Toward the Use of Fuzzy Relations in the Definition of Mathematical Morphology Operators

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Abstract
In this paper we present a definition of erosions and dilations in terms of fuzzy relations and adjoint triples. We firstly show that we can represent any algebraic erosion and dilation in such a terms and secondly, we present a set of approaches that can be covered by our definition of relational erosions and dilations.

Keywords: Fuzzy Relation, Mathematical Morphology, Fuzzy Sets, Fuzzy Mathematical Morphology, Fuzzy Concept Analysis.

1 introduction

The origins of Mathematical Morphology (initiated by G. Matheron [15] and J. Serra [20, 21]), rely on set theory, integral geometry and linear algebra. The basic operators namely, erosion and dilation, modify sets in the Euclidean Plane by means of translations of specific sets, called structuring elements. In subsequent works, this “crisp” Mathematical Morphology was extended to more general environments in order to deal with greyscale or color images [3, 2, 8, 10, 7, 5]. This methodology is used in general contexts related to activities such as information extraction in digital images, noise elimination or pattern recognition.

In another (more theoretical) plane of the evolution of Mathematical Morphology, we can find some pure algebraic approaches [12, 14]. In these approaches, the framework is minimized to Complete Lattice Theory and erosions and dilations are defined as operators that commute with intersections and union, respectively. It is worth to say that Algebraic Mathematical Morphology generalizes the most extensions of Mathematical Morphology because such a properties required on erosions and dilations are preserved in the most of those approaches. The main advantage of this approach is the extrapolation of results to different extensions of Mathematical Morphology. On the other hand, it has an important shortcoming: the absence of a structuring element which changes the initial spirit of Mathematical Morphology. This shortcoming was removed in [14] by the use of clodums and impulse response functions. However, these two structures are seldom used in the community of computer science.

This paper deals with the problem of giving an internal structure back to algebraic erosions and dilations. Specifically, we have substituted the role of structuring elements by fuzzy relations and adjoint triples. This is not the first approach that considers fuzzy relations in the definition of erosions and dilations. For instance, [1] considers fuzzy relations to

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connect fuzzy mathematical morphology with fuzzy concept analysis [17] and [4] considers fuzzy relations to unify the notions of openings and closings from different approaches. The novelty of our approach is first, the consideration of two different ordered structures to define fuzzy relations and fuzzy sets and secondly, that we prove that the class of relational erosions (resp. relational dilations) coincides with the class of algebraic erosions (resp. algebraic dilations). The paper is structured as follows. In Section 2 we recall briefly the Fuzzy and Algebraic Mathematical Morphology. Then, in Section 3 we present the definition of relational erosions and dilations together with the representation theorem. In Section 4 we show how particular approaches can be covered by Fuzzy Relational Mathematical Morphology. Finally in Section 5 we present conclusions and future works.

2 A little summary of Mathematical Morphology

In this section we recall the basics of Mathematical Morphology. It is worth to mention that there exists different approaches of Mathematical Morphology which can be divided in four groups namely, the binary (or set based) [15, 20], the umbra approach (or grayscale) [20, 21], the fuzzy [2, 8] and the algebraic [12, 21]. For the lack of space and the sake of presentation, we briefly describe in this paper the algebraic and the fuzzy approach.

2.1 Algebraic Mathematical Morphology

The fundamentals of mathematical morphology is based on two basic operators: erosion and dilation. These operators were introduced originally on Euclidean Spaces by means of translations and joins of subsets [15, 20]. However, in subsequent approaches [12, 21], such definitions were extended to apply to arbitrary complete lattices in order to cover broader applications. We call this approach by Algebraic Mathematical Morphology. The definition of the operators erosions and dilations in this approach is given as follows.

Definition 2.1. Let \((L_1, \leq_1)\) and \((L_2, \leq_2)\) be two complete lattices. The mapping \(e : L_1 \to L_2\) is called an algebraic erosion if for all \(X \subseteq L_1\) we have:

\[
e(\bigwedge X) = \bigwedge_{x \in X} e(x)
\]

The mapping \(d : L_2 \to L_1\) is called an algebraic dilation if for all \(Y \subseteq L_2\) we have:

\[
d(\bigvee Y) = \bigvee_{y \in Y} d(y)
\]

So, roughly speaking, every algebraic erosion commutes with infimum and every algebraic dilation with supremum. Note that the definition above takes into account the case where \(X\) and \(Y\) are empty. That means that algebraic erosions assign the greatest element of \(L_1\) to the greatest element of \(L_2\) and algebraic dilations assign the least element of \(L_2\) to the least element of \(L_1\). In order to describe some relations between algebraic erosions and dilations is convenient to recall the notion of adjoint pair.

