

A new fuzzy Monte Carlo method for solving SLAE with ergodic fuzzy Markov chains

Maryam Gharehdaghi^{1*}, Behrouz Fathi Vajargah²

(1) *Iran Banking Institute, Tehran, Iran*

(2) *University of Guilan, Rasht, Iran*

Copyright 2015 © Maryam Gharehdaghi and Behrouz Fathi Vajargah. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper we introduce a new fuzzy Monte Carlo method for solving system of linear algebraic equations (SLAE) over the possibility theory and max-min algebra. To solve the SLAE, we first define a fuzzy estimator and prove that this is an unbiased estimator of the solution. To prove unbiasedness, we apply the ergodic fuzzy Markov chains. This new approach works even for cases with coefficients matrix with a norm greater than one.

Keywords: Fuzzy algebra, Fuzzy Monte Carlo, Fuzzy transition possibility, Ergodic fuzzy Markov chains, Simulation.

1 Introduction

Consider a system of linear algebraic equations $B \circ x = f$. The elements of the coefficient matrix B are fuzzy numbers; i.e. positive numbers in $[0, 1]$. In this paper, fuzzy Monte Carlo approach is used to solve the system. Based on possibility theory and max-min algebra and by using a fuzzy estimator and ergodic fuzzy Markov chains, as generated in [5], the system of equations is successfully solved. In max-min algebra, the min and max operators play the role of multiplication and addition, respectively [6],[8]. In the classic form as the suffices condition to solve the system $x = A \otimes x \oplus f$, the sum of the coefficient of each row of matrix A must be less than one [1]. In our fuzzy system, this is not the case as operators are max-min operators and therefore it sufficient that the maximum coefficient of each row to be less than one. Hence, the main reason for using fuzzy logic is to solve systems with coefficient matrix norm greater than one for which classic Monte Carlo method does not work.

2 Basic Notations and Definitions

In order to make the exposition clear we introduce the underlying notations for the under study linear system. By a max-min fuzzy algebra F we mean a linear ordered set $([0, 1], \leq, \oplus, \otimes)$ with binary operations \oplus =maximum and \otimes =minimum. For any natural number $n > 0$, $L(n)$ denotes set of all n -dimensional column vector over F , and $L(m, n)$ denotes the set of all $m \times n$ matrices over F . For $x, y \in L(n)$, we write $x \leq y$, if $x_i \leq y_i$ holds for all i , and we write $x < y$, if $x \leq y$ and $x \neq y$.

In this paper, we consider a system of linear equations of the form

$$B \circ x = f,$$

*Corresponding author. Email address: gharehdaghi@ibi.ac.ir

where the matrix $B \in L(n, n)$ and the vector $f \in L(n)$ are given, and the vector $x \in L(n)$ is unknown. The matrix operations use the operations \oplus and \otimes instead of the addition and multiplication formally in the same way as is done over a field.

Definition 2.1. Let S be a nonempty set (a space of elementary events), and let \mathbf{A} be a class of subsets of S that contains \emptyset and S (the set of combinations of events). A completely additive possibility measure on the class of sets \mathbf{A} is understood to be a function $Poss : \mathbf{A} \rightarrow [0, 1]$ that satisfies:

$$\begin{aligned} Poss(S) &= 1, \\ Poss(\emptyset) &= 0, \\ 0 \leq Poss(A) &\leq 1, \text{ for any } A \in \mathbf{A}, \text{ and} \\ Poss(\cup_i A_i) &= \sup_i Poss(A_i), \text{ for each family } \{A_i\} \in \mathbf{A} \text{ [4].} \end{aligned}$$

