Numerical Solution of Impulsive Fuzzy Initial Value Problem by Modified Euler’s Method

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Abstract
In this paper we present numerical solution of first order impulsive fuzzy differential equation by modified Euler’s method for the first time. The algorithm is discussed in detail and the result of the numerical solution is given by solving the numerical example and the graph is plotted finally.

Keywords: Impulsive fuzzy differential equation, Modified Euler’s method.

1 Introduction

Impulsive differential equations appear to represent a natural framework for mathematical modeling of several real world phenomena. Impulsive differential equations are useful tools to modeling of evolution processes that are subjected to sudden changes in their state. But, many impulsive differential equations can’t be solved analytically or their solving is more complicated. Furthermore, we often for solving too practical problems need not solution of impulsive differential equation analytically, and just need numerical values of solution. In addition, when a case of real world phenomena is transformed into the deterministic differential equations with initial value, we usually can’t be sure that this modeling is perfect. Thus, we consider an impulsive fuzzy differential equation, and present a new numerical algorithm for solving this equation.

We consider First-order impulsive fuzzy differential equations [11]

\[ y'(t) = f(t, y(t)), \quad t \in I = [0, T], \quad t \neq t_k, \quad k = 1, ..., m, \]  \hspace{1cm} (1.1)

\[ y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, ..., m \]  \hspace{1cm} (1.2)

\[ y(t_0) = a, \]  \hspace{1cm} (1.3)

Where \( E^n \) is the set of triangular fuzzy number, with \( f: I \times E^n \to E^n \) and \( I_k: E^n \to E^n, \quad k = 1, ..., m \) are given functions. \( t_0 = 0 < t_1 < \cdots < t_m < t_{m+1} = T, \quad a \in E^n. \) And \( y(t_k^+) \) and \( y(t_k^-) \) represent the left and right limits of \( y(t) \) at \( t = t_k. \)

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2 Preliminaries

In this section, we first present the existence of fuzzy solutions for initial value problems for first order ordinary impulsive fuzzy differential equation, and after that we introduce definitions and preliminary facts which are used throughout this paper.

We are concerned with the existence of fuzzy solutions for problem (1.1)-(1.3), in order to define the solution of (1.1)-(1.3) the following space will be used [11]

\[
PC = \left\{ y : [0, T) \rightarrow E^n : y \in C(J_k, E^n), \lim_{t \to t_k^+} y(t) = y(t_k), \right. \\
\left. \text{and } \lim_{t \to t_k^-} y(t) \text{ exists, } k = 1, ..., m \right\}
\]

Here \( J_k = (t_k, t_{k+1}], \), \( k = 0, ..., m \) with \( t_0 = 0 \) and \( t_{m+1} = T \).

**Definition 2.1.** ([11]) A function \( y \in C^1(J \setminus \{t_k\}, E^n) \cap PC \) is said to be a solution of (1.1)-(1.3) if \( y \) satisfies the equation \( y'(t) = f(t, y(t)) \) on \( J, t \neq t_k, k = 1, ..., m \) and the conditions \( y(t_k^+) = I_k(y(t_k^-)), k = 1, ..., m \) and \( y(0) = a \).

**Theorem 2.1.** ([11]) Assume that

**H1** There exists a continuous non-decreasing function \( \psi : [0, \infty) \rightarrow (0, \infty) \) and a continuous function \( p : J \rightarrow \mathbb{R}_+ \) such that

\[
d_{\infty}(f(t, y), \bar{0}) \leq p(t)\psi(d_{\infty}(y, \bar{0})) \quad \text{for } t \in J \quad \text{and each } y \in E^n
\]

with

\[
\int_{t_k}^{t_{k+1}} p(s)ds \leq \int_{t_k}^{T} \frac{du}{\psi(u)}, \quad k = 0, ..., m \quad \text{and} \quad I_0 = a
\]

**H2** For each \( t \in J_k, k = 0, ..., m \) the set

\[
\left\{ I_k(y(t_k)) + \int_{t_k}^{t} f(s, y(s))ds : y \in A_k \right\}
\]

is a total bounded subset of \( E^n \), where

\[
A_k = \left\{ y \in C(J_k, E^n) : d_{\infty}(y(t), \bar{0}) \leq a_k(t), t \in J_k \right\}
\]

\[
a_k(t) = M_k^{-1} \left( \int_{t_k}^{t} p(s)ds \right)
\]

and

\[
M_k(z) = \int_{d_{\infty}(I_k(y(t_k)), \bar{0})}^{z} \frac{du}{\psi(u)}
\]

Then the IVP (1.1)-(1.3) has at least one fuzzy solution on \([0, T]\).