Definition 2.2. A pair \((\varepsilon, \delta)\) of mappings \(\varepsilon : L_1 \to L_2\) and \(\delta : L_2 \to L_1\) between complete lattices \((L_1, \leq_1)\) and \((L_2, \leq_2)\) is an adjoint pair if for every \(X \in L_1\) and \(Y \in L_2\) we have

\[
Y \leq \varepsilon(X) \quad \text{if and only if} \quad \delta(Y) \leq X
\]

The naming \(\varepsilon\) and \(\delta\) chosen in the previous definitions is not casual, since the operators introduced in Definition 2.1 can be related in terms of adjointness. The following two results (Theorems 2.1 and 2.2) were proven in [12], somehow rediscovering facts well-known in category theory. Firstly, we have the following result:

Theorem 2.1. If \((\varepsilon, \delta)\) is an adjoint pair. Then, \(\varepsilon\) is an algebraic erosion and \(\delta\) is an algebraic dilation.

On the other hand, the converse can be written in the following sense:
Theorem 2.2. Let \( e : L_1 \to L_2 \) be an algebraic erosion. Then, there exists exactly one algebraic dilation \( d_e : L_2 \to L_1 \) such that \((e, d_e)\) forms an adjoint pair. Specifically, such an algebraic dilation can be determined by the expression:

\[
\delta_e(Y) = \bigwedge \{ Z \in L_2 \mid Y \leq e(Z) \}
\]

for every \( Y \in L_2 \).

Similarly, for every algebraic dilation \( \delta : L_2 \to L_1 \) there exists exactly one algebraic erosion \( \varepsilon_\delta : L_1 \to L_2 \) such that \((\varepsilon_\delta, \delta)\) forms an adjoint pair. Moreover, such an algebraic erosion is determined by the expression:

\[
\varepsilon_\delta(X) = \bigvee \{ Z \in L_1 \mid \delta(Z) \leq X \}
\]

for every \( X \in L_1 \).

It is worth mentioning that Mathematical Morphology is not just about considering algebraic erosions and dilations; many other notions and operations are used as well, for instance *openings*, *closings* and *Hit-Miss transformations*, among others, are also object of study in this theory. But for the sake of simplicity, we restrict the introduction of preliminary notions to a minimum and will not go further in this section.

2.2 Fuzzy Mathematical Morphology

Note that the algebraical approach of Mathematical Morphology does not provide any internal structure in the operators erosion and dilation. This feature is not suitable for applied researches, which commonly make use of the structure established by Fuzzy Mathematical Morphology. Fuzzy dilations and erosions are defined between fuzzy sets on affine spaces (although this restriction can be reduced to groups) with a residuated lattice structure in the set of truth values. Let us recall that a complete residuated lattice is a 4-tuple \((L, \cdot, \cdot, \cdot)\) such that:

1. \((L, \cdot)\) is a complete bounded lattice, with top and bottom element 1 and 0, respectively.
2. \((L, \cdot, 1)\) is a commutative monoid with unit element 1.
3. \((\cdot, \cdot)\) forms and adjoint pair, i.e.

\[
z \leq (y \to x) \text{ if and only if } y \cdot z \leq x
\]

The definition of erosions and dilations in fuzzy mathematical morphology is given with the help of a *structuring elements*. Structuring elements are just fuzzy sets which determine “a shape” or “form” used to modify other fuzzy geometrical sets.

Definition 2.3. [3] Let \( A = \mathbb{R}^2 \) (or \( A = \mathbb{Z}^2 \)) and let \((L, \leq, \&\), \(\to\)) be a residuated lattice. The fuzzy erosion and dilation of \( X \in L^A \) by the structuring element \( S \in L^A \) are defined by:

\[
\varepsilon_S(X)(x) = \bigwedge \{ S(a-x) \to X(a) \mid a \in A \}
\]

and

\[
\delta_S(X)(x) = \bigvee \{ S(x-a)\&X(a) \mid a \in A \}
\]

respectively.

