = \mu(r-x) \text{ for all } r \in R $$

is termed as the fuzzy coset determined by $x$ and $\mu$. The set of all fuzzy cosets of $\mu$ in $R$ is a ring under the binary operations

$$ (x + \mu) + (y + \mu) = (x + y) + \mu \quad , \quad (x + \mu)(y + \mu) = (xy + \mu) \quad \forall x, y \in R $$

and it is denoted by $R/\mu$. We call it the fuzzy quotient ring of $R$ induced by the fuzzy ideal $\mu$.

Definition 2.6. [9] Ring $R$ is called graded by $(G, o)$ if there exists a family of $\{R_g\}_{g \in G}$ from subgroups of $R$, which $R = \bigoplus_{g \in G} R_g$, and for every $g, g' \in G$, $R_gR_{g'} \subseteq R_{g \circ g'}$. 

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Definition 2.7. [1] Fuzzy ring $\mu$ on $R$ is graded by natural number when fuzzy groups $\mu_i, i = 0, 1, 2, \cdots$ are existed on $R$ such that,
- $\mu = \bigoplus_{i \in \mathbb{N}} \mu_i,$
- $\mu_i \mu_j \subseteq \mu_{i+j}$ for all $i, j = 0, 1, 2, \cdots$.

As a result of $\mu_0 \mu_0 \subseteq \mu_0, \mu_0$ is a fuzzy ring of $R$.

3 graded fuzzy ring

Definition 3.1. Let $\mu$ be a fuzzy ring on $R$ and $(G, o)$ be a group (semigroup). We say $\mu$ is a graded fuzzy ring by $G$, if there exists a family $\{\mu_g\}_{g \in G}$ of fuzzy subgroups on $R$ that $\mu = \bigoplus_{g \in G} \mu_g$, and for every $g, g' \in G$, we have $\mu_g \mu_{g'} \subseteq \mu_{gg'}$.

Example 3.1. Supposes that R is a graded ring by G on $\{R_g\}_{g \in G}$ from its subgroups. Then we define $\mu_g : R_g \rightarrow [0, 1]$ by $\mu_g(x) = 1$ if $x \in R_g$ and if not $\mu_g(x) = 0$. Obviously, it can be shown that fuzzy ring $\mu : R \rightarrow [0, 1]$ defined by $\mu(x) = 1$ is a graded fuzzy ring by $G$ on family of $\{\mu_g\}_{g \in G}$ from fuzzy subgroup on $R$.

The example above shows how our definition from graded fuzzy ring by group has been extended.

Theorem 3.1. Let $\{\mu_g \mid g \in G\}$ be a family of fuzzy groups on $R$ and $\mu$ be a fuzzy ring on $R$ with $\mu = \sum_{g \in G} \mu_g$. Then $\mu$ is a graded fuzzy ring by group $G$ if and only if for every $\alpha \in [0, 1], \mu^{x>\alpha} = \{x \in R \mid \mu(x) > \alpha\}$ be a graded ring by $G$.

Proof. Let $\{\mu_g\}_{g \in G}$ be a family of fuzzy subgroup on $R$ that $\mu = \bigoplus_{g \in G} \mu_g$ and for every $g, g' \in G$, $\mu_g \mu_{g'} \subseteq \mu_{gg'}$.

We attribute $\alpha \in [0, 1]$. For every $x \in \mu^{x>\alpha}$, there is $\{x_g\}_{g \in G}$ that $\mu(x) = \inf \{\mu_g(x_g) \mid g \in G\}$ and $x = \sum_{g \in G} x_g$ and $\mu_g(x_g) > \alpha$. Therefore $\mu^{x>\alpha} = \sum_{g \in G} \mu_g^{x>\alpha}$.

