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Research Article

Fuzzy Itô Integral Driven by a Fuzzy Brownian Motion

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Abstract
In this paper we take into account the fuzzy stochastic integral driven by fuzzy Brownian motion. To define the metric between two fuzzy numbers and to take into account the limit of a sequence of fuzzy numbers, we invoke the Hausdorff metric. First this fuzzy stochastic integral is constructed for fuzzy simple stochastic functions, then the construction is done for fuzzy stochastic integrable functions.

Keywords: Fuzzy random variable, Fuzzy stochastic process, Fuzzy Brownian Motion, Fuzzy function, Fuzzy random function, Fuzzy Itô Integral.

1 Introduction

The notions of Fuzzy Stochastic Integral with respect to a crisp Brownian Motion have been introduced by Kim and Kim in [7] and since then they were successfully used in the topic of set valued fuzzy differential equations [8, 11, 15, 4]. In [12] Malinowski and Michta exploit the properties of set-valued stochastic trajectory integrals to define a notion of fuzzy stochastic Lebesgue-Stieltjes trajectory integral. Meanwhile they consider a notion of fuzzy stochastic trajectory integral with respect to martingales (crisp martingales). The notion of Fuzzy Stochastic Itô integral given in[12] allowed the authors of the paper to define stochastic fuzzy differential equations driven by Brownian Motion. Some other results containing stochastic fuzzy Integral have been published [13, 14, 5].
In this paper, we define the stochastic Integral of a fuzzy process [20, 17, 21] with respect to a fuzzy Brownian Motion [6, 10] and give its properties. The fuzzy Itô Integral with respect to a fuzzy Brownian motion is a natural tool in the study of the theory of fuzzy stochastic differential equations driven by a fuzzy Brownian Motion.
The paper is organized as follows. In section 2, we give some preliminaries on fuzzy random variables and fuzzy Brownian Motions. In section 3, we introduce the new version of the notion of fuzzy stochastic integral with respect to a fuzzy Brownian Motion, for the first time, by using simple fuzzy functions. The more general case of measurable fuzzy functions is also considered.

2 Preliminaries and notations

We give different definitions and elementary concepts of fuzzy arithmetic, fuzzy random variables and fuzzy Brownian motions that will be used in the next section. The reader is referred to [19, 9, 18, 1, 22, 23] for more details.

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Definition 2.1. *Let $X = \mathbb{R}$ be a Universal set, we define*

$$\mathscr{F}(\mathbb{R}) = \{\tilde{u} : \mathbb{R} \longmapsto [0,1] : \tilde{u}\, satisfies\, (i)\, to\, (iii)\},$$

where (i) $\tilde{u}$ is normal,(ii) $\tilde{u}$ is convex and (iii) $\tilde{u}$ is upper semi continuous. We call a fuzzy number, any $\tilde{u} \in \mathscr{F}(\mathbb{R})$.

Definition 2.2. *(α-level set)*
Let $\tilde{u} \in \mathscr{F}(\mathbb{R})$ be a fuzzy number, we call a α-level set of $\tilde{u}$, for every $\alpha \in (0,1]$, the set

$$[\tilde{u}]_\alpha = \{x \in \mathbb{R} : \tilde{u}(x) \geq \alpha, \alpha \in (0,1]\} = \left[\tilde{u}^L_\alpha, \tilde{u}^U_\alpha\right], \tag{2.1}$$

where $[\tilde{u}]^L_\alpha = \inf_{x\in\mathbb{R}} \{x \in [\tilde{u}]_\alpha\}$ and $[\tilde{u}]^U_\alpha = \sup_{x\in\mathbb{R}} \{x \in [\tilde{u}]_\alpha\}$.
The support set of $\tilde{u}$ is given by $[\tilde{u}]_{0_+} = Supp\,\tilde{u} = Cl\,\{x \in \mathbb{R} : \tilde{u}(x) > 0\}$.

Definition 2.3. *(Metric of Hausdorff) Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ be two intervals, the Hausdorff's metric between them is defined as follows:*

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\right\}. \tag{2.2}$$

According to Puri and Ralescu [17] one can define a metric on $\mathscr{F}(\mathbb{R})$ as follows:

Definition 2.4. *(Metric on $\mathscr{F}(\mathbb{R})$) Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers, a metric between $\tilde{a}$ and $\tilde{b}$ is given by*

$$d_{\mathscr{F}} = \sup_{\alpha\in(0,1]} d_H(\tilde{a}_\alpha, \tilde{b}_\alpha). \tag{2.3}$$

Theorem 2.1. *[2] Let $\widetilde{F}$ and $\widetilde{G}$ be two fuzzy numbers. Then $\forall\alpha \in [0,1]$ we have*

$$d_H\left(\widetilde{F}_\alpha, \widetilde{G}_\alpha\right) = \max\left\{\left|\widetilde{F}^L_\alpha - \widetilde{G}^L_\alpha\right|, \left|\widetilde{F}^U_\alpha - \widetilde{G}^U_\alpha\right|\right\}. \tag{2.4}$$

A fuzzy number $\tilde{a}$ of $\mathbb{R}$ can be also defined by its memberships function $\mu_{\tilde{a}} : \mathbb{R} \longmapsto [0,1]$. Between two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ we define by "'$\odot$"'any binary operation $\oplus$,$\ominus$ or $\otimes$ such that[3]

$$\mu_{\tilde{a}\odot\tilde{b}}(z) = \sup_{x\circ y=z} min\left\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\right\}, \tag{2.5}$$

for $\odot = \oplus, \ominus$ or $\otimes$ and $\circ = +, -$ or $\times$.
Between two closed intervals $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ and $\widetilde{b}_\alpha = [\tilde{b}^L_\alpha, \tilde{b}^U_\alpha]$ we define by "'$\odot_{int}$"'any binary operation $\oplus_{int}$, $\ominus_{int}$ or $\otimes_{int}$ such that[3]

$$\widetilde{a}_\alpha \odot_{int} \widetilde{b}_\alpha = \left\{z \in \mathbb{R} | z = x \circ y, \exists x \in \tilde{a}_\alpha, \exists y \in \tilde{b}_\alpha\right\}. \tag{2.6}$$