It is not difficult to check that fuzzy erosions and dilations generalize the original definition given for “crisp” sets [15]. For the lack of space, in this section we do not include examples of fuzzy erosions and dilations but the reader is referred to [8, 3] for some examples of applicability of this approach. The following result says that fuzzy erosions and dilations are algebraic erosions and dilation, respectively.

Theorem 2.3. Let \( e : L_1 \to L_2 \) be a fuzzy erosion (resp, a fuzzy dilation) then, \( e \) is an algebraic erosion (resp. an algebraic dilation).
In other words, the theorem above says that fuzzy erosion commutes with infimum and fuzzy dilation with supremum. The main advantage of this approach with respect to the algebraic one is that it provides a certain representation to the operators erosion and dilation. This fact simplifies considerably the definition of erosions and dilation for specific tasks. Perhaps this explains why Fuzzy Mathematical Morphology is more popular in applied researches than Algebraic Mathematical Morphology. However, it is appropriate to point out that this fuzzy approach is not able to represent the whole set of algebraic erosions and dilations. The reason is in the following assumptions required by the Fuzzy Mathematical Morphology approach:

1. Fuzzy sets must be defined on affine spaces. This restriction can be reduced to a group where the operator + is well-defined. Note that this restriction is irrelevant for image processing however, it is very strong in other applied fields.

2. The domain and codomain of fuzzy morphological operators must be the same lattice. This is due to two differentiable reasons:
   a. The use of fuzzy sets as structuring elements and the translation of them implies that the codomain of erosions has to coincide with the domain of dilations (an vice-versa). So fuzzy dilations transforms fuzzy sets to fuzzy sets in the same universe. By this restriction Fuzzy Mathematical Morphology becomes inapplicable to image compression or to other tasks where a modification of the universe is required (for instance as F-transforms does).
   b. The use of residuated lattices implies that the set of truth values (of fuzzy sets) in the domain and codomain, must be the same. This fact disallow to define fuzzy dilations between the set of color images and grayscale images, where the domain of the values are different despite they are defined on the same universe.

3 Fuzzy Relational Mathematical Morphology

Our goal in this section is to define a family of mathematical morphology operators based on fuzzy sets able to represent to whole set of algebraic erosions and dilations. Still inspired by the original definitions of dilations and erosions between crisp sets, we need a set of operators to model intersection, union and inclusion between fuzzy sets. For this purpose we consider the operators given by adjoint triples, which is a generalization of residuated lattices used in Fuzzy Mathematical Morphology. Note that this generalization allows us to use different set of truth values.

**Definition 3.1** ([18]). Let \((P_1, \leq_1), (P_2, \leq_2), (P_3, \leq_3)\) be three posets. We say that the mappings &: \(P_1 \times P_2 \rightarrow P_3, \setminus_\vee: P_2 \times P_3 \rightarrow P_1, \text{ and } /: P_1 \times P_3 \rightarrow P_2\) form an adjoint triple among \(P_1, P_2\) and \(P_3\) if:

\[
x \leq_1 y \setminus_\vee z \text{ if and only if } x \& y \leq_3 z \text{ if and only if } y \leq_2 x / z
\]

for all \(x \in P_1, y \in P_2\) and \(z \in P_3\).

Note that every adjoint pair in a residuated lattice is an adjoint triple where the operator & is commutative. Therefore Gödel, product and Łukasiewicz t-norms, together with their residuated implications, can be seen as examples of adjoint triples. But in contrast to adjoint pairs, adjoint triples allow us to consider different lattices, as the following example shows.

**Example 3.1.** Let \([0, 1]_m\) be a regular partition of \([0, 1]\) in \(m\) pieces, for example \([0, 1]_2 = \{0, 0.5, 1\}\) divides the unit interval in two pieces. Consider the discretization of the Gödel t-norm represented by the operator &\(^*_G\): \([0, 1]_{20} \times [0, 1]_{20} \rightarrow [0, 1]_{100}\) defined, for each \(x \in [0, 1]_{20}\) and \(y \in [0, 1]_{100}\), as:

\[
x \& y = \left\lfloor \frac{100 \cdot \min\{x, y\}}{100} \right\rfloor
\]

where \(\lfloor \cdot \rfloor\) is the ceiling function.