Now suppose that for every $g, g' \in G, 0 \neq x \in \mu_g^{x>\alpha} \cap \mu_{g'}^{x>\alpha}$. Then $\{\mu_g \cap \mu_{g'}\}(x) > \alpha$. Hence $\mu_g \cap \mu_{g'} \neq 1(0)$ that is a contradiction by the definition of graded fuzzy ring. So $x = 0$ and $\mu^{x>\alpha} = \bigoplus_{g \in G} \mu_g^{x>\alpha}$.

If $x_g \in \mu_g^{x>\alpha}$ and $x'_g \in \mu_{g'}^{x>\alpha}$, then
$$\mu_{gg'}(x_g x'_g) \geq \mu_g \mu_{g'}(x_g x'_g) \geq \inf \{\mu_g(x_g), \mu_{g'}(x'_g)\} > \alpha.$$ This Show that $x_g x'_g \in \mu_{gg'}^{x>\alpha}$.

Conversely, let $\mu^{x>\alpha}$ be a graded ring by $G$ for every $\alpha \in [0, 1]$. Then for $\alpha = 0$ we result $\mu^* = \bigoplus_{g \in G} \mu_g^*$ and $\mu_g^* \mu_g^* \subseteq \mu_g^{x>\alpha}$ by definition.

Now for $g, g' \in G$ and $0 \neq x \in R$, we have $x \not\in \mu_g^* \cap \mu_{g'}^*$. It result $x \not\in \mu_g^*$ or $x \not\in \mu_{g'}^*$. In other words $\mu_g(x) = 0$ or $\mu_{g'}(x) = 0$. This show that $\mu_g \cap \mu_{g'} = 1(0)$.

For all $x \in R$
$$\mu_g \mu_{g'}(x) = \sup \{\inf \{\mu_{g_i}(x_{g_i}), \mu_{g'_{i_i}}(x_{g'_{i_i}})\} \mid i = 1, \cdots, n\} = \sum_{i=1}^n x_{g_i} x_{g'_{i_i}}, n \in \mathbb{N}.$$ If $\mu_g \mu_{g'}(x) = 0$, it is clear that $\mu_g \mu_{g'}(x) \subseteq \mu_{gg'}^{x>\alpha}(x)$ and if $\mu_g \mu_{g'}(x) \neq 0$, then there is $n \in \mathbb{N}$ that $x = \sum_{i=1}^n x_{g_i} x_{g'_{i_i}}$, and for all $0 \leq i \leq n$, we have $x_{g_i} \in \mu_{g_i}^*$ and $x_{g'_{i_i}} \in \mu_{g'_{i_i}}^*$.

Let $\mu_g (x_{g_i}) = \alpha_i$ and $\mu_{g'} (x_{g'_{i_i}}) = \alpha'_{i_i}$. We set $\alpha = \inf \{\alpha_i, \alpha'_{i_i} \mid i = 1, \cdots, n\}$. Then for every $0 \leq i \leq n$, it results that $x_{g_i} \in \mu_{g_i}^{x>\alpha}$ and $x_{g'_{i_i}} \in \mu_{g'_{i_i}}^{x>\alpha}$ and so $x = \sum_{i=1}^n x_{g_i} x_{g'_{i_i}} \in \mu_{gg'}^{x>\alpha}$. This show that $\mu_{gg'}^* \subseteq \mu_{gg'}^{x>\alpha}$. The proof is completed.

Corollary 3.1. For a graded fuzzy ring $\mu = \bigoplus_{g \in G} \mu_g$, $\mu^{x>\alpha}$ is a unital subring of $\mu^{x>\alpha}$ if and only if $\mu^{x>\alpha}$ is unital.

Proof. Since for every $x \in R, g \in G$, we have $\mu_g(x) \leq \mu(x)$, if $\mu_g^{x>\alpha}$ is a unital subring, then $\mu(1) \geq \mu_g(1) \geq \alpha$, and so $\mu^{x>\alpha}$ is unital. Unlike the above corollary is clear by [9], proposition 1.1.1.
**Theorem 3.2.** Let $R$ be a graded ring by $G$ with family of ideals $\{R_g\}_{g \in G}$ and $\mu$ be a fuzzy ideal on $R$. If $1 > sup\{\mu(x) \mid x \notin \mu_s\}$, then we have $R/\mu$ is a graded ring by $G$ in the isomorphism.