Definition 2.5. *(Hukuhara difference of fuzzy numbers, [19]) Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers, the Hukuhara difference or H-difference is defined by $\tilde{a} \ominus_H \tilde{b} = \tilde{c}$ if and only if $\tilde{a} = \tilde{b} \oplus_H \tilde{c}$ exists, it is unique and its α-levels are $\left[\tilde{a} \ominus_H \tilde{b}\right]_\alpha = \left[\tilde{a}^L_\alpha - \tilde{b}^L_\alpha, \tilde{a}^U_\alpha - \tilde{b}^U_\alpha\right]$.*

Clearly, $\tilde{a} \ominus_H \tilde{a} = \{0\}$.

Definition 2.6. *(Interval-valued stochastic process, GUO [6])*
A random interval-valued stochastic process $\{\bar{X}(t), t \in [0,T], 0 < T < \infty\}$ on probability space $(\Omega, \mathscr{A}, P)$ is the mapping from Ω to $L(\mathbb{R}) \subset K(\mathbb{R})$

$$\begin{aligned} \bar{X}_t : \Omega &\longrightarrow K(\mathbb{R}) \\ \omega &\longrightarrow \bar{X}_t = [\bar{X}^L_t, \bar{X}^U_t], \end{aligned} \tag{2.7}$$

for all $t \in [0,T]$ uniformly, where

$$L(\mathbb{R}) = \left\{[\bar{a}^L, \bar{a}^U], \bar{a}^L \leq \bar{a}^U, \bar{a}^L, \bar{a}^U \in \mathbb{R}\right\}. \tag{2.8}$$

Definition 2.7. *(Interval-valued Gaussian process, GUO [6])*
A family $\{\bar{X}(t), t \in [0,T], 0 < T < \infty\}$ is an interval-valued Gaussian process on probability space $(\Omega, \mathscr{A}, P)$ if and only if

$$\bar{X}(t,\omega) = [\bar{X}^L(t,\omega), \bar{X}^U(t,\omega)], \tag{2.9}$$

where both $\bar{X}^L(t,\omega)$ and $\bar{X}^U(t,\omega)$ are Gaussian processes such that

$$P[\bar{X}^L(t,\omega) \leq \bar{X}^U(t,\omega)] = 1. \tag{2.10}$$

Definition 2.8. *(Interval-valued Brownian motion, GUO [6])*
A family $\{\bar{B}(t), t \in [0,T], 0 < T < \infty\}$ is an interval-valued Brownian motion on probability space $(\Omega, \mathscr{A}, P)$ if and only if

$$\bar{B}(t,\omega) = [\bar{B}^L(t,\omega), \bar{B}^U(t,\omega)], \tag{2.11}$$

where both $\bar{B}^L(t,\omega)$ and $\bar{B}^U(t,\omega)$ are Brownian motions such that

$$P[\bar{B}^L(t,\omega) \leq \bar{B}^U(t,\omega)] = 1. \tag{2.12}$$

Definition 2.9. *(Fuzzy stochastic process, GUO [6])*
A fuzzy family $\left\{\tilde{X}(t), t \in [0,T], 0 < T < \infty\right\}$ is called a fuzzy stochastic process on probability space $(\Omega, \mathscr{A}, P)$ if and only if for all $\alpha \in [0,1]$, the process

$$\tilde{X}_\alpha(t,\omega) = [\tilde{X}^L_\alpha(t,\omega), \tilde{X}^U_\alpha(t,\omega)], \tag{2.13}$$

is an interval-stochastic process on $(\Omega, \mathscr{A}, P)$ and

$$\tilde{X}(t,\omega) = \bigcup_{\alpha\in[0,1]} \tilde{X}_\alpha(t,\omega). \tag{2.14}$$

Definition 2.10. *(Fuzzy Gaussian process, GUO [6])*
A fuzzy stochastic process $\left\{\tilde{X}(t), t \in [0,T], 0 < T < \infty\right\}$ is a fuzzy Gaussian process on probability space $(\Omega, \mathscr{A}, P)$ if and only if for all $\alpha \in [0,1]$, the family $\left\{\tilde{X}_\alpha(t), t \in [0,T], 0 < T < \infty\right\}$ is an interval-valued Gaussian process.

Definition 2.11. *(Fuzzy Brownian Motion, GUO [6])*
A fuzzy stochastic process $\left\{\tilde{B}(t), t \in [0,T], 0 < T < \infty\right\}$ is a fuzzy Brownian Motion on probability space $(\Omega, \mathscr{A}, P)$ if and only if for all $\alpha \in [0,1]$, the process

$$\tilde{B}_\alpha(t,\omega) = [\tilde{B}^L_\alpha(t,\omega), \tilde{B}^U_\alpha(t,\omega)], \tag{2.15}$$

is an interval-Brownian Motion on $(\Omega, \mathscr{A}, P)$ and

$$\tilde{B}(t,\omega) = \bigcup_{\alpha\in[0,1]} \tilde{B}_\alpha(t,\omega). \tag{2.16}$$

Remark 2.1. **(i)** *For all $t \in [0, T]$ and $\omega \in \Omega$, $\tilde{B}(t,\omega)$ is a fuzzy number.*

(ii) *A nonnegative fuzzy stochastic process $\left\{\tilde{B}(t), t \in [0,T]\right\}$ is a nonnegative fuzzy Brownian motion if and only if $\left\{\tilde{B}^i_\alpha(t,\omega), t \in [0, T]\right\}$ is a non negative Brownian Motion where $i \in \{L,U\}$ and $\alpha \in [0, 1]$.*

