Dynamical control of accuracy in the fuzzy Runge-Kutta methods to estimate the solution of a fuzzy differential equation

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Abstract
In this paper, a reliable scheme is proposed to solve fuzzy differential equations by fuzzy Runge-Kutta method of order m. For this purpose, the stochastic arithmetic and CESTAC method are applied to validate the results. In order to implement the C++ codes, the CADNA library is used. In this case, the optimal step size is found. The examples illustrate the efficiency and importance of using the stochastic arithmetic in place of the floating-point arithmetic.

Keywords: Fuzzy Runge-Kutta method (FRK), Stochastic arithmetic, CESTAC method, CADNA Library, Fuzzy differential equations (FDEs), Fuzzy numbers.

1 Introduction

One of the important topics in fuzzy mathematics and its applications is to solve fuzzy differential equations (FDEs) based on the definition of fuzzy derivative [7], [14], [15], [26], [27], [30]. One of the fundamental difference schemes to solve FDEs is Runge-Kutta methods which play the important role in numerical methods of solving ordinary differential equations [8], [16], [19], [21], [23], [36], [38]. In recent years, some works were done to solve FDEs, for example, Abbasbandy and Allahviranloo [6], Ghazanfari and Shakerami [20], Jayakumar etal. [24], Palligkinis etal. [31], Pederson and Sambandham [32],[33] worked on fuzzy differential equations via Runge-Kutta methods by converting the FDEs to two crisp ordinary differential equations. In all of these works, the results have been provided by the packages like Matlab based on the floating-point arithmetic. In this case, the validity and accuracy of the results are compared with the exact solution. Only the FDEs with having the exact solution are solved. The floating-point arithmetics and the mathematical packages are not able to rely the results and detect any instabilities during the run of the program. Also, in this arithmetic, it is not possible to find the optimal step size and it is necessary to consider a tolerance of accuracy like ε in the termination criterion. It is important computationally, it is possible to validate the results and control the variations of the step size in iterative methods and find the optimal results to ensure that based on the machine accuracy the final solution, the number of iterations and the step size are optimized and can not be improved in the sequel of performing the code. There are two kinds of errors in numerical methods for solving ordinary differential equations. The error of the method (or truncation error) which depends on the order of the method and decreases when the step size is decreased. On the contrary, the round-off error which generally increases (due to error propagation) when the step size is decreased more than a certain value. Hence, a

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2 Preliminaries

In section 2, the basic definitions of fuzzy numbers and fuzzy sets are presented, also brief description of fuzzy arithmetic. Applications) library [25]. By applying the new arithmetic, the optimal iterate of the fuzzy Runge-Kutta methods can be dynamically computed to solve a given FDEs and its accuracy can be estimated based on the discrete stochastic arithmetic which is called the stochastic arithmetic [39]-[42]. In order to implement this arithmetic, it must be written des arrondis de calculs.) method [10]-[12],[17] and the CADNA (Control of Accuracy and Debugging for Numerical Algorithms) library [25]. By applying the new arithmetic, the optimal iterate of the fuzzy Runge-Kutta methods can be dynamically computed and its accuracy can be estimated. Also, the optimal step size is determined, the optimal solution at a given point is evaluated and the results are validated. Until now some works were done on the stochastic arithmetic to control the accuracy and validate the results of the numerical algorithms such as [1]-[5],[18],[28],[29].

In this work, by using the CADNA library, the optimal solution and step size of the Runge-Kutta method of order m are dynamically computed to solve a given FDEs and its accuracy can be estimated based on the discrete stochastic arithmetic.

In section 2, the basic definitions of fuzzy numbers and fuzzy sets are presented, also brief description of fuzzy arithmetic to validate the results of the numerical algorithms such as [1]-[5],[18],[28],[29].

3. A crisp number \( a \) is a closed bounded interval which is denoted by \([a_1,a_2]\). The graph of a crisp number is a triangle based on the interval \([a_1,a_2]\) and vertex at \(x = a_2\).

