Journal of Fuzzy Set Valued Analysis 2016 No.2 (2016) 140-147

International Scientific Publications and Consulting Services

Available online at www.ispacs.com/jfsva

Volume 2016, Issue 2, Year 2016 Article ID jfsva-00297, 8 Pages

doi:10.5899/2016/jfsva-00297

Research Article

Extension of artin rees lemma for fuzzy module

Alireza Sedighi[1*], Mohammad Hossein Hosseini[1]

(1) *Department of mathematics, University of Birjand, Birjand, Iran*

Abstract

In this article we introduce μ-filtered fuzzy module with a family of fuzzy submodules. It shows the relation between μ-filtered fuzzy modules and crisp filtered modules by level sets and some of it's results. It is gained that the quotient on μ-filtered fuzzy module is too. After that we aim associated fuzzy graded module by μ-filtered fuzzy module Our goal is extension of important basic result about modules over a Noetherian ring known as Artin-Rees lemma to fuzzy term.

Keywords: μ-fuzzy filtered module, associated fuzzy graded ring, μ-stable filtration, Artin-Rees Lemma.

1 Introduction

Lotfi A. Zadeh [21] in 1965 introduced the notion of a fuzzy set. Rosenfield applied this concept in the group theory [18]. In commutative algebra W. J. Liu [11] opened the way towards the development of fuzzy algebraic structures by introducing the notions of fuzzy normal subgroup, fuzzy subring and the product of fuzzy sets. Liu [20] introduced the notion of a fuzzy ideal of a ring. N. Kuroki [9] demonstrated the utility of the notion of the fuzzy set in the more general setting of semigroups. The concepts of fuzzy fields and fuzzy linear spaces were introduced by S. Nanda [16]. Malik and Mordeson presented direct sum of fuzzy rings and fuzzy ideals [14].
In commutative algebra the concept of fuzzy modules and L-modules were introduced by Negoita and Ralescu [17] and Mashinchi and Zahedi [15] respectively. In the recent years, fuzzy algebra are extended by Acar [2], Inan and Ozturk[8], Shao and Liao[19], Chen [4] and Gunduz [7].
A filtered algebra is a generalization of the notion of a graded algebra. Examples appear in many branches of mathematics, especially in homological algebra and representation theory[1].
The Artin-Rees lemma is a basic result about modules over a Noetherian ring, along with results such as the Hilbert basis theorem. It was proved in the 1950s in independent works by the mathematicians Emil Artin and David Rees. A special case was known to Oscar Zariski prior to their work. One consequence of the lemma is the Krull intersection theorem. The result is also used to prove the exactness property of completion [3].
In this article we effort to introduce filtered fuzzy ring and μ-filtered fuzzy module in order to extend the subjects of commutative algebra. In the other part of the paper, the relationship between μ-filtered fuzzy module and its crisp form are investigated. We prove a fuzzy module is μ-filtered if and only if all its level sets be filtered module. In following it is gained that the quotient on μ-filtered fuzzy module is too. After that we aim associated fuzzy graded module by μ-filtered fuzzy module. In the following with the definition of μ-stable for fuzzy ideal μ, we express and prove the fuzzy version of Artin-Rees lemma.

*Corresponding author. Email address: sedighi.phd@birjand.ac.ir, Tel:+989391926284

2 Preliminaries and notations

In the all parts of this article, R is a commutative ring and unital. Let μ be a fuzzy subset in R that $\mu(0)=1$. If for every $x,y\in R$, $\mu(x-y)\geq min\{\mu(x),\mu(y)\}$, then μ is a fuzzy group of R and we call fuzzy group μ a fuzzy ring (fuzzy ideal) of R if $\mu(xy)\geq min\{\mu(x),\mu(y)\}$ ($\mu(xy)\geq max\{\mu(x),\mu(y)\}$).
Let R be an ordinary ring and M be a R-module. we adopt the concept of fuzzy modules, which was introduce by Negoita and Ralescu [17], as follows.
(M,ν) is called a fuzzy R-module if the is a map

$$\nu : M \to [0,1]$$

satisfying the following conditions:

1. $\nu(a+b)\geq min\{\nu(a),\nu(b)\}(\forall a,b\in M)$;
2. $\nu(0)=1$;
3. $\nu(ra)\geq \nu(a)(\forall a\in M, r\in R)$.