**Proof.** If we consider

$$\chi_{R_g} = \begin{cases} 1 & x \in R_g \\ 0 & x \notin R_g \end{cases}$$

we can prove that $R = \bigoplus_{g \in G} R_g$ and $1 = \bigoplus_{g \in G} \chi_{R_g} \mu$. Since $\mu$ is a fuzzy ideal on $R$, based on [[12], proposition 3.4.], $1, \mu = \bigoplus_{g \in G} \chi_{R_g} \mu$. Now let $\mu_g = \chi_{R_g} \mu$ for every $g \in G$, then we can consider $1 > sup\{\mu(x) \mid x \notin \mu_s\}$, by [7], Theorem 1.22.] $\mu_s = \bigoplus_{g \in G} \mu_{g_s}$, and $R/\mu \cong \bigoplus_{g \in G} R_g/\mu_{g_s}$.

We set $S = \bigoplus_{g \in G} R_g/\mu_{g_s}$. Then we claim that $S$ become a ring with the following product operation.

For all $x_g + \mu_{g_s}, y_g + \mu_{g_s} \in R_g/\mu_{g_s}$ and $\lambda_g \in R_g$, we have

$$(x_g + \mu_{g_s})(\lambda_g + \mu_{g_s}) = (x_g \lambda_g + x_g \mu_{g_s} + \mu_{g_s').$$

Since $R$ is a graded ring, if for $x_{1g}, x_{2g} \in R_g$ and $y_{1g'}, y_{2g'} \in R_{g'}$, we have $x_{1g} - x_{2g} \in \mu_{g_s}$ and $y_{1g'} - y_{2g'} \in \mu_{g_s'}$, then we obtain

$$(x_{1g} - x_{2g})y_{1g'} \in R_g/\mu_{g_s} \subseteq R_g/\mu_{g_s'} \mu_s = \mu_{g_s'},$$

and

$$x_{1g}(y_{1g'} - y_{2g'}) \in R_g/\mu_{g_s} \subseteq R_g/\mu_{g_s'} \mu_s = \mu_{g_s'}.$$

This show that $x_{1g}y_{1g'} - x_{2g}y_{2g'} \in \mu_{g_s'}$. Hence our claim is correct. On the other hand, $S$ is a graded fuzzy ring by group $G$, because $x_{1g}y_{1g'} + \mu_{g_s'} \in R_g/\mu_{g_s'}$. Therefore the proof will be completed.

**Corollary 3.2.** Let $R$ be a graded ring with family of ideals. Then for every finite value fuzzy ideal $\mu$, we gain $R/\mu$ is graded ring.

**Proof.** It is clear by theorem 3.2.

**Corollary 3.3.** Let $R$ be a graded Artinian ring with family of ideals. Then $R/\mu$ is graded ring for every fuzzy ideal $\mu$ on $R$.

**Proof.** If $R$ be an Artinian ring. Then every fuzzy ideal on $R$ is finite value by [[16], theorem 3.2] and corollary 3.2.

**Proposition 3.1.** Let $\mu = \bigoplus_{h \in H} \mu_h$ be a graded fuzzy ring by group $G$ and $H$ be an arbitrary subgroup of $G$. Then $\mu_H = \bigoplus_{h \in H} \mu_h$ is a graded fuzzy ring by group $H$ and $\mu_H \subseteq \mu$.