4. A triangular fuzzy number, \( v \), is defined by three numbers \( a_1 \), \( a_2 \), and \( a_3 \) for which

\[
(1) v > 0 \text{ if } a_1 > 0; \quad (2) v \geq 0 \text{ if } a_1 \geq 0; \quad (3) v < 0 \text{ if } a_3 < 0; \quad \text{and} \quad (4) v \leq 0 \text{ if } a_3 \leq 0.
\]

Let \( E \) be a set of all upper semi-continuous normal convex fuzzy numbers with bounded \( r \)-level intervals. It means that if \( v \in E \) then the \( r \)-level set

\[
[v]_r = \{ s \mid v(s) \geq r \}, 0 < r \leq 1,
\]

is a closed bounded interval which is denoted by \([v]_r = [v_1(r),v_2(r)]\).
Definition 2.3. For arbitrary fuzzy numbers \( u = (\mu(r), \bar{\mu}(r)) \) and \( v = (\nu(r), \bar{\nu}(r)) \) the quantity

\[
D_H(u, v) = \sup_{0 \leq r \leq 1} \max\{|\mu(r) - \nu(r)|, |\bar{\mu}(r) - \bar{\nu}(r)|\},
\]

is the distance between \( u \) and \( v \). and the \( D_H \) is Hausdorff distance.

The function \( D(u, v) \) is a metric on \( E^1 \). This metric function is equivalent to the one used by Puri and Ralescu [34] and Kaleva [26],[27]. It is shown [35] that \( (E^1, D) \) is a complete metric space.

Definition 2.4. A function \( f : \mathbb{R} \rightarrow E^1 \) is called a fuzzy function. If for arbitrary fixed \( t_0 \in \mathbb{R} \) and \( \varepsilon > 0 \), a \( \delta > 0 \) such that

\[
|t - t_0| < \delta \Rightarrow D(f(t), f(t_0)) < \varepsilon,
\]

exists, \( f \) is said to be continuous.

The concept of fuzzy differentiation was introduced by Dubois and Prade [14], Puri and Ralescu [34] proposed two approaches for finding the fuzzy derivative. The first, based on the H-difference notation regards \( E^1 \) as a universe. The second approach was also suggested by Goetschel and Voxman [22].

Suppose that \( y : I \rightarrow E^1 \) is a fuzzy function. The parametric form of \( y(t) \) is represented by

\[
y(t)_r = [y_1(t, r), y_2(t, r)], t \in I, r \in (0, 1],
\]

where \( I \) is a real interval. The Seikkala derivative \( y'(t) \) of a fuzzy function \( y(t) \) is defined by [37]

\[
y'(t)_r = [y'_1(t, r), y'_2(t, r)], t \in I, r \in (0, 1],
\]

provided that this equation defines a fuzzy number. In the sequel, the Seikkala derivative is considered.

2.1 Fuzzy Cauchy problem

Consider the following fuzzy initial value problem (FIVP)[20],[24],[30]

\[
y' = f(t, y(t)), y(0) = y_0, t \in I = [0, T].
\]

where \( f \) is a continuous mapping from \( E^1 \) into \( E^1 \) and \( y_0 \in E^1 \) with \( r \)-levels sets

\[
y(r)_t = [y_1(0; r), y_2(0; r)], r \in (0, 1].
\]

The extension principle of Zadeh leads to the following definition of \( f(y(t)) \) when \( y(t) \) is a fuzzy number:

\[
f(y(t))_s = \sup_{\tau} \{y(\tau) | s = f(\tau)\}, s \in R.
\]

From this, it follows that

\[
[f(y(t))_r] = [f_1(y; r), f_2(y; r)], r \in (0, 1],
\]

where

\[
f_1(y; r) = \min\{f(u) | u \in [y_1(t; r), y_2(t; r)]\},
\]

\[
f_2(y; r) = \max\{f(u) | u \in [y_1(t; r), y_2(t; r)]\}.
\]