Definition 2.2. The Fuzzy algebraic norm infinity of the matrix $B \in L(m, n)$ is defined by

$$\|B\|_{\oplus} = \bigoplus_{i=1}^m \bigoplus_{j=1}^n |b_{ij}|. \tag{2.1}$$

Also, \odot -product (Fuzzy inner product) of the matrices $A \in L(k, j), B \in L(j, n)$ is defined by

$$C = A \odot B, \tag{2.2}$$

$$c_{kn} = \bigoplus_{j=1}^{|j|} (a_{kj} \otimes b_{jn}). \tag{2.3}$$

Definition 2.3. Fuzzy determinant of a fuzzy matrix $A \in L(n, n)$ is defined by

$$Fdet(A) = \max_{(h_1, h_2, \dots, h_n)} \{ \min \{ a_{1h_1}, a_{2h_2}, \dots, a_{nh_n} \} \}, \tag{2.4}$$

where matrix is for all permutations (h_1, h_2, \dots, h_n) of indices $\{1, 2, \dots, n\}$. Also, matrix A is a nonsingular if and only if all rows of A are linear independent and all columns of A are also linear independent [9].

3 Fuzzy Markov Chains

Let $S = \{1, 2, \dots, n\}$. A finite fuzzy set for a fuzzy distribution on S is defined by a mapping x from S to $[0, 1]$ represented by a vector $x = \{x_1, x_2, \dots, x_n\}$, with $0 \leq x_i \leq 1, i \in S$. Here, x_i is the membership degree that a state i has regarding a fuzzy set $S, i \in S$. All relations and compositions are defined by fuzzy algebra.

Now, a fuzzy transition possibility matrix $Poss$ is defined in a metric space $S \times S$ by a matrix $\{poss_{ij}\}_{i,j=1}^m$ with $0 \leq poss_{ij} \leq 1, i, j \in S$.

We note that it does not need elements of each row of the matrix $Poss$ to sum up to one. This fuzzy matrix $Poss$ allows to define all relations among the m states of the fuzzy Markov chain at each time instant t , as follows [2],[7].

Definition 3.1. At each instant $t, t = 1, 2, \dots, n$, the state of system is described by the fuzzy set $x^{(t)} \in \mathcal{F}(S)$. The transition law of a fuzzy Markov chain is given by the fuzzy relational matrix $Poss$ at instant $t, t = 1, 2, \dots, n$, as follows:

$$x_j^{(t+1)} = \bigoplus_{i \in S} \{x_i^{(t)} \otimes poss_{ij}\}, \quad j \in S, \tag{3.5}$$

$$x^{(t+1)} = x^{(t)} \circ Poss, \tag{3.6}$$

where i and $j, i, j = 1, 2, \dots, m$ are the initial and final states of the transition and $x^{(0)}$ is the initial distribution. Also,

$$poss_{ij}^t = \bigoplus_{k \in S} \{poss_{ik} \otimes poss_{kj}^{t-1}\}, \quad i, j \in S, \quad (3.7)$$

$$Poss^t = Poss \circ Poss^{t-1}. \quad (3.8)$$

Thomason in [9] shows that the powers of a fuzzy matrix are stable over the max-min operator. More information about powers of a fuzzy matrix can be found in [6],[8]. Now, a stationary distribution of a fuzzy matrix is defined as follows.

Definition 3.2. Let the powers of the fuzzy transition matrix $Poss$ converge in τ steps to a non-periodic solution, then the associated fuzzy Markov chain is called aperiodic fuzzy Markov chain and $Poss^* = Poss^\tau$ is its stationary fuzzy transition matrix.

Definition 3.3. A fuzzy Markov chain is called strong ergodic if it is aperiodic and its stationary transition matrix has identical rows.

A fuzzy Markov chain is called weakly ergodic if it is aperiodic and its stationary transition matrix is stable with non-identical rows.

We make use of ergodic fuzzy Markov chains generated in [5] to solve the system of linear equations.

Note. From now on to simplify the notations, we use \prod and \prod_{ij} for $Poss$ and $poss_{ij}$, respectively.