Theorem 2.2. *(Shoumei [10])*
Let $\left\{\tilde{B}(t), t \geq 0\right\}$ be a fuzzy stochastic process such that $\tilde{B}(t) = \tilde{0}$. Then $\left\{\tilde{B}(t,\omega), t \geq 0\right\}$ is a fuzzy Brownian motion if and only if it is Gaussian and for all $\alpha \in [0,1]$

(i) $E\left[\tilde{B}^i_\alpha(t,\omega)\right] = 0$ *for all* $t \geq 0, i \in \{L,U\}$,

(ii) $E\left[\tilde{B}^i_\alpha(t,\omega).\tilde{B}^i_\alpha(s,\omega)\right] = t \wedge s$ *for all* $s,t \geq 0, i \in \{L,U\}$,

(iii) $E\left[\tilde{B}^i_\alpha(t,\omega).\tilde{B}^j_\alpha(s,\omega)\right] = 0$ *for all* $s,t \geq 0, i,j \in \{L,U\}$ *and* $i \neq j$.

Theorem 2.3. *(Shoumei [10])*
Let $\tilde{B}(t) = \tilde{0}$, $\left\{\tilde{B}(t), t \geq 0\right\}$ be a fuzzy Brownian motion. Then we have

(i) $\left\{\tilde{B}(t+t_0), t \geq 0\right\}$ *is a fuzzy Brownian motion for all* $t_0 \geq 0$*,*

(ii) $\left\{\tilde{v} \oplus \tilde{B}(t), t \geq 0\right\}$ *is a fuzzy Brownian motion for all* $\tilde{v} \in \mathscr{F}_c(\mathbb{R})$*,*

(iii) $\left\{\frac{1}{\sqrt{\lambda}} \otimes \tilde{B}(\lambda t), t \geq 0\right\}$ *is a fuzzy Brownian motion for all* $\lambda > 0$*,*

(iv) $\left\{t \otimes \tilde{B}(\frac{1}{t}), t \geq 0\right\}$ *is a fuzzy Brownian motion .*

Definition 2.12. *(Fuzzy Martingale, PURI [17])*
The sequence $\left\{\tilde{X}_n, \mathscr{F}_n\right\}_n$ of fuzzy random variables and σ-algebra is a fuzzy martingale if for each $n \geq 1$:

(i) $\tilde{X}_n$ *is $\tilde{\mathscr{F}}_n$-measurable and* $E\left\|Supp\tilde{X}_n\right\| < \infty$*,*

(ii) $E(\tilde{X}_{n+1} \mid \mathscr{F}_n) = \tilde{X}_n$.

If property (ii) is replaced by

(ii') $E(\tilde{X}_{n+1} \mid \mathscr{F}_n) \geq \tilde{X}_n$ $(E(\tilde{X}_{n+1} \mid \mathscr{F}_n) \leq \tilde{X}_n)$.

Then $\left\{\tilde{X}_n, \mathscr{F}_n\right\}_n$ is called a fuzzy sub-martingale (surmartingale), respectively.

Theorem 2.4. *(Shoumei [10])*
let $\mathscr{F}_t = \sigma(\tilde{B}(s) : s \leq t)$ and $\left\{\tilde{B}(t) : t \geq 0\right\}$ is a fuzzy Brownian motion. Then $\left\{\tilde{B}(t), \mathscr{F}_t : t \geq 0\right\}$ is a fuzzy martingale.

3 Fuzzy Itô Integral

In this section we define a fuzzy counterpart of stochastic Integral or Itô Integral. The fuzzy stochastic Integral considered here is driven by a fuzzy Brownian motion.

Definition 3.1. *(Fuzzy Simple function)*
A fuzzy simple function is a fuzzy function of the form

$$\tilde{\Phi}(t,\omega) = \bigoplus_{j\geq 0} \tilde{E}_j(\omega) \otimes 1_{[j2^{-n},(j+1)2^{-n}]}(t) \tag{3.17}$$

where 1_A denotes the indicator function , n is a natural number and $\tilde{E}_j(\omega)$ a fuzzy number.

We define the fuzzy Itô integral of a fuzzy simple function driven by a fuzzy Brownian motion as follows.

Definition 3.2. *The fuzzy Itô integral of a fuzzy simple function defined by $\tilde{\Phi}(t,\omega)$ with respect to a fuzzy Brownian motion is defined on $[S,T]$ by*

$$\int_S^T \tilde{\Phi}(t,\omega) d\tilde{B}(t,\omega) = \bigoplus_{j\geq 0} \tilde{E}_j(\omega) \otimes [\tilde{B}(t_{j+1},\omega) \ominus_H \tilde{B}(t_j,\omega)], \tag{3.18}$$

where $t_k = t_k^{(n)} = \begin{cases} k2^{-n} & \text{if } S \leq k2^{-n} \leq T \\ S & \text{if } k2^{-n} < S \\ T & \text{if } k2^{-n} > T \end{cases}$

Remark 3.1. *In (3.18), $\tilde{B}(t_j)$ and $\tilde{E}_j$ are fuzzy numbers for all $j \geq 0$. By stability of $\oplus, \otimes$ and $\ominus_H$ in $\mathscr{F}_c(\mathbb{R})$, the fuzzy Itô Integral of fuzzy simple function with respect to a fuzzy Brownian Motion is a fuzzy number.*

By the definition of α-level set of fuzzy number (3.17) we can have:

$$\tilde{\Phi}^L_\alpha(t,\omega)=\sum_{j\geq 0}(\tilde{E}_j(\omega))^L_\alpha 1_{[j2^{-n},(j+1)2^{-n}]}(t) \tag{3.19}$$

and

$$\tilde{\Phi}^U_\alpha(t,\omega)=\sum_{j\geq 0}(\tilde{E}_j(\omega))^U_\alpha 1_{[j2^{-n},(j+1)2^{-n}]}(t). \tag{3.20}$$