The mapping \( f(y) \) is a fuzzy function and the derivative \( f'(y) \) [37] is defined by

\[
[f'(y(t))_r] = [f'_1(y; r), f'_2(y; r)], t \in I, r \in (0, 1],
\]

provided that this equation determines the fuzzy number \( f'(y) \in E^1 \), where

\[
f'_1(y; r) = \min\{f'(u) | u \in [y_1(t; r), y_2(t; r)]\},
\]

\[
f'_2(y; r) = \max\{f'(u) | u \in [y_1(t; r), y_2(t; r)]\}.
\]
Sufficient conditions for the existence of a unique solution to Eq. (2.5) is that \( f \) satisfies the Lipschitz condition

\[
\|f(y) - f(z)\| \leq L\|y - z\|, \quad L \geq 0.
\]  

(2.10)

We may replace Eq. (2.5) by the equivalent system

\[
\begin{align*}
y'(t; r) &= f_i(y(t)) = f_1(y;r) = F(y(t;r), \tilde{y}(t;r)), y(0, r) = y_0(r), \\
y'(t; r) &= \tilde{F}(y(t)) = f_2(y;r) = G(y(t;r), \tilde{y}(t;r)), \tilde{y}(0, r) = \tilde{y}_0(r),
\end{align*}
\]

for \( r \in (0, 1] \).

2.2 Fuzzy Runge-Kutta method

We consider fuzzy initial value problem (FIVP) (5) with step-size \( h = T/N \) and grid points

\[
t_j = jh, \quad j = 0, 1, ..., N.
\]

(2.12)

where \( \dot{\cdot} \) denotes Seikkala differentiation.

Let the exact solution \( [Y(t)]_r = [Y_1(t;r), Y_2(t;r)] \) be approximated by \([y(t)]_r = [y_1(t;r), y_2(t;r)]\). For Runge-Kutta of order \( m \), let [20],[24]

\[
\begin{align*}
y_1(t_{n+1}; r) - y_1(t_n; r) &= \sum_{i=1}^{m} w_i k_i(t_n, y(t_n; r)), \\
y_2(t_{n+1}; r) - y_2(t_n; r) &= \sum_{i=1}^{m} w_i k_i(t_n, y(t_n; r)),
\end{align*}
\]

(2.13)

where the \( w_i \) are constants and

\[
[k_i(t, y(t; r))]_r = [k_{i;1}(t, y(t, r)), k_{i;2}(t, y(t, r))], \quad i = 1, 2, ..., m
\]

(2.14)

Equation (2.13) is to be exact for powers of \( h \) through \( h^m, m \geq 1 \). Therefore, the truncation error \( T_m \) can be written as [6]

\[
T_m(h) = \gamma_m h^{m+1} + O(h^{m+2}),
\]

(2.15)

If the term \( O(h^{m+2}) \) is small compared with \( \gamma_m h^{m+1} \) as we expect it will be if \( h \) is small, then the bound on \( \gamma_m h^{m+1} \) will usually be a bound on the error as a whole.

3 Main idea

Let \( \tilde{x} = (\tilde{x}(r), \bar{x}(r)) \) be a fuzzy number in \( E^1 \). Then, \( \tilde{x} \) is represented as \( X = (X(r), \bar{X}(r)), 0 \leq r \leq 1 \) in the computer. It can be shown that:

\[
\begin{align*}
\frac{X(r)}{\bar{X}(r)} &= \tilde{x}(r) - e_1 2^{-p} g, \\
\frac{X(r)}{\bar{X}(r)} &= \bar{x}(r) - e_2 2^{-p} \bar{g},
\end{align*}
\]