4 A New Fuzzy Monte Carlo Method for Solving SLAE

In this section, we introduce a new fuzzy Monte Carlo method for solving SLAE. Suppose

$$B \circ x = f, \quad (4.9)$$

where $B \in L(n, n)$ is a given fuzzy matrix, $f^T = (f_1, \dots, f_n)$ is a known fuzzy vector and $x^T = (x_1, \dots, x_n)$ is the fuzzy solution vector that we are looking for. If we consider a fuzzy matrix $A \in L(n, n)$ such that $B = I - A$, then the linear system (4.9) is converted to

$$x = A \otimes x \oplus f. \quad (4.10)$$

In Monte Carlo calculations for solving linear systems, we use the fuzzy maximum norm of matrix A given by Definition 2.2 with

$$\|A\|_{\oplus} = \bigoplus_i \bigoplus_j |a_{ij}| \leq 1 \quad i, j = 1, 2, \dots, n. \quad (4.11)$$

The iterative form of Eq. (4.10) is

$$x^{(k+1)} = A \otimes x^{(k)} \oplus f \quad k = 0, 1, 2, \dots, \quad (4.12)$$

with $x^{(0)} = 0$ and $A^0 = I$. This implies that

$$x^{(k+1)} = \bigoplus_{m=0}^k A^m \otimes f \quad k = 0, 1, 2, \dots, \quad (4.13)$$

converges for any nonsingular fuzzy matrix A with [8]

$$\lim_{k \rightarrow \infty} x^{(k)} = \lim_{k \rightarrow \infty} \bigoplus_{m=0}^k A^m \otimes f = x \quad k = 0, 1, 2, \dots \quad (4.14)$$

Generally, the required elements of the fuzzy matrix A are well-defined and randomly selected. These elements can be selected via the following fuzzy Markov chain,

$$x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_k$$

where $x_i, i = 1, 2, \dots, k$ belongs to the state space $S = \{1, 2, \dots, n\}$ (defined in Section 3). Also notice that for each $\alpha, \beta \in S$, we have $\Pi_\alpha = \prod(x_0 = \alpha)$ and $\prod(x_{j+1} = \beta | x_j = \alpha) = \Pi_{\alpha\beta}$ which are the initial fuzzy distribution and one step fuzzy transition possibility of fuzzy Markov chain, respectively. Possibilities $\Pi_{\alpha\beta}$ define a transition possibility matrix $\Pi = [\Pi_{\alpha\beta}]$. Moreover, we have $\bigoplus_{\beta=1}^n \Pi_{\alpha\beta} = 1$, for all $\alpha = 1, 2, \dots, n$. This holds for any $\Pi_{\alpha\beta}$ in ergodic fuzzy Markov chains.

Set

$$\omega_0 = 1, \quad \omega_m = \frac{a_{x_0x_1} \otimes a_{x_1x_2} \otimes \dots \otimes a_{x_{m-1}x_m}}{\prod_{x_0x_1} \otimes \prod_{x_1x_2} \otimes \dots \otimes \prod_{x_{m-1}x_m}}. \quad (4.15)$$

Let a fuzzy estimator $\eta_k(h) = \frac{h_{x_0}}{\prod_{x_0}} \bigoplus_{m=0}^k [\omega_m \otimes f_{x_m}]$, where $h = \underbrace{(0, 0, \dots, 0, 1, 0, \dots, 0)}_i \in R^n$ that the distribution $(\Pi_1, \dots, \Pi_n)^t$ is acceptable for vector h . Now, we are ready to prove the main result of this paper.

Theorem 4.1. Under above assumptions for each $h \in R^n$, we have

$$E[\eta_k(h)] = h \odot x^{(k+1)} \text{ where } \eta_k(h) = \frac{h_{x_0}}{\prod_{x_0}} \bigoplus_{m=0}^{\infty} \omega_m \otimes f_{x_m}.$$