For this operator, the corresponding residuated implication operators \(\setminus_G: [0, 1]_{100} \times [0, 1]_{20} \rightarrow [0, 1]_{20}\) and /\(^*_G\): \([0, 1]_{20} \times [0, 1]_{100}\)
Now we can define the morphological operators dilation and erosion based on the underlying structure given by Definition 3.2. That is:

**Definition 3.2** with respect to Definition 2.3. For the sake of presentation, in the rest of the paper we assume the sets on different universes and with different set of truth values. This increases considerably the expressiveness of the concept and the relation used to define the corresponding erosion and dilation. Note moreover that relational structuring elements by fuzzy relations, thus we use also the terminology structural elements.

For all $X \in L_1^A$, and the Relational Fuzzy Dilation with respect to $R$, $\delta_R: L_1^B \rightarrow L_1^A$, as

$$\delta_R(Y)(a) = \bigvee_{b \in B} R(a,b) \& Y(b)$$

for all $Y \in L_2^B$.

Considering adjoint triples removes a restriction imposed by Fuzzy Mathematical Morphology concerning with the truth value of fuzzy sets (item (2b) at the end of Section 2.2). To avoid the others two restrictions, we substitute structuring elements by $P$-fuzzy relations. Let us recall that given a poset $P$, a $P$-fuzzy relation between to sets $A$ and $B$ is a mapping $R: A \times B \rightarrow P$. The value $R(a,b)$ represents the degree of the statement “$a$ is related with $b$ (by $R$)”.

**Theorem 3.1.** The pair $(\varepsilon_R, \delta_R)$ forms an adjoint pair.

**Corollary 3.1.** Relational Fuzzy Dilations and Relational Fuzzy Erosions are Algebraic Dilations and Algebraic Erosions, respectively.

As we say at the beginning of this section and in opposite to Fuzzy Mathematical Morphology, Fuzzy Relational Mathematical Morphology is able to represent the whole set of adjoint pairs, i.e., algebraic erosions and dilations. Thus, Definition 3.2 provides an structure of every algebraic erosion (resp, dilation)

**Theorem 3.2.** Let $\varepsilon: L_1^A \rightarrow L_1^B$ and $\delta: L_1^B \rightarrow L_1^A$ be two (algebraic) erosion and dilation respectively such that $(\varepsilon, \delta)$ is and adjoint pair. Then, there exists a $P$-fuzzy relation $R$ and an adjoint triple $(\lessdot_G, \&_G, \lor_G)$ between $P, L_2$ and $L_1$ such that $\varepsilon = \varepsilon_R$ and $\delta = \delta_R$. 

$$[0,1]^{100} \rightarrow [0,1]^8$$

are defined as:

$$a \lessdot_G b = \frac{[20 \cdot (a \lessdot G b)]}{20}$$

$$c \lor_G b = \frac{[8 \cdot (c \lor_G b)]}{8}$$

where $\lfloor \_ \rfloor$ is the floor function.

The tuple $(\&_G, \lessdot_G, \lor_G)$ is an adjoint triple; it is remarkable that the operator $\&_G$ is neither commutative nor associative, as this is not required in the definition of adjoint triple.
4 Approaches covered by Fuzzy Relational Mathematical Morphology

In this section we show some examples of relational mathematical morphology operators to illustrate firstly that is able to cover different approaches and that the use of the relation is not vain.

4.1 Fuzzy Mathematical Morphology as a specific case of Relational Mathematical Morphology.

It is easy to see that Fuzzy Mathematical Morphology is a specific case of Relational Fuzzy Mathematical Morphology. Specifically, consider a residuated lattice \((L, \leq, \&; \rightarrow)\) and \(A = \mathbb{R}^2\) (or \(A = \mathbb{Z}^2\)). Then the fuzzy erosion with respect to the structuring element \(S\) coincides with the relational fuzzy erosion with respect to the \(L\)-fuzzy relation \(R_S\) defined as:

\[
R_S: A \times A \rightarrow L \\
(x, y) \mapsto S(y - x)
\]

Therefore, every example given in \([8, 3, 5]\) can be considered also as particular cases of Fuzzy Relational Mathematical Morphology. It is convenient to mention that in mathematical morphology is common the use of “flat” structuring elements. Let us recall that a flat structuring element is a fuzzy set \(S\) such that \(S(x) \in \{0, 1\}\) for all \(x \in \mathbb{U}\). Note that \(\{0, 1\}\)-fuzzy relational mathematical morphology characterizes the morphological operators that can be defined from flat structuring elements.