Theorem 3.1. *Let $\tilde{B}_t$ be a fuzzy Brownian Motion, the following assertions are trues:*

(i) *If $\tilde{\Phi}(t,\omega)$ is a non negative fuzzy simple function, then*

$$(\int_S^T \tilde{\Phi}(t,\omega)d\tilde{B}(t,\omega))_\alpha=\left[\int_S^T \tilde{\Phi}^L_\alpha(t,\omega)d\tilde{B}^L_\alpha(t,\omega),\int_S^T \tilde{\Phi}^U_\alpha(t,\omega)d\tilde{B}^U_\alpha(t,\omega)\right], \tag{3.21}$$

(ii) *If $\tilde{\Phi}(t,\omega)$ is a non positive fuzzy simple function, then*

$$(\int_S^T \tilde{\Phi}(t,\omega)d\tilde{B}(t,\omega))_\alpha=\left[\int_S^T \tilde{\Phi}^L_\alpha(t,\omega)d\tilde{B}^U_\alpha(t,\omega),\int_S^T \tilde{\Phi}^U_\alpha(t,\omega)d\tilde{B}^L_\alpha(t,\omega)\right]. \tag{3.22}$$

Proof. The proof follows from Theorem 4.1 in Wu [2]. Indeed, let's show (ii).

$$\begin{aligned}\left(\int_S^T \tilde{\Phi}(t,\omega)d\tilde{B}(t,\omega)\right)^U_\alpha &=\left(\bigoplus_{j\geq 0}\tilde{E}_j(\omega)\otimes[\tilde{B}(t_{j+1},\omega)\ominus_H \tilde{B}(t_j,\omega)]\right)^U_\alpha\\ &=\sum_{j\geq 0}\left(\tilde{E}_j(\omega)\otimes[\tilde{B}(t_{j+1},\omega)\ominus_H \tilde{B}(t_j,\omega)]\right)^U_\alpha\\ &=\sum_{j\geq 0}\left(\tilde{E}_j(\omega)\right)^U_\alpha[\tilde{B}(t_{j+1},\omega)\ominus_H \tilde{B}(t_j,\omega)]^L_\alpha\\ &=\sum_{j\geq 0}\left(\tilde{E}_j(\omega)\right)^U_\alpha[\tilde{B}^L_\alpha(t_{j+1},\omega)-\tilde{B}^L_\alpha(t_j,\omega)]\\ &=\int_S^T \tilde{\Phi}^U_\alpha(t,\omega)d\tilde{B}^L_\alpha(t,\omega).\end{aligned}$$

Similarly, for the lower bound case we have:

$$\left(\int_S^T \tilde{\Phi}(t,\omega)d\tilde{B}(t,\omega)\right)^L_\alpha=\int_S^T \tilde{\Phi}^L_\alpha(t,\omega)d\tilde{B}^U_\alpha(t,\omega).$$

□

Corollary 3.1. *Suppose that $\tilde{\Phi}(t,\omega)$ is a simple positive or negative fuzzy function integrable with respect to a fuzzy Brownian motion. Then the followings assertions are true.*

(i) *If $\tilde{\Phi}(t,\omega)$ is positive then $\int_S^T \tilde{\Phi}(t,\omega)d\tilde{B}(t,\omega)$ is a positive fuzzy number.*

(ii) *If $\tilde{\Phi}(t,\omega)$ is negative then $\int_S^T \tilde{\Phi}(t,\omega)d\tilde{B}(t,\omega)$ is a negative fuzzy number.*

Proof. The result follows from Definition 3.2 and Remark 3.1 immediately. □

Definition 3.3. *Let $V_F=V_F(S,T)$ be the class of all fuzzy functions*

$$\tilde{F}(t,\omega):[0,\infty]\times\Omega\longmapsto\mathscr{F}_c(\mathbb{R}),$$

such that:

(i) $(t,\omega) \longmapsto \tilde{F}(t,\omega)$*is* $\mathscr{B}\times\mathscr{F}_t$*- measurable, where* $\mathscr{B}$ *denotes the Borel* σ*-algebra on* $[0,\infty)$*,*

(ii) $\tilde{F}(t,\omega)$ *is* $\mathscr{F}_t$*-adapted,*

(iii) $E\left[\int_S^T \left|\left|\tilde{F}(t,\omega\right|\right|^2 dt\right] < \infty$.

Where $\left|\left|\tilde{F}(t,\omega)\right|\right| = \sup_{\alpha\in(0,1]} d_H\left(\tilde{F}_\alpha(t,\omega),\tilde{0}_\alpha\right)$

Theorem 3.2. *Let* $\tilde{F}\in V_F(S,T)$*, its fuzzy Itô integral is defined by*

$$I[\tilde{F}](\omega) = \int_S^T \tilde{F}(t,\omega)d\tilde{B}(t,\omega). \tag{3.23}$$

Proof. To prove this identity, we first consider the fuzzy Itô integral of a fuzzy simple function as in definition 3.2 and show that $\tilde{F}$ can be approximated by a sequence $(\tilde{\Phi}_n)_{n\geq 1}$ of such fuzzy simple function as follows

$$\int_S^T \tilde{F}(t,\omega)d\tilde{B}(t,\omega) = \lim_{n\to\infty}\int_S^T \tilde{\Phi}_n(t,\omega)d\tilde{B}(t,\omega). \tag{3.24}$$

□

In the following Lemma, we give the property of isometry for the fuzzy Itô integral.