2.1. Fuzzy Cauchy Problem

The numerical algorithm for solving impulsive fuzzy initial value problem is different from numerical algorithm for solving fuzzy initial value problem only at the pulse point, where we have to apply the operators concern with the particular point. For this reason, at first we consider fuzzy initial value problem in the following form

\[
\begin{align*}
y'(t) &= f(t, y(t)) \\
y(t_0) &= y_0
\end{align*}
\]

(2.4)

Where \( y \) is a fuzzy function of \( t \), \( f(t, y) \) is a fuzzy function of the crisp variable \( t \) and the fuzzy variable \( y \), \( y' \) is the fuzzy derivative of \( y \) and \( y(t_0) = y_0 \) is a triangular fuzzy number. Therefore, we have a fuzzy Cauchy problem. [18] We consider that, the fuzzy function \( y \) is \( y = [y_1, y_2] \). It means that the \( r \)-level set of \( y(t) \) for \( t \in [t_0, T] \) is
\[ y(t_0) = [y_1(t_0), y_2(t_0)] \]

By using the extension principle of Zadeh, we have the membership function
\[ f(t, y(t)) = \sup \{ y(t)(s) : s \in f(t, \tau) \}, \quad s \in R \]

So \( f(t, y(t)) \) is a fuzzy number. From this it follows that
\[ [f(t, y(t))]_r = [f_1(t, y(t)); f_2(t, y(t)); r] \quad r \in (0, 1] \]

Where
\[ f_1(t, y(t); r) = \min \{ f(t, u) | u \in [y_1(t), y_2(t)] \} \]
\[ f_2(t, y(t); r) = \max \{ f(t, u) | u \in [y_1(t), y_2(t)] \} \]

We define
\[ f_1(t, y(t); r) = F(t, y_1(t), y_2(t)) \]
\[ f_2(t, y(t); r) = G(t, y_1(t), y_2(t)) \]

Now, we consider the initial value problem (2.4), it is known that, the sufficient conditions for the existence of a unique solution to (2.4) are \( f \) to be continuous function satisfying the Lipschitz condition of the following form
\[ \| f(t, x) - f(t, y) \| \leq L \| x - y \|, \quad L > 0 \]

### 2.2. Modified Euler’s Method

We consider modified Euler’s method as following:

First we replace the interval \([t_0, T]\) by a set of discrete equally spaced points,
\[ t_0 < t_1 < t_2 < \cdots < t_n = T, h = \frac{T-t_0}{N}, \quad t_i = t_0 + ih, \quad i = 0, 1, ..., N, n \geq 0. \]

[18] One-step modified Euler’s method is the form
\[ y(t_{n+1}) = y(t_n) + \frac{h}{2} \left[ f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1})) \right] \]

With initial value \( y(t_0) = y_0 \).

Let \( y^{(0)}(t_{n+1}) = y(t_n) + hf(t_n, y(t_n)) \) be a good initial guess of the solution \( y(t_{n+1}) \), and define
\[ y^{j+1}(t_{n+1}) = y(t_n) + \frac{h}{2} \left[ f(t_n, y(t_n)) + f \left( t_{n+1}, y^{(j)}(t_{n+1}) \right) \right], \quad j = 0, 1, ... \] \hspace{1cm} (2.5)

Where (2.5) is known as iterative solution of modified Euler’s method relation and \( j \) is index of iteration.

### 2.3. Modified Euler’s Method for Numerical Solution of Fuzzy Cauchy Problem

Let \( Y = [Y_1, Y_2] \) be the exact solution and \( y = [y_1, y_2] \) be the approximated solution of the initial value equation (2.4) by using the one-step modified Euler’s method. Let,
\[ [Y(t)]_r = [Y_1(t, r), Y_2(t, r)], \quad [y(t)]_r = [y_1(t, r), y_2(t, r)] \]

The exact and approximated solution at \( t_n \) is denoted by \([Y(t_n)]_r = [Y_1(t_n, r), Y_2(t_n, r)], [y(t_n)]_r = [y_1(t_n, r), y_2(t_n, r)], (0 \leq n \leq N) \) respectively. [18] The grid points at which the solution is calculated are
\[ h = \frac{T-t_0}{N}, \quad t_i = t_0 + ih \quad 0 \leq i \leq N \]