**Proof.** As regards remark 2.1, $\mu_H$ is a fuzzy subgroup on $R$. It is sufficient to show that $\mu_H(xy) \geq \min\{\mu_H(x), \mu_H(y)\}$ for all $x, y \in R$.

$$\mu_H(xy) = \sup\{\inf\{\mu_h(z_h) \mid xy = \sum_{h \in H} z_h \} \mid z_h = x_h y_h \}$$

$$\geq \sup\{\inf\{\min\{\mu_{h'}(x_{h'}), \mu_{h''}(y_{h''}) \} \mid xy = \sum_{h' h'' = h} x_{h'} y_{h''} \} \mid h \in H\}$$

$$\geq \min\{\sup\{\inf\{\mu_{h'}(x_{h'}) \mid x = \sum_{h'} x_{h'} \}, \sup\{\inf\{\mu_{h''}(y_{h''}) \mid y = \sum_{h''} y_{h''} \}\} \mid y \leq \sum_{h''} y_{h''}\}$$

$$= \min\{\mu_H(x), \mu_H(y)\}.$$
From the induction of gradation $\mu$ by $G$, we result $\mu_h \mu_g \subseteq \mu_{hg'}$, for every $h, h' \in H$. Now, for all $x \in R$ we have,

$$
\mu_H(x) = \sup \{\inf \{\mu_h(x_h)\} \mid x = \sum_{h \in H} x_h\} \\
\leq \sup \{\inf \{\mu_g(x'_g)\} \mid x = \sum_{g \in G - H} x'_g + \sum_{g \in H} x'_g\} = \mu(x).
$$

\[\square\]

4 Examples of graded fuzzy ring and applications

Example 4.1. Suppose $\mu$ be a fuzzy ring on $R$. For every group $G$, we set $\mu_e = \mu$ and $\mu_g = 1_{\{0\}}$ for $e \neq g \in G$. Then $\mu$ is a graded fuzzy ring called trivial gradation.

We remind if $G$ is a group and $R$ is a ring, then we present the set of maps $f : G \rightarrow R$ with finite support as $\mathfrak{I}$. Scalar product $\alpha f$ is defined as $x \mapsto \alpha \cdot f(x)$, for $\alpha \in R$, and also for every $f, g \in \mathfrak{I}$, we introduce,

$$
\begin{align*}
\forall f + g : G \rightarrow R & \quad \forall f \cdot g : G \rightarrow R \\
x \mapsto f(x) + g(x) & \quad x \mapsto \sum_{u \in G} f(u)g(u)
\end{align*}
$$

It is shown that $(\mathfrak{I}, +, \cdot)$ forms a ring.

Now, let $\mu$ be a fuzzy ring on $R$. Then we define $\nu$ and $\nu_x$ fuzzy subsets on $\mathfrak{I}$ for $x \in G$ as

$$
\nu(f) = \inf \{\mu(f(x)) \mid x \in G\}, \\
\nu_x(f) = \begin{cases} 
\mu(f(x)) & \text{if } f(y) = 0 \text{ for all } y \neq x \\
0 & \text{if } f(y) \neq 0 \text{ for some } y \neq x.
\end{cases}
$$

Remark 4.1. It must be paid attention that $\nu(f)$ and $\nu_x(f)$ are well defined, because $f \in \mathfrak{I}$ is a map with finite support.

Proposition 4.1. The fuzzy set $\nu$ is a fuzzy ring on $\mathfrak{I}$ and $\nu_x$ is a fuzzy subgroup, for all $x \in G$.

Proof. For all $f, g \in \mathfrak{I}$,

$$
\nu(f - g) = \nu(f) - \nu(g) = \inf \{\mu((f - g)(x)) \mid x \in G\} \\
= \inf \{\nu(f(x)) \mid x \in G\} - \inf \{\nu(g(x)) \mid x \in G\}
$$

and

$$
\nu(fg) = \nu(f) \cdot \nu(g) = \inf \{\mu(f(x)g(x)) \mid x \in G\} \\
= \inf \{\nu(f(x)) \cdot \nu(g(x)) \mid x \in G\} \\
= \nu(f) \cdot \nu(g)
$$