4.2 F-Transforms as Fuzzy Relational Morphologic Operators.

F-Transforms were defined in \([19]\) with the aim of summarizing the information represented by a function into a discrete set of vectors. The core of this technique is to group the universe in a set of fuzzy sets, called the partition of the universe. Formally the definition of fuzzy partition is given as follows.

**Definition 4.1.** Let \(\mathbb{U}\) be a universe of discourse. A set of fuzzy sets \(A_1, \ldots, A_n\) is called a fuzzy partition of \(\mathbb{U}\) if for every \(u \in \mathbb{U}\) there exists \(i \in \{1, \ldots, n\}\) such that \(A_i(u) > 0\).

There are two phases of the F-transform namely, direct and inverse. Moreover, there are also two types of F-direct transforms denoted by \(F^+\) and \(F^\#\). To define F-transforms on complete lattices is needed also to fix a residuated lattice \((L, \leq, \&; \rightarrow)\).

**Definition 4.2.** Let \(f: \mathbb{U} \rightarrow L\) be a mapping and let \(A_1, \ldots, A_n\) be a fuzzy partition of \(\mathbb{U}\) then, the direct \(F^+\)-transform is the vector \(F^+_k(f) = (F^+_1, \ldots, F^+_n)\) defined component wise by

\[
F^+_k = \bigvee_{x \in \mathbb{U}} A_k(x) \& f(x),
\]

and the direct \(F^\#\)-transform is the vector \(F^\#_k(f) = (F^\#_1, \ldots, F^\#_n)\) defined component wise by

\[
F^\#_k = \bigwedge_{x \in \mathbb{U}} A_k(x) \rightarrow f(x)
\]

Note that both direct F-transforms are elements in \(L^n\). For each direct F-transform it is defined a respective inverse F-transform.

**Definition 4.3.** Let \(f: \mathbb{U} \rightarrow L\) be a mapping, let \(A_1, \ldots, A_n\) be a fuzzy partition of \(\mathbb{U}\) and let \(F^+\) and \(F^\#\) the respective direct F-transforms. Then, the inverse F-transforms \(f^+\) and \(f^\#\) are the mappings from \(\mathbb{U}\) to \(L\) defined by:

\[
f^+ = \bigwedge_{k=1}^n A_k(x) \rightarrow F^+_k
\]

and,

\[
f^\# = \bigvee_{k=1}^n A_k(x) \& F^\#_k
\]
These four operators introduced above can be considered as particular cases of Fuzzy Relational Morphology. Specifically, according with the notation used in Definitions 3.2 and 4.2, let us consider:

- as adjoint triple the residuated lattice \((L, \leq, \&), \rightarrow)\),
- \(A = \{1, \ldots, n\}\),
- \(B = \mathcal{U}\) and,
- the \(L\)-relation between \(\{1, \ldots, n\}\) and \(\mathcal{U}\) given by
  \[
  R_{F} : \{1, \ldots, n\} \times \mathcal{U} \rightarrow L, \\
  (k, x) \mapsto A_{k}(x).
  \]

Then, we have that \(\delta_{R}(f) = F^\dagger(f)\) and \(\varepsilon_{R} \circ \delta_{R}(f) = f^\dagger\). Reciprocally, if we consider

- as adjoint triple the residuated lattice \((L, \leq, \&), \rightarrow)\),
- \(A = \mathcal{U}\),
- \(B = \{1, \ldots, n\}\) and,
- the \(L\)-relation between \(\{1, \ldots, n\}\) and \(\mathcal{U}\) given by
  \[
  R_{F} : \mathcal{U} \times \{1, \ldots, n\} \rightarrow L, \\
  (x, k) \mapsto A_{k}(x).
  \]

Then, \(\varepsilon_{R}(f) = F^\dagger(f)\) and \(\delta_{R} \circ \varepsilon_{R}(f) = f^\dagger\). To summarize, the four operators defined on F-transforms can be considered special operators of Fuzzy Relational Mathematical Morphology.