By using the modified Euler method, we obtain
\[ Y_1(t_{n+1};r) = Y_1(t_n; r) + \frac{h}{2} F[t_n, Y_1(t_n; r), Y_2(t_n; r)] \\
+ \frac{h}{2} F[t_{n+1}, Y_1(t_n; r) + hF[t_n, Y_1(t_n; r), Y_2(t_n; r)], Y_2(t_n; r)] \\
+ hG[t_n, Y_1(t_n; r), Y_2(t_n; r)] + h^3 A_1(r) \]
and
\[ Y_2(t_{n+1}; r) = Y_2(t_n; r) + \frac{h}{2} G[t_n, Y_1(t_n; r), Y_2(t_n; r)] \\
+ \frac{h}{2} G[t_{n+1}, Y_1(t_n) + hF[t_n, Y_1(t_n; r), Y_2(t_n; r)], Y_2(t_n; r) + hG[t_n, Y_1(t_n; r), Y_2(t_n; r)]] \\
+ h^3 A_2(r) \]

Where \( A = [A_1, A_2], A_r = [A_1(r), A_2(r)] \)
and
\[ [A]_r = \left[ \frac{1}{4} f'(\xi_2, Y(\xi_2)) \cdot f_y(t_{n+1}, \xi_3) - \frac{1}{12} f''(\xi_1, Y(\xi_1)) \right] \]

Also we have
\[ y_1(t_{n+1}; r) = y_1(t_n; r) + \frac{h}{2} F[t_n, y_1(t_n; r), y_2(t_n; r)] \\
+ \frac{h}{2} F[t_{n+1}, y_1(t_n; r) + hF[t_n, y_1(t_n; r), y_2(t_n; r)], y_2(t_n; r)] \\
+ hG[t_n, y_1(t_n; r), y_2(t_n; r)] \]
and
\[ y_2(t_{n+1}; r) = y_2(t_n; r) + \frac{h}{2} G[t_n, y_1(t_n; r), y_2(t_n; r)] \\
+ \frac{h}{2} G[t_{n+1}, y_1(t_n) + hF[t_n, y_1(t_n; r), y_2(t_n; r)], y_2(t_n; r) + hG[t_n, y_1(t_n; r), y_2(t_n; r)]] \]

Where \( y_1(t_n; r) \) and \( y_2(t_n; r) \) converge to \( Y_1(t; r) \) and \( Y_2(t; r) \), respectively whenever \( h \to 0 \).

3 Numerical Algorithm for Solving IFIVP

In this section, we present algorithm for solving first order impulsive fuzzy differential equations with initial value. If function \( y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{pmatrix} \) is solution of impulsive fuzzy differential equations (2.4), by the numerical algorithm for Solving Impulsive Fuzzy Differential Equations, it is possible to obtain values \( y(t_z) = \begin{pmatrix} y_1(t_z) \\ y_2(t_z) \\ \vdots \\ y_m(t_z) \end{pmatrix} \) for fixed value \( t_z \) of parameter \( t \). Where \( t_z > t_0 \), at the moment \( t = t_z \).

The algorithm include of following steps:

**Step One:** At \( t = t_0 \) we apply the modified Euler’s method to the function \( y \) with considering \( y = y_0 \) from initial condition (2.4). The algorithm applies until the first pulse point.
Step Two: At the pulse point $t = t_k$ impulsive operator $I_k$ brings rapidly changes of function $y$ that moment is

$$[J(t_k)]_r = [J_1(t_k, r), J_2(t_k, r)], \quad 0 < r \leq 1$$

Where

$$J_1(t_k, r) = \left\{ I_k \left( y_1(t_k, r) + \sum_{t_0 < t_0 < t_k} J_1(t, r) \right) \right\}$$

$$J_2(t_k, r) = \left\{ I_k \left( y_2(t_k, r) + \sum_{t_0 < t_0 < t_k} J_2(t, r) \right) \right\}$$

Step Three: We solve the function $y$ of argument $t$ taking it from half-segment $(t_k, t_{k+1})$ by modified Euler’s method.

Step Four: We repeat step two and step three until we encounter with the desire $y(t_2)$ that has to be found.

Step Five: We add to the function $y$ the summation of all pulses.

$$Y_1(t, r) := y_1(t, r) + \sum_{t_0 < t_k < t_k} J_1(t_k, r)$$

$$Y_2(t, r) := y_2(t, r) + \sum_{t_0 < t_k < t_k} J_2(t_k, r)$$

$$[y(t)]_r = [y_1(t, r), y_2(t, r)]$$

is the approximated solution and $[J(t_k)]_r = [J_1(t_k, r), J_2(t_k, r)]$

is pulse in point $t_k$, we have:

$$Y = [y + \sum_{t_0 < t_k < t_k} J_1(t_k, r), y_2(t, r) + \sum_{t_0 < t_k < t_k} J_2(t_k, r)]$$

We apply modified Euler’s method on each half-segment $(t_k, t_{k+1})$.