Definition 2.1. *[14] Let μ be a fuzzy subset of R. We define,*

$$\mu^* = \{x\in R \mid \mu(x)>0\},$$
$$\mu^{x>\alpha} = \{x\in R \mid \mu(x)>\alpha\}.$$

Proposition 2.1. *[14] Let μ be a fuzzy ring (fuzzy ideal), then μ^* are subrings (ideals) of R.*

Definition 2.2. *[13] Let ζ and ν be fuzzy module on M such that $\zeta\subseteq\nu$. Then obviously ζ^* and ν^* are submodule of M and $\zeta^*\subseteq\nu^*$. Thus ζ^* is a submodule of ν^*. Now define ν/ζ on quotient module ν^*/ζ^* as follow:*

$$\nu/\zeta(x+\zeta^*) = sup\{\nu(y) \mid y\in x+\zeta^*\} \quad x\in\nu^*.$$

Then ν/ζ is fuzzy module on ν^*/ζ^* and called the quotient of ν with respect to ζ.

Definition 2.3. *[13] Let f be a mapping from X into Y, and μ,ν fuzzy subsets on X,Y, respectively. The fuzzy subset $f(\mu)$ on y, defined by $\forall y\in Y$,*

$$f(\mu)(y) = \begin{cases} sup\{\mu(x) \mid x\in X, f(x)=y\} & if \quad f^{-1}(y)\neq\emptyset \\ 0 & otherwise \end{cases}$$

Definition 2.4. *[14] Let μ and ν be two fuzzy subsets on R. Fuzzy subset $\mu\nu$ on R is defined as following,*

$$\mu\nu(x) = sup\{inf\{\mu(y),\nu(z)\}|yz=x\}.$$

Proposition 2.2. *[14] Let μ and ν be fuzzy rings (fuzzy ideal) on R. Then $\mu\nu$ is fuzzy ring (fuzzy ideal) on R.*

Definition 2.5. *Let μ be a fuzzy ring on R and ν is fuzzy module on M then $\mu\nu$ is defined as*

$$\mu\nu(x) = sup\{inf\{\mu(r),\nu(y)\}|ry=x\}, \qquad r\in R, x,y\in M.$$

Proposition 2.3. *If μ be a fuzzy ring on R and ν is fuzzy module on M then $\mu\nu$ is a fuzzy module.*

Proof. It is clear that $\mu\nu$ is fuzzy group by proof of proposition above. Now, for every $x\in M$ and $r\in R$ we have,

$$\begin{aligned}\mu\nu(rx) = sup\{inf\{\mu(s),\nu(y)\}|sy=rx\} &= sup\{inf\{\mu(s'),\nu(ry')\}|s'y'=x\} \\ &\geq sup\{inf\{\mu(s'),\nu(y')\}|s'y'=x\} \\ &= \mu\nu(x).\end{aligned}$$

This show that $\mu\nu$ is a fuzzy module. □

Definition 2.6. *[10] Let μ be a fuzzy ideal of R and let $x \in R$. Then the fuzzy subset $x+\mu$ of R defined by*

$$(x+\mu)(r) = \mu(r-x) \quad \text{for all} \quad r \in R$$

is termed as the fuzzy coset determined by x and μ. The set of all fuzzy cosets of μ in R is a ring under the binary operations

$$(x+\mu)+(y+\mu) = (x+y)+\mu \quad \text{and}$$
$$(x+\mu)(y+\mu) = (xy+\mu) \quad \forall x,y \in R$$

and it is denoted by R/μ. We call it the fuzzy quotient ring of R induced by the fuzzy ideal μ.