### 4.3 Fuzzy Concept Analysis and Fuzzy Relational Mathematical Morphology

The main task of Fuzzy Concept Analysis is the representation via complete lattices of data in relational tables [1, 9, 13, 16, 18]. The construction of such complete lattice is given by using two operators called possibility and necessity. The basic structure of this theory (according to [18]) is the fuzzy property-oriented frame, which resembles the notion of adjoint triple.

**Definition 4.4.** Given two complete lattices \((L_{1}, \leq_{1})\) and \((L_{2}, \leq_{2})\), a poset \((P, \leq)\) and one adjoint triple with respect to \(P, L_{2}, L_{1}\), \((\&), \kappa\), a fuzzy property-oriented frame is the tuple

\[
(L_{1}, L_{2}, P, \leq_{1}, \leq_{2}, \&), \kappa,
\]

The notion of fuzzy property-oriented context is defined analogously to the one given in [18], and is given below.

**Definition 4.5.** Let \((L_{1}, L_{2}, \leq_{1}, \leq_{2}, P, \&), \kappa\) be a fuzzy property-oriented frame. A context is a tuple \((A, B, R)\) such that \(A\) and \(B\) are non-empty sets (usually interpreted as attributes and objects, respectively), \(R\) is a \(P\)-fuzzy relation \(R : A \times B \rightarrow P\).

Let us fix a fuzzy property-oriented frame \((L_{1}, L_{2}, \leq_{1}, \leq_{2}, P, \&), \kappa\) and a context \((A, B, R)\). The mappings \(g_{\uparrow \Pi} : L_{2}^{B} \rightarrow L_{1}^{A}\) and \(f_{\uparrow \Pi} : L_{1}^{A} \rightarrow L_{2}^{B}\), called possibility and necessity operators, respectively, are defined as follows

\[
g_{\uparrow \Pi}(a) = \bigvee_{b \in B} R(a, b) \& g(b), \\
\]

\[
f_{\uparrow \Pi}(b) = \bigwedge_{a \in A} f(a) \kappa R(a, b),
\]

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for \( g \in L^B_2 \) and \( f \in L^A_1 \).

These definitions generalize the classical possibility and necessity operators \([11]\), as well as F-transforms considered in the previous section, and are the basic operators of Fuzzy Concept Analysis. The reader can easily check that operators \( g^\cap \) and \( f^\cup \) are the dilation \( \delta_g \) and erosion \( \epsilon_g \) of the fuzzy relational mathematical morphology, respectively. Thus, \((^\cap, ^\cup)\) is an adjoint pair and, therefore, \( ^\cap : L^B_2 \to L^B_2 \) is a closure operator and \( ^\cup : L^A_1 \to L^A_1 \) is an interior operator.

In this environment, a concept is a pair of mappings \( \langle g, f \rangle \), with \( g \in L^B, f \in L^A \), such that \( g^\cap = f \) and \( f^\cup = g \) (formally called fuzzy property-oriented concept). The set of all these concepts forms a lattice with the ordering \( \langle g_1, f_1 \rangle \preceq \langle g_2, f_2 \rangle \) if and only if \( g_1 \leq g_2 \) (or equivalently \( f_2 \leq f_1 \)).

### 4.4 Reducing relational tables.

In the previous sections we have seen that Fuzzy Relational Mathematical Morphology cover at least three different frameworks namely, Fuzzy Mathematical Morphology, F-transforms and Fuzzy Concept Analysis. In this section we show that fuzzy relational erosions and dilations can also be used to other tasks. Below we show that fuzzy relational erosions and dilations can be used to reduce the size of relational tables. This kind of reduction are useful in Data Mining \([6]\), where the construction of a rule base system has an exponential complexity with respect to the size of the data. For the sake of the presentation, let us denote by \( \mathcal{R}(A, B) \) the set of L-fuzzy relational tables between \( A \) and \( B \). The reduction presented in this section transforms one relational table \( R \in \mathcal{R}(A, B) \) to another \( \overline{R} \in \mathcal{R}(A, \overline{B}) \) with \( |B| \geq |\overline{B}| \). Thus, the reduction just reduces the number of objects in the relational database. To proceed by fuzzy relational morphology operators, we need to define a structuring relation \( S \) between elements in \( B \) and \( \overline{B} \), i.e., between the original and new objects. Thus, fixed an adjoint pair \( (\& \rightarrow) \) on \( L \), we can define a dilation and a erosion between \( \mathcal{R}(A, B) \) and \( \mathcal{R}(A, \overline{B}) \) by

\[
\delta_S : \mathcal{R}(A, B) \to \mathcal{R}(A, \overline{B}) \quad R \mapsto \delta_{\text{dil}}(R)(a, \overline{b}) = \bigvee_{b \in B} S(b, \overline{b}) \& R(a, b)
\]

\[
\epsilon_S : \mathcal{R}(A, B) \to \mathcal{R}(A, \overline{B}) \quad R \mapsto \epsilon_{\text{er}}(R)(a, \overline{b}) = \bigwedge_{b \in B} S(b, \overline{b}) \to R(a, b)
\]