Proof. We have:

$$\begin{aligned} E[\eta_k(h)] &= \bigoplus_{x_0=1}^n \bigoplus_{x_1=1}^n \dots \bigoplus_{x_k=1}^n \left(\frac{h_{x_0}}{\prod_{x_0}} \bigoplus_{m=0}^k \omega_m \otimes f_{x_m} \right) \otimes \Pi_{x_0} \otimes \Pi_{x_0x_1} \otimes \dots \otimes \Pi_{x_{k-1}x_k} \\ &= \bigoplus_{x_0=1}^n \bigoplus_{x_1=1}^n \dots \bigoplus_{x_k=1}^n \frac{h_{x_0}}{\prod_{x_0}} \bigoplus_{m=0}^k \left(\frac{a_{x_0x_1} \otimes a_{x_1x_2} \otimes \dots \otimes a_{x_{m-1}x_m}}{\prod_{x_0x_1} \otimes \prod_{x_1x_2} \otimes \dots \otimes \prod_{x_{m-1}x_m}} \right) \otimes \\ &\quad f_{x_m} \otimes \Pi_{x_0} \otimes \Pi_{x_0x_1} \otimes \dots \otimes \Pi_{x_{k-1}x_k} \\ &= \bigoplus_{m=0}^k \bigoplus_{x_0=1}^n \bigoplus_{x_1=1}^n \dots \bigoplus_{x_m=1}^n \bigoplus_{x_{m+1}=1}^n \dots \bigoplus_{x_k=1}^n h_{x_0} \otimes a_{x_0x_1} \otimes a_{x_1x_2} \otimes \dots \otimes a_{x_{m-1}x_m} \otimes \\ &\quad \Pi_{x_mx_{m+1}} \otimes \dots \otimes \Pi_{x_{k-1}x_k}. \end{aligned}$$

Since for all $i = 1, 2, \dots, n$, we have $\bigoplus_{j=1}^n \Pi_{ij} = 1$ and

$$\bigoplus_{x_{m+1}=1}^n \Pi_{x_mx_{m+1}} = 1, \dots, \bigoplus_{x_k=1}^n \Pi_{x_{k-1}x_k} = 1.$$

Thus we get:

$$E[\eta_k(h)] = \bigoplus_{m=1}^k \bigoplus_{x_0=1}^n \bigoplus_{x_1=1}^n \dots \bigoplus_{x_k=1}^n h_{x_0} \otimes a_{x_0x_1} \otimes a_{x_1x_2} \otimes \dots \otimes a_{x_{m-1}x_m} \otimes f_{x_m}. \quad (4.16)$$

Finally notice that the product of order $m \geq 2$ of the fuzzy matrix A is given by

$$A^m = A \circ A \circ \dots \circ A = \left[\bigoplus_{x_0} \bigoplus_{x_1} \dots \bigoplus_{x_m} a_{x_0x_1} \otimes \dots \otimes a_{x_{m-1}x_m} \right]_{x_0, \dots, x_m=1, \dots, n}. \quad (4.17)$$

Therefore, using Eq. (4.16) and Eq. (4.17) we obtain

$$E[\eta_k(h)] = h \odot \left(\bigoplus_{m=1}^k A^m \otimes f \right) = h \odot x^{(k+1)} \quad (4.18)$$

□

Now, consider N paths $x_0^{(m)} \rightarrow x_1^{(m)} \rightarrow \dots \rightarrow x_k^{(m)}, m = 1, 2, \dots, N$, on the fuzzy possibility matrix of ergodic fuzzy Markov chains. By Theorem 4.1, we define the fuzzy Monte Carlo estimated solution by

$$\hat{\Theta}(h) = \frac{1}{N} \sum_{s=1}^N \eta_k^{(s)}(h) \approx x_i. \tag{4.19}$$

Using Strong Law of Large Numbers over fuzzy variable [3],[4] it can be easily proved that the above fuzzy Monte Carlo estimation $\hat{\Theta}(h)$ is a consistent estimator of $E[\eta_k(h)] = h \odot x^{(k+1)}$.

5 Simulation

Since our new Monte Carlo algorithm is applicable to find the solution of systems of linear algebraic equations with norm greater than one, we are going to present its performances. At first, we simulate ergodic fuzzy Markov chains 500 times. Then using above algorithm the SLAE is solved. We employ MATLAB software for our simulations.