Lemma 3.1. *(Itô isometry)*
Let $\tilde{\Phi}(t,\omega)$ *be a bounded fuzzy simple function, then*

$$E[(\int_S^T \tilde{\Phi}(t,\omega)d\tilde{B}(t,\omega))^2] = E[\int_S^T \tilde{\Phi}(t,\omega)^2 dt]. \tag{3.25}$$

Proof. We must show that for each $\alpha\in(0,1]$,

$$[E[(\int_S^T \tilde{\Phi}(t,\omega)d\tilde{B}(t,\omega))^2]]_\alpha = [E[\int_S^T \tilde{\Phi}(t,\omega)^2 dt]]_\alpha.$$

We first consider the case $\tilde{\Phi}(t,\omega)\succeq\tilde{0}$ for each $t\in[0,\infty[$ and $\omega\in\Omega$.
By the definition of the expectation, the Theorem 3.1 and the classical Itô isometry we have that:
$[E[(\int_S^T \tilde{\Phi}(t,\omega)d\tilde{B}(t,\omega))^2]]_\alpha$

$$\begin{aligned}
&= [E[(\int_S^T \tilde{\Phi}^L_\alpha(t,\omega)d\tilde{B}^L_\alpha(t,\omega))^2],(\int_S^T \tilde{\Phi}^U_\alpha(t,\omega)d\tilde{B}^U_\alpha(t,\omega))^2]\\
&= [E[(\int_S^T \tilde{\Phi}^L_\alpha(t,\omega)d\tilde{B}^L_\alpha(t,\omega))^2],E[\int_S^T \tilde{\Phi}^L_\alpha(t,\omega)d\tilde{B}^U_\alpha(t,\omega))^2]]\\
&= [E[(\int_S^T \tilde{\Phi}^L_\alpha(t,\omega)^2 dt],E[\int_S^T \tilde{\Phi}^U_\alpha(t,\omega)^2 dt]]\\
&= [E[\int_S^T \tilde{\Phi}(t,\omega)^2 dt]]_\alpha.
\end{aligned}$$

□

Similarly we can show the result for a non positive simple fuzzy function.
In the sequel, we shall use this Itô isometry property to define the fuzzy Itô integral of $V_F(S,T)$-fuzzy functions. We first prove some useful results.

Definition 3.4. *A fuzzy function* $\tilde{G}:\mathbb{R}\longmapsto\mathscr{F}_c(\mathbb{R})$ *is bounded if and only if for every* $\alpha\in(0,1]$ *and* $i\in\{L,U\}$*,* $\tilde{G}^i_\alpha$ *is a real bounded fuzzy function.*

Theorem 3.3. *Let $\tilde{G} \in V_F(S,T)$ be a bounded fuzzy function and let $\tilde{G}(\cdot,\omega)$ be continuous for each ω. There exists a sequence of fuzzy simple functions $\tilde{\Phi}_n$ such that*

$$\lim_{n\to\infty} E\int_S^T \left|\left|\left|\tilde{G}(t,\omega) \ominus_H \tilde{\Phi}_n(t,\omega)\right|\right|\right|^2 dt = 0. \tag{3.26}$$

Proof. We define

$$\tilde{\Phi}_n(t,\omega) = \bigoplus_{j\geq 0} \tilde{G}(t_j,\omega) \otimes 1_{[t_j,t_{j+1})}(t).$$

Clearly $\tilde{\Phi}_n(t,\omega)$ is a fuzzy simple function for all $n \geq 1$. Hence the definition we have

$$\left(\tilde{\Phi}_n\right)^i_\alpha(t,\omega) = \sum_{j\geq 0} \tilde{G}^i_\alpha(t_j,\omega) 1_{[t_j,t_{j+1})}(t),$$

for each $\alpha \in [0,\ 1]$ and $i \in \{L,U\}$, where the function $\tilde{G}^i_\alpha(\cdot,\omega)$ are continuous. The Itô isometry property for bounded function $\tilde{G}^i_\alpha$ leads to (see [16])

$$\lim_{n\to\infty} \int_S^T \left(\tilde{G}^i_\alpha - \left(\tilde{\Phi}_n\right)^i_\alpha\right)^2 dt = 0$$

$\forall \omega \in \Omega, \alpha \in [0,1]$ and $i \in \{L,U\}$. That is

$$\lim_{n\to\infty} E\left[\int_S^T \left(\tilde{G}^i_\alpha - \left(\tilde{\Phi}_n\right)^i_\alpha\right)^2 dt\right] = 0,$$

$\forall \omega \in \Omega, \alpha \in [0,1]$ and $i \in \{L,U\}$.

Thus

$\lim_{n\to\infty} E\int_S^T \left|\left|\left|\tilde{G} \ominus_H \tilde{\Phi}_n\right|\right|\right|^2 dt$

$$\begin{aligned}
&= \lim_{n\to\infty} E\int_S^T \sup_{\alpha\in[0,1]} \max\left(\left|\tilde{G}^L_\alpha - \left(\tilde{\Phi}_n\right)^L_\alpha\right|^2, \left|\tilde{G}^U_\alpha - \left(\tilde{\Phi}_n\right)^U_\alpha\right|^2\right) dt \\
&\leq \sup_{\alpha\in[0,1]} \max\left(\lim_{n\to\infty} E\int_S^T \left|\tilde{G}^L_\alpha - (\tilde{\Phi}_n)^L_\alpha\right|^2 dt, \lim_{n\to\infty} E\int_S^T \left|\tilde{G}^U_\alpha - (\tilde{\Phi}_n)^U_\alpha\right|^2 dt\right) \\
&= \sup_{\alpha\in[0,1]} \max(0,0) \\
&= 0.
\end{aligned}$$

□

Corollary 3.2. *Let $\tilde{H} \in V_F(S,T)$ be a bounded fuzzy function, then there exists a sequence of fuzzy bounded functions $\tilde{G}_n \in V_F(S,T)$ such that $\tilde{G}_n(\cdot,\omega)$ is continuous for every $\omega \in \Omega$ and $n \geq 1$, and we have*

$$\lim_{n\to\infty} E\int_S^T \left|\left|\left|\tilde{H} \ominus_H \tilde{G}_n\right|\right|\right|^2 dt = 0. \tag{3.27}$$

Proof. Suppose that $\left|\left|\left|\tilde{H}(t,\omega)\right|\right|\right| \leq M$ for every $t \in [S,T]$ and $\omega \in \Omega$. For every n, let's consider Ψ_n a real positive continuous function such that

(i) $\Psi_n(x) = 0$ for every $x \leq \frac{-1}{n}$ and $x \geq 0$

(ii) $\int_\infty^{-\infty} \Psi_n(x)dx = 1$.