(3.16)

where, \( e_1 \) and \( e_2 \) are the signs of \( x(r) \) and \( \bar{x}(r) \) respectively and \( 2^{-p} g \) and \( 2^{-p} \bar{g} \) are the lost part of the mantissa due to round-off error and \( E_1 \) and \( E_2 \) are the binary exponents of the results. In single precision case, \( P = 24 \) and \( -1 \leq g, \bar{g} \leq 1 \). In the CESTAC method, \( \alpha \) and \( \bar{\alpha} \) in (3.16) are considered as random variables uniformly distributed on \([-1, 1]\). In order to find samples for the obtained random variables, we perturb the last mantissa bit (or previous bits) of the values \( X(r) \) and \( \bar{X}(r) \), \( 0 \leq r \leq 1 \). The algorithm of fuzzy CESTAC method is as follows where \( d \) is a small positive value like \( 10^{-2} \) and \( T_\beta \) is the value of \( T \) distribution with \( N - 1 \) degree of freedom and confidence interval \( 1 - \beta \). If \( N = 3 \) and \( \beta = 0.05 \) then \( T_\beta = 4.303 \).
Algorithm 1:
For \( r = 0(d)1 \) do the following steps [18]:

1) Find \( N \) samples for \( \bar{x}(r) \) and \( \bar{x}(r) \) as
\[
\bar{x}_1(r), \bar{x}_2(r), \ldots, \bar{x}_N(r),
\]
and
\[
\bar{x}'_1(r), \bar{x}'_2(r), \ldots, \bar{x}'_N(r),
\]
by means of the perturbation of the last bit of the mantissa,

2) Compute
\[
\bar{x}_{\text{ave}}(r) = \frac{1}{N} \sum_{i=1}^{N} \bar{x}_i(r),
\]
and
\[
\bar{x}'_{\text{ave}}(r) = \frac{1}{N} \sum_{i=1}^{N} \bar{x}'_i(r),
\]

3) Compute
\[
\bar{s}^2(r) = \frac{1}{N-1} \sum_{i=1}^{N} (\bar{x}_i(r) - \bar{x}_{\text{ave}}(r))^2,
\]
and
\[
\bar{s}'^2(r) = \frac{1}{N-1} \sum_{i=1}^{N} (\bar{x}'_i(r) - \bar{x}'_{\text{ave}}(r))^2,
\]

4) Compute
\[
C_{\bar{x}_{\text{ave}}(r), \bar{x}(r)} = \log_{10} \frac{\sqrt{N} |\bar{x}_{\text{ave}}(r)|}{\tau B \bar{s}(r)},
\]
and
\[
C_{\bar{x}'_{\text{ave}}(r), \bar{x}(r)} = \log_{10} \frac{\sqrt{N} |\bar{x}'_{\text{ave}}(r)|}{\tau B \bar{s}(r)},
\]
as the common significant digits between the exact values \( \bar{x}(r) \) and \( \bar{x}(r) \) and the approximate values \( \bar{x}_{\text{ave}}(r) \) and \( \bar{x}'_{\text{ave}}(r) \) respectively,

5) If
\[
C_{\bar{x}_{\text{ave}}(r), \bar{x}(r)} \leq 0 \text{ or } \bar{x}_{\text{ave}}(r) = 0,
\]
and
\[
C_{\bar{x}'_{\text{ave}}(r), \bar{x}(r)} \leq 0 \text{ or } \bar{x}'_{\text{ave}}(r) = 0,
\]
then write \( \hat{x} = @ .0 \). The implementation of FRK method based on the fuzzy CESTAC method is mentioned in the following algorithm.
Algorithm 2:
1- Let \( n = 1 \), \( i = 1 \) and \( d = 0.1 \).
2- Let \( h = T / 2^n \).
3- For \( r = 0(d)1 \) do the following steps based on the fuzzy CESTAC method:
   3.a) Find \( y_1^{n+1} \) and \( y_2^{n+1} \) by using FRK method mentioned in (2.13),(2.14).
   3.b) Let
   \[
   D_n^{(r)} = |y_1^{n+1} - y_1^n|,
   \]
   and
   \[
   D_n^{(i)} = |y_2^{n+1} - y_2^i|,
   \]
   and put the maximum value of them as \( D_{\max}[i] \).
   3.c) \( i = i + 1 \).
4- Find the maximum element of the array \( D_{\max}[i] \) and call it \( maxdis \) which is the approximation of Hausdorff distance in the stochastic arithmetic. If \( maxdis = @.0 \) then go to step 5 else put \( n = n + 1 \) and go to step 2,
5- Print \( n \) as the optimal iteration and \( (y_1^n, y_2^n) \approx (Y_1(t_j), Y_2(t_j)) \) as the approximate solution of the FIVP (2.5).