Example 5.1. Consider the linear system:

$$\begin{pmatrix} 1.00 & 0.20 & 0.30 \\ 0.70 & 0.80 & 0.50 \\ 0.40 & 0.90 & 0.70 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.40 \\ 0.70 \\ 0.90 \end{pmatrix}.$$

The exact solution is $(0.0621, 0.2000, 0.9931)^t$ and if we use our fuzzy Monte Carlo method by generating ergodic fuzzy Markov chains, the solution after 500 iterations is $(0.0623, 0.2001, 0.9940)$. Results for some matrix dimensions are given in Table 1.

Table 1: Result of simulation

Size	Number of EFMC	Classic norm	Fuzzy norm	Max error
$m=5$	827	2.40	1	10^{-4}
$m=50$	1530	18.90	1	10^{-3}
$m=100$	4876	49.60	1	10^{-2}
$m=1000$	20000	396	1	10^{-2}

As shown in Table 1, the maximum estimation error is 10^{-4} for the matrix size of 5 and the number of ergodic fuzzy Markov chains (EFMC) is equal to 827. Even if the matrix size gets close to 1000, the maximum estimation error is 10^{-2} . Obviously in our algorithm, for each desired matrix of any dimension, using more number of fuzzy Markov chains improves the accuracy of results. However, in this article it is not our main goal, but we will show in our next work that somehow we may use less fuzzy Markov chains, but we still have enough performances and accuracies in our algorithm.

6 Conclusion

In this paper, we present a new fuzzy Monte Carlo method to solve the system $B \circ x = f$ (where $\|A\|_\infty > 1$) over the max-min algebra. It is proved that the proposed fuzzy estimator is an efficient solution. Simulation results reveal that the estimated solution is a good approximation of real solution.

We emphasize that our approach works only for ergodic fuzzy Markov chains, and it is not stable for periodic fuzzy Markov chains.

References

- [1] V. N. Alexandrov, Efficient parallel Monte Carlo methods for matrix computation, *Mathematics and Computers in Simulation*, 47 (1998) 113-122.
[http://dx.doi.org/10.1016/S0378-4754\(98\)00097-4](http://dx.doi.org/10.1016/S0378-4754(98)00097-4)
- [2] K. E. Avrachenkov, E. Sanchez, Fuzzy Markov chains, IPMU, Spain, (2000) 1851-1856.
- [3] R. Badard, The law of large numbers for fuzzy processes and the estimation problem, *Information Sciences*, 28 (1982) 161-178.
[http://dx.doi.org/10.1016/0020-0255\(82\)90046-9](http://dx.doi.org/10.1016/0020-0255(82)90046-9)
- [4] A. S. Bychkov, E. V. Ivanov, Convergence of sequences of fuzzy perceptive elements defined on different possibility spaces, *Cybernetics and Systems Analysis*, 46 (2) (2010).
<http://dx.doi.org/10.1007/s10559-010-9207-0>
- [5] B. Fathi, M. Gharehdaghi, Reducing periodicity of Fuzzy Markov Chains Based on Simulation Using Halton Sequence, *Communications in statistics, Simulation and Computation*, (2014).
<http://www.tandfonline.com/doi/full/10.1080/03610918.2013.854904>
- [6] M. Gavalec, Computing orbit period in max-min algebra, *Discrete Applied Mathematics*, 100 (2000) 49-65.
[http://dx.doi.org/10.1016/S0166-218X\(99\)00174-2](http://dx.doi.org/10.1016/S0166-218X(99)00174-2)
- [7] D. Kalenatic, J. Figueroa-Garcia, C. A. Lopez, Scalarization of Type-1 Fuzzy Markov Chains, LNCS, Springer-Verlag, (2010) 110-117.
http://dx.doi.org/10.1007/978-3-642-14922-1_15
- [8] K. G. Peeva, Resolution of Composite intuitionistic fuzzy relational equations, *NIFS*, 6 (2000) 15-24.
- [9] M. G. Thomason, Convergence of powers of a fuzzy matrix, *J. Math. Anal. Appl.*, 57 (1977) 476-480.
[http://dx.doi.org/10.1016/0022-247X\(77\)90274-8](http://dx.doi.org/10.1016/0022-247X(77)90274-8)