Definition 2.7. *[6] A filtered ring R is a ring R together with a family $\{R_n\}_{n\geq 0}$ of subgroups of R satisfying the conditions (i) $R_0 = R$ (ii) $R_{n+1} \subseteq R_n$ for all $n \geq 0$ (iii) $R_nR_m \subseteq R_{n+m}$ for all $m,n \geq 0$.*

Example 2.1. *[6] Let I be an ideal in R and let $R_n = I^n$, $n \geq 0$. Then $\{R_n\}$ is a filtration on R called I-adic filtration.*

Definition 2.8. *[6] Let R be a filtered ring. A filtered R-module M is an R-module M together with a family $\{M_n\}_{n\geq 0}$ of R-submodules of M satisfying (i) $M_0 = M$ (ii) $M_{n+1} \subseteq M_n$ for all $n \geq 0$ (iii) $R_mM_n \subseteq M_{n+m}$ for all $m,n \geq 0$.*

Example 2.2. *[6] Let M be a filtered R-module and N an R-submodule of M. The filtration $\{M_n\}$ on M induces a filtration $\{N_n\}$ on N where $N_n = N \cap M_n, n \geq 0$.*

Definition 2.9. *[5] Fuzzy ring μ on R is graded by natural number when fuzzy groups μ_i, $i = 0,1,2,\cdots$, are existed on R that,*

- $\mu = \oplus_{i\in I}\mu_i$,
- $\mu_i\mu_j \subseteq \mu_{i+j}$ *for all* $i,j = 0,1,2,\cdots$.

As a result of $\mu_0\mu_0 \subseteq \mu_0$, μ_0 is a fuzzy ring of R.

Definition 2.10. *[5] Let μ be a fuzzy graded ring on R. A fuzzy R-module ν is called fuzzy graded R-module if ν can be expressed as a direct sum of fuzzy groups $\{\nu_n\}$ i.e. $\nu = \bigoplus_{n=0}^{\infty}\nu_n$, such that $\mu_i\nu_j \subseteq \nu_{i+j}$, $i,j = 0,1,\cdot$.*

3 filtration on fuzzy ring

Definition 3.1. *Let R be an unital ring and μ a fuzzy ring of R. μ is called filtered fuzzy ring, if μ with $\{\mu_n\}_{n\geq 0}$ of fuzzy groups on R include the following conditions,*

i. $\mu_0 = \mu$;

ii. $\mu_{n+1} \subseteq \mu_n(\mu_n \subseteq \mu_{n+1})$, *for all* $n \geq 0$;

iii. $\mu_n\mu_m \subseteq \mu_{n+m}$, *for all* $n,m \geq 0$.

Example 3.1. *Suppose μ be a fuzzy ring on R. Then μ is filtered by $\{\mu_n\}_{n\geq 0}$ filtration such that $\mu_n = \mu^n$. This filtration is called μ-adic filtration.*

Example 3.2. *Let μ be a filtered fuzzy ring on R. Then $\nu \subseteq \mu$ fuzzy ring of μ is filtered by $\nu_n = \mu_n \cap \nu$ filtration.*

Example 3.3. *Let ν be a discrete valuation on quotient field of R. Then $\mu_n(x) = 1 - n^{-\nu(x)}$ is an increasing filtration on fuzzy ring 1_R.*

Definition 3.2. *Let μ be a fuzzy filtered ring by $\{\mu_n\}$ on ring R and ν a fuzzy module on R-module M. ν is called μ-filtered fuzzy module, if ν with $\{\nu_n\}_{n\geq 0}$ of fuzzy module on M include the following conditions,* *i.* $\nu_0 = \nu$;

ii. $\nu_{n+1} \subseteq \nu_n$, *for all* $n \geq 0$;

iii. $\mu_n \nu_m \subseteq \nu_{n+m}$, *for all* $n,m \geq 0$.