Note that, according to Definition 3.2, the structuring relation used in \( \delta_{\text{dil}} \) is the dual relation of \( S \), i.e., \( S^\text{op} \) defined by \( S^\text{op}(\overline{b}, b) = S(b, \overline{b}) \). But for the sake of presentation, in the formula above we have omitted \( S^\text{op} \) and we have written directly the expression of \( \delta_{\text{dil}} \) in terms of \( S \).

The role of the \( L \)-fuzzy relation \( S \) is to group elements of the original set of objects \( B \) into elements of another (smaller in size) set \( \overline{B} \). This grouping can be done in many different ways, for instance by clustering, by an expert decision or by taking information of the original relational table. The following example shows a way to define the \( L \)-fuzzy relation \( S \) for a specific case.

**Example 4.1.** Let us consider a discretization of the unit interval \([0, 1]\) in 6 elements, i.e., \( L = \{0, 0.2, 0.4, 0.6, 0.8, 1\} \) and the \( L \)-relational table below puts into the correspondence types of cars (objects) with features (attributes).
Let us assume that the only information we have is the one provided by the table above. Then, a possible way of grouping objects is by focusing on one attribute and to group objects according to the value of a such attribute. For instance, we can create two groups of objects according to a high value of its attribute $b$ and those with a low value of the same attribute $b$. So, to represent what a high and a low value means, we consider the following two $L$-fuzzy sets on $L$.

<table>
<thead>
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<th>HighPower</th>
<th>BigSpace</th>
<th>HighConsume</th>
<th>Expensive</th>
<th>Sport</th>
<th>Familiar</th>
</tr>
</thead>
<tbody>
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<td>0.2</td>
<td>1</td>
<td>0.8</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>1</td>
<td>1</td>
<td>0.8</td>
<td>1</td>
<td>0.6</td>
</tr>
<tr>
<td>$c$</td>
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<td>0.8</td>
<td>0.4</td>
<td>0.6</td>
<td>0.2</td>
</tr>
<tr>
<td>$d$</td>
<td>0.8</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.2</td>
</tr>
<tr>
<td>$e$</td>
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<td>0.2</td>
<td>0.6</td>
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</tr>
<tr>
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<td>0.8</td>
<td>0.2</td>
<td>0</td>
</tr>
<tr>
<td>$g$</td>
<td>0.8</td>
<td>0.2</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>$h$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$i$</td>
<td>0.6</td>
<td>1</td>
<td>0.4</td>
<td>0.6</td>
<td>0</td>
</tr>
<tr>
<td>$j$</td>
<td>0.6</td>
<td>1</td>
<td>0.6</td>
<td>0.6</td>
<td>0</td>
</tr>
<tr>
<td>$k$</td>
<td>0.6</td>
<td>0.6</td>
<td>0.4</td>
<td>0.6</td>
<td>0</td>
</tr>
<tr>
<td>$l$</td>
<td>0.2</td>
<td>0.4</td>
<td>0.2</td>
<td>0.2</td>
<td>0</td>
</tr>
<tr>
<td>$m$</td>
<td>0.8</td>
<td>0</td>
<td>0.8</td>
<td>1</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Note that $L_1$ represent high values and $L_2$ low values. Note as well that we could consider three fuzzy sets to represent high, medium and low values. Now let us consider the attribute Sport and let us group cars into be sport or not. Obviously, one car $b \in B$ is highly sport if and only if the value $R(b, Sport)$ is high, i.e., $b$ is sport in degree $L_1(R(b, Sport))$. Conversely, one car $b \in B$ is non sport in degree $L_2(R(b, Sport))$. Thus we can consider the structuring relation $S$ that relates cars with their degree of sport, that is