Remark 3.1. We shall choose $s$ ($s \in N$) nodes from the half-segment on which we are solving, i.e.

$$(t_k, t_{k+1}) = \left( t_k^{[0]}, t_{k+1}^{[0]} \right) \cup \left( t_k^{[1]}, t_{k+1}^{[1]} \right) \cup \ldots \cup \left( t_k^{[s-1]}, t_{k+1}^{[s]} \right)$$

Remark 3.2. The extension principle of Zadeh leads to the following definition of $I_k(y(t_k))$ when $y(t_k)$ is a fuzzy number:

$$[I_k(y(t_k))]_r = [I_{k1}(y(t_k, r)), I_{k2}(y(t_k, r))]$$

It follows that

$$[I_k(y(t_k))]_r = [\min \{I_k(u(t_k)) : u \in [y_1(t_k, r), y_2(t_k, r)] \}, \max \{I_k(u(t_k)) : u \in [y_1(t_k, r), y_2(t_k, r)] \}]$$

Remark 3.3. In step two, on the one hand for $k = 1$ we have $I_1(y(t_1)) = I_1(y(t_1)) + \sum_{t_0 < t_0 < t_1} J(t_1)$, and $J(t_1)$ is a fuzzy number, so $f(t_k), k = 2, \ldots, m, are fuzzy numbers. On the other hand, $y(t_k), k = 1, \ldots, m, is fuzzy number too.$

4 Numerical Result

In this section we present a numerical example, in order to see the accuracy of our numerical solution. The numerical results show that for smaller step-size $h$ we obtain smaller errors. The approximated solution by Modified Euler’s method is plotted in figure 1.
Consider the first order impulsive fuzzy initial value problem

\[ y'(t) = y(t), \quad t \in [0,1], \]
\[ y(t_k^+) = 0.01 \ y(t_k^-), \]
\[ y(0) = (0.75 + 0.25r, 1.125 - 0.125r) \]

Where \( 0 \leq r \leq 1 \),

And by considering \( t_2 = 1 \)

Now we apply the numerical algorithm for solving equation (4.6)-(4.8), defined earlier. By using modified Euler’s method, on each half-segment \((t_k, t_{k+1}]\) we have

\[ y_1(0; r) = 0.75 + 0.25r, \quad y_2(0; r) = 1.125 - 0.125r, \]

and

\[ y_1^{(0)}(t_{i+1}; r) = y_1(t_i; r) + hy_1(t_i; r) \]
\[ y_2^{(0)}(t_{i+1}; r) = y_2(t_i; r) + hy_2(t_i; r) \]

Where \( i = 0,1,...,N-1, h = \frac{1}{N} \). Now, using these equations as an initial guess for following iterative solutions, respectively.

\[ y_1^{(j)}(t_{i+1}; r) = y_1(t_i; r) + \frac{h}{2} \left[ y_1(t_i; r) + y_1^{(j-1)}(t_{i+1}; r) \right] \]
\[ y_2^{(j)}(t_{i+1}; r) = y_2(t_i; r) + \frac{h}{2} \left[ y_2(t_i; r) + y_2^{(j-1)}(t_{i+1}; r) \right] \]

Where \( j = 1,2,..., \), iterative index \( j \) increase until we encounter with the desired values to be found, for \( i = 1,...,N \).

And for the pulse points \( t = t_k \), we computing

\[ J_1(t_k, r) = \left\{ I_k \left( y_1(t_k, r) + \sum_{t_0 < t_s < t_k} f_1(t_s, r) \right) \right\} \]
\[ J_2(t_k, r) = \left\{ I_k \left( y_2(t_k, r) + \sum_{t_0 < t_s < t_k} f_2(t_s, r) \right) \right\} \]

Where \( s \in N \) and \( k = 1,...,m \).

Figure 1 shows the results are obtained by using numerical algorithm.

![Graph of numerical solution](image-url)
We note that impulsive operator $I_k$ influence on error too. Finally we have $Y_1(1, r) \approx y_1^{(2)}(1, r)$ and $Y_2(1, r) \approx y_2^{(2)}(1, r)$.

5 Conclusion

In this work, we have applied iterative solution of modified Euler’s method for numerical solution of first order impulsive fuzzy differential equations for the first time. This algorithm that introduced in this paper, is a new idea for solving large category of impulsive fuzzy initial value problem such as applying another numerical method instead of modified Euler’s method.

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