3.1 The use of the CADNA library for solving FIVP

According to the previous results, by using the Fuzzy CESTAC method and the CADNA library, we propose to use \( D_H(y^n, y^{n+1}) = @.0 \) as the stopping criterion. For solving FIVP in the stochastic arithmetic we use the algorithm 2 by applying Fuzzy Runge-Kutta method with the initial value \( Y^{(0)} = (y_1^{(0)}, y_2^{(0)}) \). The programs have been written in C++ and executed on a Linux machine using CADNA library. Suppose \( y^n = (y_1^n(h), y_2^n(h)) \), \( y^{n+1} = (y_1^{n+1}(\frac{h}{2}), y_2^{n+1}(\frac{h}{2})) \) are the approximate values of \( Y(t_j) = (Y_1(t_j), Y_2(t_j)) \) in two successive iterations obtained from the Fuzzy Runge-Kutta method, respectively. For the termination criterion, we consider the Hausdorff distance to be an informatical zero (\( @.0 \)).

The following theorem is proved to show the accuracy of FRK to compute the solution of a given FIVP at a point. It is proved, the common significant digits of two sequential results is almost equal to the common significant digits between the approximate and the exact solution. In this theorem, the notation \( C_{a,b} \) means the common significant digits between two distinct real numbers \( a \) and \( b \) in \( \mathbb{R} \) which is defined as \([1]-[5],[17]\)

\[
C_{a,b} = \log_{10} \left| \frac{a + b}{2(a - b)} \right| = \log_{10} \left| \frac{a}{a - b} - \frac{1}{2} \right|. \tag{3.17}
\]

**Theorem 3.1.** Suppose \((y_1^n(h), y_2^n(h))\), \((y_1^n(\frac{h}{2}), y_2^n(\frac{h}{2}))\) are the approximate solutions of \((Y_1(t_j), Y_2(t_j))\) of the FIVP (2.5) in two successive steps of the Fuzzy Runge-Kutta method of the order \( m \). Then

\[
C_{y_1^n(h), y_1^n(\frac{h}{2})} = C_{y_2^n(h), Y_1(t_j)} + \log \frac{2^m + 1}{2m + 1}, \tag{3.18}
\]

\[
C_{y_2^n(h), y_2^n(\frac{h}{2})} = C_{y_2^n(h), Y_2(t_j)} + \log \frac{2^m + 1}{2m + 1}. \tag{3.19}
\]
Proof. If Eq. (3.18) is proved, then Eq. (3.19) can be proved similarly. According to Eq. (2.15), with step size $h$ the local truncation error

$$T_m(h) = y_i^h(t_j) - Y_i(t_j) = \gamma_n h^{m+1} + O(h^{m+2}),$$

(3.20)

and by step size $\frac{h}{2}$ we have

$$T_m(\frac{h}{2}) = y_i^\frac{h}{2}(\frac{h}{2}) - Y_i(t_j) = \gamma_n (\frac{h}{2})^{m+1} + O((\frac{h}{2})^{m+2}).$$

(3.21)

From Eqs. (3.20) and (3.21),

$$y_i^h(t_j) - y_i^\frac{h}{2}(\frac{h}{2}) = y_i^h(t_j) - (y_i^\frac{h}{2}(\frac{h}{2}) - Y_i(t_j)) = \gamma_n h^{m+1} - \gamma_n (\frac{h}{2})^{m+1} + O(h^{m+2})$$

hence

$$y_i^h(t_j) - y_i^\frac{h}{2}(\frac{h}{2}) = (1 - \frac{1}{2^{m+1}})\gamma_n h^{m+1} + O(h^{m+2}).$$