<table>
<thead>
<tr>
<th>$x$</th>
<th>$L_1(x)$</th>
<th>$x$</th>
<th>$L_2(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Note that as we are considering the relational table above as structuring relation, the set of new object is $\overline{B}$ =

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\text{SportCar}(x)$</th>
<th>$\text{NonSportCar}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$b$</td>
<td>0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>$c$</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>$d$</td>
<td>0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>$e$</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>$f$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$g$</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>$h$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$i$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$j$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$k$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$l$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$m$</td>
<td>0.8</td>
<td>0.2</td>
</tr>
</tbody>
</table>
So, the dilation $\delta(R)$ and the erosion $\varepsilon(R)$ of the relational tables $R$ defined by using the Gödel connectives ($\&_G$, $\rightarrow_G$) are

<table>
<thead>
<tr>
<th></th>
<th>HighPower</th>
<th>BigSpace</th>
<th>HighConsume</th>
<th>Expensive</th>
<th>Familiar</th>
</tr>
</thead>
<tbody>
<tr>
<td>SportCars</td>
<td>1</td>
<td>0.6</td>
<td>1</td>
<td>1</td>
<td>0.6</td>
</tr>
<tr>
<td>NonSportCars</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

and

<table>
<thead>
<tr>
<th></th>
<th>HighPower</th>
<th>BigSpace</th>
<th>HighConsume</th>
<th>Expensive</th>
<th>Familiar</th>
</tr>
</thead>
<tbody>
<tr>
<td>SportCars</td>
<td>1</td>
<td>0.2</td>
<td>0.8</td>
<td>0</td>
<td>0.8</td>
</tr>
<tr>
<td>NonSportCars</td>
<td>0</td>
<td>0.2</td>
<td>0</td>
<td>0.2</td>
<td>0</td>
</tr>
</tbody>
</table>

The above relational tables can be interpreted as follows. The dilation $\delta(R)$ represents somehow the possibility of a car to be sport (in some degree) and to have another attributes. That is, $\delta(R)(a,\text{SportCar})$ represents an upper bound of the value $R(a,x) \& \text{SportCar}(x)$ for a car $x \in B$. Reciprocally, the erosion $\varepsilon(R)$ represents somehow the necessity of a car that is sport (in some degree) to have another attribute. In other words, $\varepsilon(R)(a,\text{SportCar})$ determines a lower bound of $\text{SportCar}(x) \rightarrow R(a,x)$ for a sport car $x \in B$. Similarly for non sport cars. So from the interpretability above, we can infer from tables $\delta(R)$ and $\varepsilon(R)$ that a sport car must have a high Power (because $\varepsilon(R)(\text{SportCar,HighPower}) = \varepsilon(R)(\text{SportCar,HighPower}) = 1$), must have a quite High Consume of oil and must be quite expensive ($\varepsilon(R)(\text{SportCar,HighConsume}) = 0.8$ and $\varepsilon(R)(\text{SportCar,Expensive}) = 0.8$). Moreover, a sport car is not in fully contradictory with being familiar as well, since $\delta(R)(\text{SportCar,Familiar}) = 0.6$. On the other hand, such tables show that there is no pattern for a non sport car according to the parameters HighPower, BigSpace, HighConsume and Expensive, as the respective parameters vary from 0 (or 0.2) in Table $\varepsilon(R)$ to 1 in Table $\delta(R)$.

5 Conclusions and Future works.

In this paper we have presented a definition of erosions and dilations in terms of fuzzy relations and adjoint triples. We have seen that every algebraic erosion and dilation can be defined in terms of relational erosions and dilations, respectively. Moreover, we have shown that those fuzzy relational morphology operators generalize operators used in different approaches like F-transforms, Fuzzy Concept Lattices or Fuzzy Mathematical Morphology. Finally we have illustrated with an example that the notion of relational erosion and dilation can be used to define novel operators.

As future work we want to research the invariance with respect to different transformations in Fuzzy Relational Mathematical Morphology like it is done in Mathematical Morphology with respect to translations.
http://dx.doi.org/10.1007/s00500-002-0208-4


http://dx.doi.org/10.1016/j.ins.2007.08.011

http://dx.doi.org/10.1117/1.482677

http://dx.doi.org/10.1007/978-1-4615-5473-8_4

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http://dx.doi.org/10.1016/j.fss.2005.11.012
