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Research Article

Fixed Point of Generalized Eventual Cyclic Gross in Fuzzy Norm Spaces for Contractive Mappings

S. A. M. Mohsenialhosseini[1]*, H. Mazaheri[2]

(1) *Faculty of Mathematics, Vali-e-Asr University, Rafsanjan, Iran*

(2) *Faculty of Mathematics, Yazd University, Yazd, Iran*

Abstract

We define generalized eventual cyclic gross contractive mapping in fuzzy norm spaces, which is a generalization of the eventual cyclic gross contractions. Also we prove the existence of a fixed point for this type of contractive mapping on fuzzy norm spaces.

Keywords: Fuzzy norm spaces, generalized eventual cyclic, contractive mapping.
MSC (2000): 46A32, 46M05, 41A17.

1 Introduction

In this paper, starting from the article of Jin Liang and Sheng-Hua Yan [4], we study generalized eventual cyclic gross contractive mapping on fuzzy norm spaces, and we give some fuzzy fixed points of such mapping.
Fuzzy set was defined by Zadeh [12]. Katsaras [8], who while studying fuzzy topological vector spaces, was the first to introduce in 1984 the idea of fuzzy norm on a linear space. In 1992, Felbin [7] defined a fuzzy norm on a linear space with an associated metric of the Kaleva and Seikkala type [9]. A further development along this line of inquiry took place when in 1994, Cheng and Mordeson [6] evolved the definition of a further type of fuzzy norm having a corresponding metric of the Kramosil and Michalek type [10].
Chitra and Mordeson [5] introduce a definition of norm fuzzy and thereafter the concept of fuzzy norm space has been introduced and generalized in different ways by Bag and Samanta in [1], [2], [3].
Throughout this article, the symbols $\wedge$ and $\vee$ mean the **Min** and the **Max**, respectively.

Definition 1.1. *Let U be a linear space on $\boldsymbol{R}$. A function $N : U \times \boldsymbol{R} \to [0,1]$ is called fuzzy norm if and only if for every $x,u \in U$ and every $c \in \boldsymbol{R}$ the following properties are satisfied :*

$(F_{N1}):\ N(x,t) = 0$ *for every* $t \in \boldsymbol{R}^- \cup \{0\}$,

$(F_{N2}):\ N(x,t) = 1$ *if and only if* $x = 0$ *for every* $t \in \boldsymbol{R}^+$,

$(F_{N3}):\ N(cx,t) = N(x, \frac{t}{|c|})$ *for every* $c \neq 0$ *and* $t \in \boldsymbol{R}^+$,

*Corresponding author. Email address: amah@vru.ac.ir

$(F_{N4}):\ N(x+u,s+t)\geq min\{N(x,s),N(u,t)\}$ *for every* $s,t\in \boldsymbol{R}^{+}$,

(F_{N5}) : *the function* $N(x,.)$ *is nondecreasing on* $\boldsymbol{R}$*, and*

$$\lim_{t\to\infty} N(x,t)=1.$$

A pair (U,N) *is called a fuzzy norm space. Sometimes, We need two additional conditions as follows :*

$(F_{N6}):\ \forall t\in \boldsymbol{R}^{+}\ N(x,t)>0\ \Rightarrow x=0.$

(F_{N7}) : *function* $N(x,.)$ *is continuous for every* $x\neq 0$*, and on subset*

$$\{t:\ 0<N(x,t)<1\}$$

is strictly increasing.
Let (U,N) *be a fuzzy norm space. For all* $\alpha\in(0,1)$*, we define* α *norm on* U *as follows :*

$$\|x\|_{\alpha}=\wedge\{t>0:N(x,t)\geq\alpha\}\ for\ every\ x\in U.$$

Then $\{\|x\|_{\alpha}:\ \alpha\in(0,1]\}$ is an ascending family of normed on U and they are called $\alpha-norm$ on U corresponding to the fuzzy norm N on U. Some notation, lemmas and example which will be used in this paper are given below:

Lemma 1.1. *[1] Let* (U,N) *be a fuzzy norm space such that satisfy conditions* F_{N6} *and* F_{N7}*. Define the function* $N':U\times\boldsymbol{R}\to[0,1]$ *as follows:*

$$N'(x,t)=\begin{cases}\vee\{\alpha\in(0,1):\|x\|_{\alpha}\leq t\} & (x,t)\neq(0,0)\\ 0 & (x,t)=(0,0)\end{cases}$$

Then
a)N' *is a fuzzy norm on* U.
b) $N=N'$.