Example 3.4. *Let μ be a filtered fuzzy ideal by μ-adic filtration. Fuzzy module ν of M is μ-filtered with $\{\nu_n = \mu^n \nu\}$ that is called μ-adic filtration.*

Example 3.5. *Let ν be a μ-filtered fuzzy module by $\{\nu_n\}_{n\geq 0}$ in M and μ is fuzzy ring filtered by $\{\mu_n\}_{n\geq 0}$ and ζ is a fuzzy submodule of ν. Then fuzzy module $\frac{\nu}{\zeta}$ of $\frac{\nu^*}{\zeta^*}$ is μ-filtered with $\{\frac{\nu}{\zeta}\}_n(x+\zeta^*) = sup\{\nu_n(y) \mid y \in x+\zeta^*\}$, for all $x \in \nu^*$. Because, for every $x \in \nu^*$,*

i. $\{\frac{\nu}{\zeta}\}_0(x+\zeta^*) = sup\{\nu_0(y) \mid y \in x+\zeta^*\} = sup\{\nu(y) \mid y \in x+\zeta^*\} = \{\frac{\nu}{\zeta}\}(x+\zeta^*)$;

ii. $\{\frac{\nu}{\zeta}\}_{n+1}(x+\zeta^*) = sup\{\nu_{n+1}(y) \mid y \in x+\zeta^*\} \leq sup\{\nu_n(y) \mid y \in x+\zeta^*\} = \{\frac{\nu}{\zeta}\}_n(x+\zeta^*)$, *for every* $n \geq 0$.

iii. for all $n,m \geq 0$,

$$\begin{aligned}
\mu_n \left(\frac{\nu}{\zeta}\right)_m (x+\zeta^*) &= sup\{inf\{\mu_n(r), \left(\frac{\nu}{\zeta}\right)_m (y+\zeta^*)\} | ry+\zeta^* = x+\zeta^*\} \\
&\leq sup\{sup\{inf\{\mu_n(r), \nu_m(z)\} | z \in y+\zeta^*\} | ry = x\} \\
&\leq sup\{\nu_{n+m}(y) | y \in x+\zeta^*\} = \left(\frac{\nu}{\zeta}\right)_{n+m} (x+\zeta^*).
\end{aligned}$$

Hence $\mu_n \left(\frac{\nu}{\zeta}\right)_m \subseteq \left(\frac{\nu}{\zeta}\right)_{n+m}$

Example 3.6. *Let ν be a μ-filtered fuzzy module on M. Then $\zeta \subseteq \nu$ fuzzy submodule of ν is μ-filtered by filtration $\zeta_n = \nu_n \cap \zeta$.*

Proposition 3.1. *Let μ be a fuzzy filtered ring with filtration $\{\mu_n\}_{n\geq 0}$. Then ν is a μ-filtered fuzzy module with filtration $\{\nu_n\}_{n\geq 0}$ if and only if $\nu^{x>\alpha}$ be a filtered module with $\{\nu_n^{x>\alpha}\}$ for every $\alpha \in [0,1]$.*

Proof. Let ν be a μ-filtered fuzzy module by filtration $\{\nu_n\}_{n\geq 0}$ and $\alpha \in [0,1]$. Then it is clear that $\nu_0^{x>\alpha} = \nu^{x>\alpha}$. Since $\nu_{n+1} \subseteq \nu_n$ for every $n \geq 0$,

$$\begin{aligned}
x \in \nu_{n+1}^{x>\alpha} &\Longrightarrow \alpha < \nu_{n+1}(x) \leq \nu_n(x) \\
&\Longrightarrow x \in \nu_n^{x>\alpha}.
\end{aligned}$$

Therefore $\nu_{n+1}^{x>\alpha} \subseteq \nu_n^{x>\alpha}$.
Finally for every $n,m \geq 0$, $r \in \mu_n^{x>\alpha}$ and $x \in \nu_m^{x>\alpha}$, we have