We define

$$\tilde{G}_n(t,\omega) = \int_0^t \Psi_n(s-t) \otimes \tilde{H}(s,\omega)ds$$

such that

$$\left(\tilde{G}_n\right)^i_\alpha(t,\omega) = \int_0^t \Psi_n(s-t) \otimes \tilde{H}^i_\alpha(s,\omega)ds$$

$\forall \omega \in \Omega, \alpha \in [0,1]$, $t \in [S,T]$ and $i \in \{L,U\}$. Since $\left(\tilde{G}_n\right)^i_\alpha(\cdot,\omega)$ is continuous for every ω and $\left|\left(\tilde{G}_n\right)^i_\alpha(t,\omega)\right| \leq M$. By measurability of $\tilde{H}^i_\alpha(s,\omega)$, $\left(\tilde{G}_n\right)^i_\alpha$ is measurable for every $\alpha \in [0,1]$, $t \in [S,T]$ and $i \in \{L,U\}$.
Thus

$$\lim_{n\to\infty} \int_S^T \left|\tilde{H}^i_\alpha(t,\omega) - (\tilde{G}_n)^i_\alpha(t,\omega)\right|^2 dt = 0.$$

By the classical bounded convergence Theorem, we have

$$\lim_{n\to\infty} E \int_S^T \left|\tilde{H}^i_\alpha(t,\omega) - (\tilde{G}_n)^i_\alpha(t,\omega)\right|^2 dt = 0.$$

$\forall \omega \in \Omega, \alpha \in [0,1]$, $t \in [S,T]$ and $i \in \{L,U\}$. The result follows by using similar arguments as in the proof of Theorem 3.3. □

Corollary 3.3. *Let $\tilde{F} \in V_F(S,T)$, then there exists a sequence of fuzzy functions $\tilde{H}_n \in V_F(S,T)$ such that $\tilde{H}_n$ is bounded for every $n \geq 1$ and we have*

$$\lim_{n\to\infty} E \int_S^T \left|\left\|\tilde{F} \ominus_H \tilde{H}_n\right\|\right|^2 dt = 0. \tag{3.28}$$

Proof. Let $\tilde{H}_n$ such that $\left(\tilde{H}_n(t,\omega)\right)^i_\alpha = \begin{cases} -n & \text{if } \tilde{F}^i_\alpha \leq -n \\ \tilde{F}^i_\alpha & \text{if } -n \leq \tilde{F}^i_\alpha \leq n \\ n & \text{if } \tilde{F}^i_\alpha > n \end{cases}$ $\forall \omega \in \Omega, \alpha \in [0,1]$, $t \in [S,T]$ and $i \in \{L,U\}$. By the classical dominated convergence Theorem, we have

$$\lim_{n\to\infty} E \int_S^T \left(\tilde{F}^i_\alpha(t,\omega) - \left(\tilde{H}_n\right)^i_\alpha(t,\omega)\right)^2 dt = 0.$$

The result follows by using a similar development as in the proof of Theorem 3.3. □

Using the above results, for $\tilde{F} \in V_F(S,T)$, the fuzzy Itô integral w.r.t. Fuzzy Brownian motion

$$\int_S^T \tilde{F}(t,\omega)d\tilde{B}(t,\omega)$$

can be defined as.

Theorem 3.4. *(Fuzzy Itô Integral)*
Let $\tilde{F} \in V_F(S,T)$. Then the fuzzy Itô integral of $\tilde{F}$ (from S to T) w.r.t. a Fuzzy Brownian Motion $\tilde{B}(t,\omega)$ is defined by

$$\int_S^T \tilde{F}(t,\omega)d\tilde{B}(t,\omega) = \lim_{n\to\infty} \int_S^T \tilde{\Phi}_n(t,\omega)d\tilde{B}(t,\omega), \tag{3.29}$$

the limit is taken in $L^2_{\mathscr{F}_c(\mathbb{R})}(P)$ and $\left\{\tilde{\Phi}_n\right\}_{n\geq 1}$ is a sequence of fuzzy simple functions such that

$$\lim_{n\to\infty} E \int_S^T \left|\left\|\tilde{F} \ominus_H \tilde{\Phi}_n(t,\omega)\right\|\right|^2 dt = 0. \tag{3.30}$$

Proof. From the Theorems 2.2, 2.3 and 2.4, there is a sequence $\left\{\tilde{\Phi}_n\right\}_{n\geq 1}$ satisfying 3.30. Hence we are done. □

Corollary 3.4.

$$E\left[\left(\int_S^T \tilde{F}(t,\omega)d\tilde{B}(t,\omega)\right)^2\right] = E\left[\int_S^T \tilde{F}^2(t,\omega)dt\right], \tag{3.31}$$

for every $\tilde{F} \in V_F(S,T)$

Proof. From Theorem 3.4 and Lemma 3.1, we have the result. □

Example 3.1. *Suppose that* $\tilde{B}_0 = \tilde{0}$. *Then* $\int_0^t \tilde{B}(s)d\tilde{B}(s) = \frac{1}{2}\tilde{B}^2(t) \ominus_H \frac{1}{2}t$.