Furthermore,

$$\frac{y_i^\frac{h}{2}(\frac{h}{2}) + Y_i(t_j)}{2(y_i^h(t_j) - Y_i(t_j))} = \frac{y_i^\frac{h}{2}(\frac{h}{2})}{y_i^h(t_j) - Y_i(t_j)} - \frac{1}{2} = \frac{y_i^\frac{h}{2}(\frac{h}{2})}{\gamma_n h^{m+1}(1 + O(h))} + O(1).$$

From Definition (3.17),

$$C_{y_i^h(t_j), Y_i(t_j)} = \log \left| \frac{y_i^\frac{h}{2}(\frac{h}{2})}{\gamma_n h^{m+1}(1 + O(h))} + O(1) \right| = \log \left| \frac{y_i^\frac{h}{2}(\frac{h}{2})}{\gamma_n h^{m+1}} \right| + \log (1 + O(h)) = \log \left| \frac{y_i^\frac{h}{2}(\frac{h}{2})}{\gamma_n h^{m+1}} \right| + O(h).$$

(3.22)

And

$$C_{y_i^\frac{h}{2}(\frac{h}{2}), Y_i(t_j)} = \log \left| \frac{y_i^\frac{h}{2}(\frac{h}{2})}{\gamma_n h^{m+1}(1 + O(h))} + O(1) \right| = \log \left| \frac{y_i^\frac{h}{2}(\frac{h}{2})}{\gamma_n h^{m+1}} \right| + \log (1 + O(h))$$

(3.23)

According to (3.22), (3.23)

$$C_{y_i^\frac{h}{2}(\frac{h}{2}), Y_i(t_j)} = C_{y_i^\frac{h}{2}(\frac{h}{2}), Y_i(t_j)} + \log \left( \frac{2^{m+1}}{2^{m+1} - 1} \right) + O(h).$$

If the convergence zone is reached, i.e., if the term $O(h)$ becomes negligible, the significant digits common to two successive approximations $y_i^h(t_j)$ and $y_i^\frac{h}{2}(\frac{h}{2})$ are also in common with the exact result $Y_i(t_j)$, up to one bit. Indeed the term $\log \left( \frac{2^{m+1}}{2^{m+1} - 1} \right)$ decreases as $m$ increases and for $m \geq 1$ term tends to zero.
For example in the fuzzy Runge-Kutta method of the order fourth:

\[
C_{\frac{h}{2}}^{\delta}(h), y(\frac{h}{2}) = C_{\frac{h}{2}}^{\delta}(h), y(t) + \log\frac{32}{31} + O(h).
\]

Eq. (3.18) shows that, if the fuzzy Runge-Kutta method is used in order to estimate \( Y(1) \) then, for \( h \) small enough, the number of common significant digits between \( Y(h) \) and \( Y(\frac{h}{2}) \) is almost equal to the number of common significant digits between \( Y(t) \) and \( Y(\frac{h}{2}) \). Therefore, if the fuzzy CESTAC method is used then, the computations of the sequence \( y_i^n(h) \)'s are stopped when for an index like \( h_{opt}, D_H \left(y_1(h_{opt}), y_1(\frac{1}{2}h_{opt}) \right) \geq 0 \). In this case, \( Y(h_{opt}) \) is an approximation of \( Y(t) \).

4 Numerical Examples

Since, the results of the iterative methods are obtained in the floating-point arithmetic, the termination criterion depends on a positive number like \( e \). So, the final results may not be accurate or the number of iterations may increase without increasing the accuracy of the results. Therefore, the validation of the computed results is important. In this case, because of the round-off error propagation, the computer may not able to improve the accuracy of the computed solution. The fuzzy CESTAC method is an efficient method in order to estimate the accuracy of the results and find the optimal number of iterations. In the following examples, the results are obtained based on the Algorithm 2 and by applying the fuzzy Runge-Kutta method of order 4 (FRK4). The results in the tables have been provided by the CADNA library and C++ codes in double precision.