$$\nu_{n+m}(rx) \geq \mu_n \nu_m(rx) \geq inf\{\mu_n(r), \nu_m(x)\} > \alpha.$$

This show that $rx \in \nu_{n+m}^{x>\alpha}$ and so $\mu_n^{x>\alpha} \nu_m^{x>\alpha} \subseteq \nu_{n+m}^{x>\alpha}$.
Conversly we suppose $\nu^{x>\alpha}$ is filtered module for all $\alpha \in [0,1]$. Then for every $x \in M$, if $\nu_0(x) = \beta > 0$, then $x \in \nu_0^{x>0} = \nu^{x>0}$ and so $\nu = \nu_0$. In otherwise it is clear that $\nu = \nu_0$.
Now for every $n \geq 0$, let $\nu_{n+1}(x) = \alpha$, then $x \in \nu_{n+1}^{x>\beta}$ for every $\beta \in [0,\alpha)$. Therefore $x \in \nu_n^{x>\beta}$. This result $\nu_n(x) \geq$

$\alpha = \nu_{n+1}(x)$ and hence $\nu_{n+1} \subseteq \nu_n$.
We choose $n, m \geq 0$, arbitrary. For every $x \in R$, let $\mu_n \nu_m(x) = \alpha$. Then there exist $r \in R$ and $y \in M$ such that $ry = x$ and $\alpha = inf\{\mu_n(r), \nu_m(y)\}$. Therefore $r \in \mu_n^{x>\alpha}, y \in \nu_m^{x>\alpha}$.
By above text we result that $ry \in \nu_{n+m}^{x>\alpha}$ and we have

$$\nu_{n+m}(x) = \mu_{n+m}(ry) \geq \alpha = \mu_n \nu_m(x).$$

This complete the proof. □

Corollary 3.1. *Let ν be a μ-filtered fuzzy module on M. Then fuzzy submodule ζ is μ-filtered by induced filtration if and only if $\zeta^{x>\alpha}$ is a filtered module by induced filtration of $\nu^{x>\alpha}$ for every $\alpha \in [0,1]$.*

Proof. It is enough we show $\zeta_n = \zeta \cap \nu_n$ if and only if $\zeta_n^{x>\alpha} = \zeta^{x>\alpha} \cap \nu_n^{x>\alpha}$ for all $\alpha \in [0,1]$ and $n \geq 0$.
In fact, by this work we prove level sets of elements of induced filtration $\{\zeta_n\}_{n\geq 0}$ are induced filtration of level sets ν. The proof of be filtered, gained by last proposition. If $\zeta_n = \zeta \cap \nu_n$ for $n \geq 0$, then for $\alpha \in [0,1]$,

$$\begin{aligned} x \in \zeta_n^{x>\alpha} &\Longleftrightarrow \zeta_n(x) = min\{\zeta(x), \nu_n(x)\} > \alpha \\ &\Longleftrightarrow x \in \zeta^{x>\alpha} \cap \nu_n^{x>\alpha}. \end{aligned}$$

Conversly, if for every $\alpha \in [0,1]$, $\zeta_n^{x>\alpha} = \zeta^{x>\alpha} \cap \nu_n^{x>\alpha}$. Then for every $x \in M$, let $\zeta_n(x) = \eta$. Since $\eta = sup\{\beta \in [0,1] \mid \zeta_n(x) > \beta\}$, for every $\beta \in [0,\alpha)$, $x \in \zeta^{x>\beta} \cap \nu_n^{x>\beta}$. This prove that $(\zeta \cap \nu_n)(x) > \beta$ and so $(\zeta \cap \nu_n)(x) \geq \alpha = \zeta_n(x)$. Similarly $\zeta_n(x) \geq (\zeta \cap \nu_n)(x)$. Hence the proof is completed. □

Theorem 3.1. *Suppose μ be a filtered fuzzy ring on R with $\{\mu_n\}_{n\geq 0}$. We define, $fgr_n(\mu) = \dfrac{\mu_n}{\mu_{n+1}}$, and $fgr(\mu) = \oplus_{n\geq 0} fgr_n(\mu)$. Then $fgr(\mu)$ is a fuzzy graded ring.*

We call the above fuzzy graded ring, associeted fuzzy graded ring by fuzzy ring.