Proof. It is enough to show that for all $\alpha \in [0,1]$,

$$\left[\int_0^t \tilde{B}(s)d\tilde{B}(s)\right]_\alpha = \left[\frac{1}{2}\tilde{B}^2(t) \ominus_H \frac{1}{2}t\right]_\alpha.$$

First, for a positive fuzzy Brownian motion

$$\begin{aligned}\left[\int_0^t \tilde{B}(s)d\tilde{B}(s)\right]_\alpha &= \left[\int_0^T (\tilde{B}(s))_\alpha^L d\tilde{B}_\alpha^L(s), \int_0^t (\tilde{B}(s))_\alpha^U d\tilde{B}_\alpha^U(s)\right] \\ &= \left[\frac{1}{2}(\tilde{B}^2(t))_\alpha^L - \frac{1}{2}t, \frac{1}{2}(\tilde{B}^2(t))_\alpha^U - \frac{1}{2}t\right]\end{aligned}$$

by the definition of Hukuhara difference [19] we have

$$\left[\frac{1}{2}\tilde{B}^2(t) \ominus_H \frac{1}{2}t\right]_\alpha = [\min S, \max S],$$

where

$$S = \left\{\frac{1}{2}(\tilde{B}_\alpha^L(t))^2 - \frac{1}{2}t, \frac{1}{2}(\tilde{B}_\alpha^U(t))^2 - \frac{1}{2}t\right\}.$$

Since $(\tilde{B}_\alpha^L(t)) \le (\tilde{B}_\alpha^U(t))$ for every $t \ge 0$ and $\alpha \in (0,1]$ and by the fact of positivity of fuzzy Brownian motion considered, we have $(\tilde{B}_\alpha^L(t)) > 0$.
Thus $\frac{1}{2}(\tilde{B}_\alpha^L(t))^2 - \frac{1}{2}t < \frac{1}{2}(\tilde{B}_\alpha^U(t))^2 - \frac{1}{2}t$, for every $t \ge 0$ and $\alpha \in (0,1]$. Thus $\min S = \frac{1}{2}(\tilde{B}_\alpha^L(t))^2 - \frac{1}{2}t$ and $\max S = \frac{1}{2}(\tilde{B}_\alpha^U(t))^2 - \frac{1}{2}t$. □

Theorem 3.5. *(Some properties of Fuzzy Itô integral)*
Let $\tilde{F}$ *and* $\tilde{G} \in V_F(0,T)$ *and* $0 \le S < U < T$. *Then*

(i) $\int_S^T \tilde{F}d\tilde{B}(t) = \int_S^U \tilde{F}d\tilde{B}(t) \oplus \int_U^T \tilde{F}d\tilde{B}(t)$,

(ii) $\int_S^T \tilde{c} \otimes \tilde{F}d\tilde{B}(t) = \tilde{c} \otimes \int_S^T \tilde{F}d\tilde{B}(t)$, *where* $\tilde{c} \in \mathscr{F}(\mathbb{R})$,

(iii) $\int_S^T \tilde{F} \oplus \tilde{G}d\tilde{B}(t) = \int_S^T \tilde{F}d\tilde{B}(t) \oplus \int_S^T \tilde{G}d\tilde{B}(t)$,

(iv) $E\left[\int_S^T \tilde{F}d\tilde{B}(t)\right] = \tilde{0}$,

(v) $\int_S^T \tilde{F}d\tilde{B}(t)$ *is* $\mathscr{F}_T$*-measurable.*

Proof. This holds for all simple fuzzy functions, so by taking limits we obtain this for all fuzzy functions $\tilde{F}$ and $\tilde{G}$ $\in V_F(0,T)$. We have:

(i) Let $\tilde{\Phi}(t,\omega)$ be a simple fuzzy function given by

$$\tilde{\Phi}(t,\omega) = \bigoplus_{j\ge 0} \tilde{E}_j(\omega) \otimes 1_{[t_{j+1},t_j[}(t).$$

By definition

$$\int_S^T \tilde{\Phi}(t,\omega)d\tilde{B}(t,\omega) = \bigoplus_{j\geq 0} \tilde{E}_j(\omega)\otimes\left[\tilde{B}(t_{j+1},\omega)\ominus_H \tilde{B}(t_j,\omega)\right],$$

where

$$t_j = t_j^{(n)} = \begin{cases} j2^{-n} & \text{if } S\leq j2^{-n}\leq T \\ S & \text{if } j2^{-n}<S \\ T & \text{if } j2^{-n}>T \end{cases}$$

which is equivalent to $S\leq t_1^{(n)}\leq\cdots\leq t_j^{(n)}\leq\cdots\leq T$. We take now two subdivisions on $[S,T]$, the first from S to U and the second from U to T as following $t_{k_1} = t_{k_1}^{(n')} = \begin{cases} k_1 2^{-n'} & \text{if } S\leq k_1 2^{-n'}\leq T \\ S & \text{if } k_1 2^{-n'}<S \\ U & \text{if } k_1 2^{-n'}>U \end{cases}$

and

$$t_{k_2} = t_{k_2}^{(n")} = \begin{cases} k_2 2^{-n"} & \text{if } U\leq k_2 2^{-n"}\leq T \\ U & \text{if } k_2 2^{-n"}<U \\ T & \text{if } k_2 2^{-n"}>T \end{cases}$$

or once more

$S\leq t_1^{(n')}\leq\cdots\leq t_{k_1}^{(n')}\leq\cdots\leq U$ and $U\leq t_1^{(n")}\leq\cdots\leq t_{k_2}^{(n")}\leq\cdots\leq T$ which are two contiguous infinite subdivisions of interval $[S,T]$.