Let the interval \([0, T]\) is divided into \( N = 2^n \) sub-intervals, \( n = 1, 2, 3, \ldots \) the value \( n \) is increased until the termination criterion in algorithm 2 is satisfied.

Example 4.1. Consider the fuzzy initial value problem [6],[20].

\[
y'(t) = y(t), t \in I = [0, T] = [0, 1], \\
y(0) = (0.75 + 0.25e, 1.125 - 0.125e), r \in (0, 1].
\]

The exact solution is given by

\[
Y_1(t; r) = y_1(0; r)e^r, Y_2(t; r) = y_2(0; r)e^r,
\]

then

\[
Y(t; r) = \left[(0.75 + 0.25r)e^r, (1.125 - 0.125r)e^r\right], r \in (0, 1].
\]

Which at \( t = 1 \),

\[
Y(1; r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e], r \in (0, 1].
\]

By using the Runge-Kutta method of order 4, we have

\[
y_1(t_{n+1}; r) = y_1(t_n; r) \left[1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}\right], \\
y_2(t_{n+1}; r) = y_2(t_n; r) \left[1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}\right],
\]
Table 1: Numerical results of Example 4.1

<table>
<thead>
<tr>
<th>n</th>
<th>$h = \frac{1}{2n}$</th>
<th>r</th>
<th>$\gamma_1(1,r)$</th>
<th>$\gamma_2(1,r)$</th>
<th>$D_{\text{TE}}(\gamma_2(h),\nu_1^*(\frac{h}{2}))$</th>
<th>$D_{\text{TE}}(Y_1(\nu_2),\nu_1^*(\frac{h}{2}))$</th>
</tr>
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According to Table 1, it has been found the optimal step size \( h_{\text{opt}} = \frac{1}{2^{10}} \).

Also, it can be compared the Hausdorff distance between the approximate solutions \( y^n(h) \) and \( y^n(h') \) in two successive iterations with the Hausdorff distance between the exact solution \( Y(t_j) \) and approximate solution \( y^n(h) \) of Eq. (4.24) in the \( t_j = 1 \). The numerical solution in different \( r \) is observed in columns \( y_1(1;r) \) and \( y_2(1;r) \) which is optimized at \( n = 10 \).

**Example 4.2.** Consider the fuzzy initial value problem [15],

\[
y'(t) = f(y(t)) = cy(t), t \in I = [0, 1],
\]

\[
y(0) = y_0, y_0 = (8/8.5/9), c = (1/2/3).
\]  

The exact solution is

\[
Y_1(t;r) = y_1(0;r)e^{c_1t} = (8 + 0.5r)e^{(1+r)t},
\]

\[
Y_2(t;r) = y_2(0;r)e^{c_2t} = (9 - 0.5r)e^{(3-r)t}.
\]

with the initial condition \((y_1(0), y_2(0))\). The results of FRK4 method at \( t_j = 1 \) are shown in Table 2.
Table 2: Numerical results of Example 4.2

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5 Conclusion

In this paper, we observed that the use of the fuzzy CESTAC method and the CADNA library allow us to find the optimal step \( h_{\text{opt}} \) of the fuzzy Runge-Kutta methods (FRK). In other word, via the stochastic arithmetic, by determining the \( h_{\text{opt}} \), the step size in Fuzzy Runge-Kutta methods is controlled and the solution of the fuzzy initial value problem is validated. It was shown that, it is possible, by using the optimal termination criterion which uses the computational zero, to stop correctly the iterative process, to save computer time, to estimate the accuracy of the results and to find the optimal step size and the best result computationally in the fuzzy Runge-Kutta methods by applying the CADNA library. This idea can be done for other kinds of the numerical methods to solve a fuzzy differential equation.

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