Proof. The following statements show $fgr(\mu)$ is a fuzzy ring. First, by calculation $gr(\mu^*) = \oplus_{n\geq 0} gr_n(\mu^*)$ is associeted grade ring by filtered ring μ^* and we have

$$\begin{aligned} &fgr(\mu) : gr(\mu^*) \to [0,1] \\ &fgr(\mu)(\{x_n + \mu_{n+1}^*\}_{n\geq 0}) = inf_{n\geq 0} \left\{ \frac{\mu_n}{\mu_{n+1}} (x_n + \mu_{n+1}^*) \right\} \end{aligned}$$

second, for every $x + \mu_{n+1}^*, y + \mu_{m+1}^*$

$$\begin{aligned} fgr((x + \mu_{n+1}^*)(y + \mu_{m+1}^*)) = fgr(xy + \mu_{n+m+1}^*) &= \frac{\mu_{n+m}}{\mu_{n+m+1}} (xy + \mu_{n+m+1}^*) \\ &= sup\{\mu_{n+m}(z) | z \in xy + \mu_{n+m+1}^*\}. \end{aligned}$$

In other hand

$$\begin{aligned} \frac{\mu_n}{\mu_{n+1}} \frac{\mu_m}{\mu_{m+1}} (xy + \mu_{n+m+1}^*) &= sup\{inf\{sup\{\mu_n(u) | u \in r + \mu_{n+1}^*\}, \\ &\quad sup\{\mu_m(w) | w \in s + \mu_{m+1}^*\} | rs \in xy + \mu_{n+m+1}^*\}\} \\ &\leq \mu_{n+m}(rs) \leq \frac{\mu_{n+m}}{\mu_{n+m+1}} (xy + \mu_{n+m+1}^*). \end{aligned}$$

This shows that

$$\begin{aligned} fgr(\mu)((x + \mu_{n+1}^*)(y + \mu_{m+1}^*)) &= \frac{\mu_{n+m}}{\mu_{n+m+1}} (xy + \mu_{n+m+1}^*) \geq \frac{\mu_n}{\mu_{n+1}} \frac{\mu_m}{\mu_{m+1}} (xy + \mu_{n+m+1}^*) \\ &\geq inf\{\frac{\mu_n}{\mu_{n+1}} (x + \mu_{n+1}^*), \frac{\mu_m}{\mu_{m+1}} (y + \mu_{m+1}^*)\} \\ &= inf\{fgr(\mu)(x + \mu_{n+1}^*), fgr(\mu)(y + \mu_{m+1}^*)\} \end{aligned}$$

since for every $n \geq 0$, $\dfrac{\mu_n}{\mu_{n+1}}$ is fuzzy group, it is clear that $fgr(\mu)$ is fuzzy group. The definition from $fgr(\mu)$ implies that be fuzzy graded ring. □

Corollary 3.2. *Let ν be a μ-filtered fuzzy module and μ be a filtered fuzzy ring, with filtration $\{\nu_n\}_{n\geq 0}$ and $\{\mu_n\}_{n\geq 0}$ respectively. Let*

$$fgr_n(\nu) = \nu_n/\nu_{n+1} \quad \text{and} \quad fgr(\nu) = \oplus_{n\geq 0} fgr_n(\nu).$$

Then $fgr(\nu)$ has a natural $gr(\mu^*)$-fuzzy graded module structure. This module is called the associated fuzzy graded module of ν.

Proof. By definition of $fgr(\nu)$, for every $r+\mu^*_{n+1}, x+\nu^*_{m+1}$

$$\begin{aligned} fgr(\nu)((r+\mu^*_{n+1})(x+\nu^*_{m+1})) &= fgr_{n+m}(\nu)(rx+\nu^*_{n+m+1}) \\ &= sup\{\nu(z)|z \in rx+\nu^*_{n+m+1}\} \\ &\geq sup\{\nu(rs)|s \in x+\nu^*_{m+1}, r \in \mu^*_n\} \\ &\geq sup\{\nu(s)|s \in x+\nu^*_{m+1}\} \\ &= fgr_m(\nu)(x+\nu^*_{m+1}) = fgr(\nu)(x+\nu^*_{m+1}) \end{aligned}$$