since $f, g$ are maps with finite support that obtain,

$$
\nu(fg) \geq \nu(f) \cdot \nu(g)
$$

Hence $\nu$ is a fuzzy ring on $\mathfrak{I}$. Now for every $x \in G$ and $f, g \in \mathfrak{I}$, we have

$$
\nu_x(f - g) = \begin{cases} 
\mu(f(x) - g(x)) & \text{if } f(y) = g(y) \text{ for all } y \neq x \\
0 & \text{if } f(y) \neq g(y) \text{ for some } y \neq x.
\end{cases}
$$
If \( f(y) = 0 \) and \( g(y) = 0 \), then \( f(y) = g(y) \). Therefore we result
\[
\nu(f - g) = \mu(f(x) - g(x)) \geq \min\{\mu(f(x)), \mu(g(x))\}
\]
\[
= \min\{\nu(f), \nu(g)\}.
\]
If the revised definition, then we obtain that \( \nu = \bigoplus_{x \in G} \nu_x \) and \( \nu_x \nu_y \subseteq \nu_{xy} \) for all \( x, y \in G \).

**Example 4.2.** Let \( R[G] \) be a group ring relate to \( R \) as ring and \( G \) as group and \( \mu \) a fuzzy subgroup on \( R \). Then we define
\[
\nu : R[G] \rightarrow [0, 1]
\]
\[
\nu(\sum_{g \in G} f \circ g) = \inf \{\mu(f(g)) \mid g \in G\}.
\]
According to the previous description, we can easily obtain that \( R[G] \cong \mathbb{Z} \). So \( \nu \), the fuzzy subset defined on \( R[G] \) is a graded fuzzy ring by \( G \), Which we call a fuzzy group ring.

**Example 4.3.** In the previous example, if we set \( G = \mathbb{Z} \), then \( R[G] \) is Laurent polynomial ring and so the fuzzy Laurent polynomial ring obtained by our method on \( R[\mathbb{Z}] \) is graded fuzzy ring on \( \mathbb{Z} \).

**Example 4.4.** For every \( H \) subgroup of \( G \), fuzzy group ring on \( R[H] \) induced of \( \nu \), fuzzy group ring on \( R[G] \) is \( \nu_H = \bigoplus_{x \in H} \nu_x \) because of proposition 3.1.

For a ring \( R \) and a group \( G \), a fuzzy group ring on elements of \( G \) over \( R \) is notation by \( R[G^{01}] \) and defined as follows,
\[
R[G^{01}] = \left\{ \sum_{i=1}^{n} a_{i1} a_{i2} \ldots a_{ik_i} g_{\alpha_1} g_{\alpha_2} \ldots g_{\alpha_{k_i}} \mid g_{\alpha_1}, g_{\alpha_2}, \ldots, g_{\alpha_{k_i}} \in G, \gamma_{\alpha_1}, \gamma_{\alpha_2}, \ldots, \gamma_{\alpha_{k_i}} \in [0, 1], n \in \mathbb{N} \right\}
\]
For making a ring a criterion for the assesment of equality between two elements most be presented. In other word we most state the summation and production on the elements of \( R[G^{01}] \), in such an extend that the necessary condition in the definition of the ring be provided. Therefore we consider,
\[
\sum_{i=1}^{n} a_{i1} a_{i2} \ldots a_{ik_i} g_{\alpha_1} g_{\alpha_2} \ldots g_{\alpha_{k_i}} = \sum_{i=1}^{m} b_{i1} b_{i2} \ldots b_{ik_i} \gamma_{\alpha_1} \gamma_{\alpha_2} \ldots \gamma_{\alpha_{k_i}}
\]
if and only if \( n = m \) and After a suitable index,
\[
a_{i1} a_{i2} \ldots a_{ik_i} = b_{i1} b_{i2} \ldots b_{ik_i}
\]
and
\[
\gamma_{\alpha_1} \gamma_{\alpha_2} \ldots \gamma_{\alpha_{k_i}} = \h_{i1} \h_{i2} \ldots \h_{ik_i}, \forall i.
\]
The summation act on \( R[G^{01}] \) introduced as below, if set
\[
\delta = \sum_{i=1}^{n} a_{i1} a_{i2} \ldots a_{ik_i} g_{\alpha_1} g_{\alpha_2} \ldots g_{\alpha_{k_i}}
\]
and