Thus

$$\begin{aligned}\int_S^T \tilde{\Phi}(t,\omega)d\tilde{B}(t,\omega) &= \bigoplus_{j\geq 0} \tilde{E}_j(\omega)\otimes\left[\tilde{B}(t_{j+1},\omega)\ominus_H \tilde{B}(t_j,\omega)\right] \\ &= \bigoplus_{k_1\geq 0} \tilde{E_{k_1}}(\omega)\otimes\left[\tilde{B}(t_{k_1+1},\omega)\ominus_H \tilde{B}(t_{k_1},\omega)\right]\oplus \\ &\bigoplus_{k_2\geq 0} \tilde{E_{k_2}}(\omega)\otimes\left[\tilde{B}(t_{k_2+1},\omega)\ominus_H \tilde{B}(t_{k_2},\omega)\right] \\ &= \int_S^U \tilde{\Phi}(t,\omega)d\tilde{B}(t,\omega)\oplus\int_U^T \tilde{\Phi}(t,\omega)d\tilde{B}(t,\omega).\end{aligned}$$

(ii) Let $\tilde{c}\in\mathscr{F}(\mathbb{R})$

$$\begin{aligned}\int_S^T \tilde{c}\otimes\tilde{\Phi}(t,\omega)d\tilde{B}(t) &= \int_S^T \bigoplus_{j\geq 0}\tilde{c}\otimes\tilde{E}_j(\omega)\otimes 1_{[t_j,t_{j+1}[}(t)d\tilde{B}(t) \\ &= \bigoplus_{j\geq 0}\tilde{c}\otimes\tilde{E}_j(\omega)\otimes[\tilde{B}(t_{j+1},\omega)\ominus_H \tilde{B}(t_j,\omega)] \\ &= \tilde{c}\otimes\bigoplus_{j\geq 0}\tilde{E}_j(\omega)\otimes[\tilde{B}(t_{j+1},\omega)\ominus_H \tilde{B}(t_j,\omega)] \\ &= \tilde{c}\otimes\int_S^T \bigoplus_{j\geq 0}\tilde{E}_j(\omega)\otimes 1_{[t_j,t_{j+1}[}(t)d\tilde{B}(t) \\ &= \tilde{c}\otimes\int_S^T \tilde{\Phi}(t,\omega)d\tilde{B}(t).\end{aligned}$$

(iii) Let $\tilde{F}$ and $\tilde{G}$ be of the form

$$\tilde{F} = \bigoplus_{i\geq 1}\tilde{E}_i(\omega)\otimes 1_{[t_i,t_{i+1}[}(t)$$

and

$$\tilde{G} = \bigoplus_{j\geq 1}\tilde{E'}_j(\omega)\otimes 1_{[t_j,t_{j+1}[}(t)$$

Thus

$$\tilde{F} \oplus \tilde{G} = \bigoplus_{i \geq 1} \bigoplus_{j \geq 1} \tilde{E}_i(\omega) \oplus \tilde{E'}_j(\omega) \otimes 1_{[t_i, t_{i+1}[\cap [t_j, t_{j+1}[}(t)$$

Thus

$$\begin{aligned}
\int_S^T \tilde{F} \oplus \tilde{G} d\tilde{B}(t) &= \bigoplus_{i \geq 1} \bigoplus_{j \geq 1} \tilde{E}_i(\omega) \otimes [\tilde{B}(t_{i+1} \vee t_{j+1}, \omega) \ominus_H \tilde{B}(t_i \wedge t_j, \omega)] \\
&\oplus \bigoplus_{i \geq 1} \bigoplus_{j \geq 1} \tilde{E'}_i(\omega) \otimes [\tilde{B}(t_{i+1} \vee t_{j+1}, \omega) \ominus_H \tilde{B}(t_i \wedge t_j, \omega)] \\
&= \bigoplus_{i \geq 1} \tilde{E}_i(\omega) \otimes [\tilde{B}(t_{i+1}, \omega) \ominus_H \tilde{B}(t_i, \omega)] \\
&\oplus \bigoplus_{j \geq 1} \tilde{E'}_j(\omega) \otimes [\tilde{B}(t_{j+1}, \omega) \ominus_H \tilde{B}(t_j, \omega)] \\
&= \int_S^T \tilde{F} d\tilde{B}(t) \oplus \int_S^T \tilde{G} d\tilde{B}(t).
\end{aligned}$$

(iv) Let $\tilde{F} = \bigoplus_{i \geq 1} \tilde{E}_i(\omega) \otimes 1_{[t_i, t_{i+1}[}(t)$, then

$$\begin{aligned}
E[\int_S^T \tilde{F} d\tilde{B}(t)] &= E[\bigoplus_{i \geq 1} \tilde{E}_i(\omega) \otimes [\tilde{B}(t_{i+1}, \omega) \ominus_H \tilde{B}(t_i, \omega)] \\
&= \bigoplus_{i \geq 1} \tilde{E}_i(\omega) \otimes E[\tilde{B}(t_{i+1}, \omega) \ominus_H \tilde{B}(t_i, \omega)] \\
&= \bigoplus_{i \geq 1} \tilde{E}_i(\omega) \otimes \tilde{0} \\
&= \tilde{0}.
\end{aligned}$$

(v) Let Let $\tilde{F} = \bigoplus_{i \geq 1} \tilde{E}_i(\omega) \otimes 1_{[t_i, t_{i+1}[}(t)$, then

$$\int_S^T \tilde{F} d\tilde{B}(t) = \bigoplus_{i \geq 1} \tilde{E}_i(\omega) \otimes [\tilde{B}(t_{i+1}, \omega) \ominus_H \tilde{B}(t_i, \omega)].$$

Since $\tilde{B}(t_{i+1}, \omega)$ is measurable, so is $\tilde{E}_i(\omega) \otimes [\tilde{B}(t_{i+1}, \omega) \ominus_H \tilde{B}(t_i, \omega)$. Thus $\int_S^T \tilde{F} d\tilde{B}(t)$ is measurable for $\tilde{F}$ a fuzzy simple function. So by taking the limit we obtain the assertion for all measurable fuzzy functions.

□

4 Conclusion

In this paper, the fuzzy Brownian Motion is used to describe a new fuzzy Itô integral of fuzzy valued functions. First we define this fuzzy Itô integral for simple fuzzy functions. Then we use the fact that all fuzzy integrable functions can be expressed as a limit of fuzzy simple functions to define their fuzzy Itô integral.

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