It is obvious that $fgr(\nu)$ is fuzzy group and fuzzy graded. □

Definition 3.3. *Let ν be a μ-filtered fuzzy module on M with a filtration $\{\nu_n\}$ and η an fuzzy ideal in R. The filtration is called η-stable if there exists some m such that for all $n \geq m$, $\eta\nu_n = \nu_{n+1}$ and $\eta\nu_n \subseteq \nu_{n+1}$ for all $n \geq 0$.*

Example 3.7. *Every μ-adic fuzzy filteration by fuzzy ideal μ is μ-stable.*

Theorem 3.2. *(*FUZZY VERSION OF ARTIN-REES LEMMA*). Let ν be a μ-filtered fuzzy R-module with an η-stable filtration. Asuume that R is Noetherian and M is R-finitely generated. Then the fuzzy filtration induced by ν on a fuzzy submodule ζ of ν is also η-stable.*

Proof. First, we claim for every $\alpha \in [0,1]$, $(\eta\nu)^{x>\alpha} = \eta^{r>\alpha}\nu^{x>\alpha}$. For proof, we suppose $x \in (\eta\nu)^{x>\alpha}$, then $\eta\nu(x) = sup\{inf\{\eta(r), \nu(y)\}|x = ry\} > \alpha$. Since M is finitely generated, there exist $r \in R$ and $y \in M$ such that $ry = x$ with $r \in \eta^{r>\alpha}$ and $y \in \nu^{x>\alpha}$. conversly is similar. Now, let for $x \in M$, $\eta(\nu_n \cap \zeta)(x) = \alpha$, then for every $\beta \in [0,\alpha)$, $\eta(\nu_n \cap \zeta)(x) > \beta$. Therefore $x \in \eta^{x>\beta}(\nu_n^{x>\beta} \cap \zeta^{x>\beta})$ and by crisp version of Artin Rees lemma, $x \in (\eta^\beta\nu_n^\beta \cap \zeta^\beta)$. This show $\eta(\zeta_{n+1})(x) = (\nu_{n+1} \cap \zeta)(x) = (\eta\nu_n \cap \zeta)(x) > \beta$. Our assertion show $(\eta\nu_n \cap \zeta)(x) = \alpha$ and so $\eta\zeta_n(x) = \eta(\nu_n \cap \zeta)(x) = (\eta\nu_n \cap \zeta)(x) = \zeta_{n+1}(x)$. This complete the proof. □

Corollary 3.3. *Let R be a Noetherian ring, μ an fuzzy ideal on R, M a finitely generated R-module and $\zeta \subseteq \nu$ fuzzy modules on M. Then there exist some m such that $\mu^{m+k}\nu \cap \zeta = \mu^k(\mu^m\nu \cap \zeta)$ for all $k \geq 0$.*

Proof. Apply the fuzzy version of Artin-Rees lemma for the μ-adic filtration on ν. □

Example 3.8. *Let R be a Noetherian ring and μ an fuzzy ideal on R. Then we define filtration $\{\nu_n\}_n$ with $\nu_n = \mu^n\chi_R$ for characteristic function χ_R. Then the filtration is μ-stable and by 3.2 for every ideal I of R the fuzzy filtration induced by χ_R on fuzzy ideal χ_I is μ-stable and we have $\mu^{m+k}\chi_R \cap \chi_I = \mu^k(\mu^m\chi_R \cap \chi_I)$ for some m and for all $k \geq 0$.*

4 Conclusions

In [12] Murali and Makamba study the concepts of primary decompositions of fuzzy ideals and the radicals of such ideals over a commutative ring. Using such decompositions and a form of Nakayamas lemma, they prove Krulls intersection theorem on fuzzy ideals. In fact they show that if μ be a finitely generated fuzzy ideal of R such that μ is contained in the fuzzy Jacobson radical of R, Then $\bigwedge_{n=0}^{\infty}\mu^n = 0$. As mentioned before one consequence of the Artin-Rees lemma is the Krull intersection theorem and to prove the exactness property of completion. So with proving of fuzzy version of Artin-Ress lemma we can find a way for to extend Krull intersection theorem to fuzzy modules and apply that for processing of properties of fuzzy completion.