\[ \sigma = \sum_{i=1}^{n} b_{\bar{a}_1, i} \bar{a}_{1, i} \bar{a}_{2, i} \cdots \bar{a}_{l, i} \]

then we define,

\[ \delta + \sigma = \sum_{i=1}^{n} (a_{\bar{a}_1, i} \bar{a}_{1, i} \bar{a}_{2, i} \cdots \bar{a}_{l, i} + b_{\bar{a}_1, i} \bar{a}_{1, i} \bar{a}_{2, i} \cdots \bar{a}_{l, i}) \]

the multiplication "\( \cdot \)" on elements of \( R[G^{0,1}] \) according to the following method described, for every \( g, h \in G \) and \( \alpha, \beta \in [0,1] \), we define

i. \( g^\alpha h^\beta = \begin{cases} g^{\alpha+\beta} & \text{if } \alpha + \beta \leq 1 \\ g^{\alpha+\beta-1} & \text{if } \alpha + \beta > 1 \end{cases} \)

ii. \( g^0 = 1 \)

iii. \( g^\alpha h^\beta = (gh)^\alpha \).

If this multiplication apply on the elements of \( R[G^{0,1}] \), system of \( (R[G^{0,1}], +, \cdot) \) is a ring [11].

**Example 4.5.** suppose \( \mu \) be a fuzzy ring on \( R \). Then we introduced \( v \), the fuzzy set on \( R[G^{0,1}] \) as below,

\[ v : R[G^{0,1}] \to [0,1] \]

\[ v\left( \sum_{i=1}^{n} a_{\bar{a}_1, i} \bar{a}_{1, i} \bar{a}_{2, i} \cdots \bar{a}_{l, i} \right) = \inf \left\{ \mu \left( \frac{1}{\bar{a}_1, i} \bar{a}_{1, i} \bar{a}_{2, i} \cdots \bar{a}_{l, i} \right) \right\} \]

If we set \( \prod_{j=1}^{l} g_{\bar{a}_j} = g_{\bar{a}_1} \bar{a}_{2, i} \cdots \bar{a}_{l, i} \) for every \( l_i \geq 1 \), we define.

\[ v\left( \prod_{j=1}^{l} \bar{a}_{\bar{a}_j} \right) = \left\{ \begin{array}{ll} \mu \left( \frac{1}{\bar{a}_1, i} \bar{a}_{1, i} \bar{a}_{2, i} \cdots \bar{a}_{l, i} \right) & \text{for all } 1 \leq k \leq n, k \neq i \\ 0 & \text{if } a_{\bar{a}_1, i} \bar{a}_{1, i} \cdots \bar{a}_{l, i} \neq 0 \text{ for some } 1 \leq k \leq n, k \neq i \end{array} \right. \]

Similarly to what was done before, \( v \) is a fuzzy ring on \( R[G^{0,1}] \) and \( v_{\bar{a}_j} \) a fuzzy subgroup, for \( i = 1, 2, \cdots, n \) and we can prove that,

\[ v = \bigoplus_{i=1}^{\infty} v_{\bar{a}_i} \]

such that \( \prod_{j=1}^{l} g_{\bar{a}_j} = \prod_{j=1}^{l} \bar{a}_{\bar{a}_j} \bigcirc \prod_{j=1}^{l} g_{\bar{b}_j} \).
Example 4.6. We consider $S$ as standard colors system which is produced by three basic standard colors Green, Blue and Red. Binary operation of standard colors is at the table 1.