Acknowledgments

This section should come before the References. Funding information may also be included here.

References

[1] E. Abe, Hopf Algebras, Cambridge: Cambridge University Press, (1980).

[2] U. Acar, F. Koyuncu, B. Tanay, Soft sets and soft rings, Comput. Math. Appl, 59 (2010) 3458-3463.
http://dx.doi.org/10.1016/j.camwa.2010.03.034

[3] M. E. Atiyah, I. G. MacDonald, Introduction To Commutative Algebra, Addison-Wesley Publishing Company, University of Oxford, (1964).

[4] W. Chen, Fuzzy subcoalgebras and duality, Bull. Malays. Math. Sci. Soc, (2009) 283-294.

[5] E. Eslami, J. N. Mordeson, Completion and Fuzzy Power Series Subrings, Fuzzy Sets and Systems, 82 (1996) 97-102.
http://dx.doi.org/10.1016/0165-0114(95)00232-4

[6] N. S. Gopalakrishnan, Commutative algebra, Oxonian Press Pvt. University of Poona, (1984).

[7] C. Gunduz(Aras), S. Bayramov, Inverse and direct system in category of fuzzy modules, Fuzzy Sets, Rough Sets and Multivalued Operations and Applications, 3 (2011) 11-25.

[8] E. Inan, M. A. Öztürk, Fuzzy soft rings and fuzzy soft ideals, Neural Comput. Appl, (2011).
http://dx.doi.org/10.1007/s00521-011-0550-5

[9] N. Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy Sets and Systems, 5 (1981) 203-215.
http://dx.doi.org/10.1016/0165-0114(81)90018-X

[10] K. H. Lee, On fuzzy quotient rings and chain conditions, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math, 7 (1) (2000) 33-40.

[11] W. J. Liu, Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems, 8 (1982) 133-139.
http://dx.doi.org/10.1016/0165-0114(82)90003-3

[12] V. Murali, B. B. Makamba, On Krull's intersection theorem of fuzzy ideals, International journal of mathematics and mathematical sciences, 4 (2003) 251-262.
http://dx.doi.org/10.1155/S0161171203013097

[13] J .N. Mordeson, D. S. Malik, Fuzzy Commutative Algebra, World Scientific, (1998).
http://dx.doi.org/10.1142/9789812815897

[14] D. S. Malik, John N. Mordeson, Fuzzy direct sums of fuzzy rings, Fuzzy Sets and Systems, 45 (1992) 83-91.
http://dx.doi.org/10.1016/0165-0114(92)90094-K

[15] M. Mashinchi, M. M. Zahedi, On L-fuzzy primary submodules, Fuzzy Sets and Systems, 49 (1992) 231-236.
http://dx.doi.org/10.1016/0165-0114(92)90328-2

[16] S. Nanda, Fuzzy fields and fuzzy linear spaces, Fuzzy Sets and Systems, 19 (1986) 89-94.

[17] C. V. Negoita, D. A. Ralescu, Applications of Fuzzy Sets and System Analysis, Birkhauser, Basel, (1975).
http://dx.doi.org/10.1007/978-3-0348-5921-9

[18] A. Rosenfeld, Fuzzy groups, J1. Math. Anal. Appl, 35 (1971) 512-517.
http://dx.doi.org/10.1016/0022-247X(71)90199-5

[19] Y. Shao, Z. Liao, R-generalized fuzzy subnear-rings and ideals, Journal of Beihua University:Natural Science Edition, 9 (6) (2010) 25-28.

[20] L. Wang-Jin, Operation on fuzzy ideals, Fuzzy Sets and Systems, 11 (1983) 31-41.
http://dx.doi.org/10.1016/S0165-0114(83)80067-0

[21] L. A. Zadeh, Fuzzy sets, Inform. Control, 81 (1965) 338-353.
http://dx.doi.org/10.1016/S0019-9958(65)90241-X