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<td>$BR$</td>
<td>$BG$</td>
<td>$B$</td>
</tr>
</tbody>
</table>

Table 1: Table of combination act on standard color RGB

In the table 1, $RG$ is the result of combination of Red and Green. Since in the mixture of the two colors the priority and order is not important, the operation above is commutative. Obviously it can be shown that $(S, o)$ is a semigroup. Now for $\gamma_i \in [0, 1]$ that $1 \leq i \leq 3$, we mean $R^{\gamma_i}G^{\gamma_2}B^{\gamma_3}$ as a color which is a member of standard color Red $\gamma_1$, standard color Green $\gamma_2$ and standard color Blue $\gamma_3$. In this case we consider standard color Red is $R^1$, standard color Green $G^1$ and standard color Blue $B^1$.

As regards defined multiplication on $S^{[0,1]} = \{R^{\gamma_1}G^{\gamma_2}B^{\gamma_3} | \gamma_1, \gamma_2, \gamma_3 \in [0,1]\}$ in the last subject, which expressed the operation of “$o$” on $S$ in table above, is well defined.

Suppose $f : S^{[0,1]} \rightarrow \mathbb{R}$ expresses the level of different color spectrum resolution. So,

$$\mathbb{R}[S^{[0,1]}] = \left\{ \sum_{i=1}^{n} f(R^{\gamma_{i1}}G^{\gamma_{i2}}B^{\gamma_{i3}})R^{\gamma_{j1}}G^{\gamma_{j2}}B^{\gamma_{j3}} | \gamma_{i,j} \in [0,1], 1 \leq j \leq 3, 1 \leq i \leq n, n \in \mathbb{N} \right\}$$

shows the resolution level of a picture with variety of colors on colorful pixel separation. Now the situation control function of picture resolution as $\mu : f[S^{[0,1]}] \rightarrow [0, 1]$ can be set by purpose processing. For example if purpose processing be bolding picture resolution based on reddish colors, we can use the control function as $\mu_{\alpha} : f[S^{[0,1]}] \rightarrow [0, 1]$ with

$$\mu_{\alpha}(f(R^{\gamma_{11}}G^{\gamma_{22}}B^{\gamma_{33}})) = \left\{ \begin{array}{ll} \gamma_1 & \gamma_1 \geq \alpha \\ \gamma_2 & \gamma_2 < \alpha \end{array} \right.$$  

The act of this control function is vanishing the resolution of reddish colors with less tendentions of $\alpha$ and in this way the red shadow in a pixel will be omitted and the red color of picture will bold. For the whole of picture, the control function is as following,

$$\mu = \bigotimes_{i=1}^{\infty} \mu_{\alpha,R^{\gamma_{i1}}G^{\gamma_{i2}}B^{\gamma_{i3}}}, \gamma_{i,j} \in [0,1], 1 \leq j \leq 3, 1 \leq i \leq n, \alpha \in [0,1]$$

such that $\mu_{\alpha,R^{\gamma_{11}}G^{\gamma_{22}}B^{\gamma_{33}}} = \mu_{\alpha}(f(R^{\gamma_{11}}G^{\gamma_{22}}B^{\gamma_{33}}))$.

5 conclusion

The set $S^{[0,1]}$ make it possible for us to find the location of different places of a picture based on the amount and composition of the used color and applied control function by this location. As it has seen function $\mu$ is an applied example on fuzzy situation set $S^{[0,1]}$. The last function is an description of any situation, property or description of spectrum based on an essential system. The description of particles around the atom by the properties of the core like mass, volume, tempreture and etc, or the situation of not clear places in a satelitte image from a geographical place based on the quality of the nearest clear place in the image, are the options that can be processed with the function $\mu$. Actually function $\mu$ is the processing function related to marginal areas of a phenomenon.
Acknowledgments

The authors would like to thank the referee for the valuable suggestions and comments.

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