Lemma 1.2. *[1] Let* (U,N) *be a fuzzy norm space such that satisfy conditions* F_{N6} *and* F_{N7}*. and* $\{x_n\}\subseteq U$*, Then* $\lim_{n\to\infty}N(x_n-x,t)=1$ *if and only if*

$$\lim_{n\to\infty}\|x_n-x\|_{\alpha}=0$$

for every $\alpha\in(0,1)$.

Note that the sequence $\{x_n\}\subseteq U$ converges if there exists a $x\in U$ such that

$$\lim_{n\to\infty}N(x_n-x,t)=1\ for\ every\ t\in\mathbf{R}^{+}.$$

In this case x is called the limit of $\{x_n\}$.

Example 1.1. *[1] Let* V *be the Real or Complex vector space and let* N *define on* $V\times R$ *as follows :*

$$N(x,t)=\begin{cases}1 & t>|x|\\ 0 & t\leq|x|\end{cases}$$

for all $x\in V$ *and* $t\in R$*. Then* (V,N) *is a fuzzy norm space and the function* N *satisfy conditions* F_{N6} *and* $\|x\|_{\alpha}=|x|$ *for every* $\alpha\in(0,1)$.

Definition 1.2. *[11] A mapping* $T:U\to U$ *is an* $a-$*contraction if there exists* $a\in(0,1)$ *such that*

$$\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq a\wedge\{t>0:N(u-v,t)\geq\alpha\},\ \forall u,v\in U.$$

2 Fuzzy Fixed Point of Generalized Eventual Cyclic Gross Contractive Mappings

In this section we Let (U,N) be a fuzzy norm space, and let A and B be nonempty subsets of U.

Definition 2.1. *Let (U,N) be a fuzzy norm space, $T : A\cup B \to A\cup B$ is called a fuzzy generalized eventual cyclic gross contractive mapping if the following are satisfied:*

(i) $T(A) \subset B$ *and* $T(B) \subset A$,

(ii) *for some* $u \in A$,

$$f(\wedge\{t>0 : N(T^{2n}u-Tv,t)\geq\alpha\}) \quad \leq \quad f(\lambda\wedge\{t>0 : N(T^{2n-1}u-Tv,t)\geq\alpha\}$$

$$+(1-\lambda)\wedge\{t>0 : N(T^{2n}u-Tv,t)\geq\alpha\})-g(\wedge\{t>0 : N(T^{2n-1}u-Tv,t)\geq\alpha\},$$

$$\wedge\{t>0 : N(T^{2n}u-Tv,t)\geq\alpha\}), n\geq n_0\in N, \;\; v\in A. \tag{2.1}$$

where $\lambda\in[0,1)$, $f : R^+\to R^+$ *is a monotone increasing and continuous function,* $g : R^+\times R^+\to R^+$ *is a lower semi continuous mapping such that* $g(a,b)=0$ *if and only if* $a=b=0$, *and* n_0 *is sufficiently large.*

Lemma 2.1. *Let (U,N) be a fuzzy norm space, $T : A\cup B\to A\cup B$ be a generalized eventual cyclic gross contractive mapping and $\lambda\in(\frac{1}{2},1]$. Then $\{T^n x\}$ is a Cauchy sequence for every $x\in A\cup B$.*

Proof. For every $x\in A\cup B$, let

$$n\geq n_0, \;\; y=T^{2n-2}x. \tag{2.2}$$

Then, (2.1) and the monotone increasing property of f imply that

$$\wedge\{t>0 : N(T^{2n}u-T^{2n-1}u,t)\geq\alpha\} \quad \leq \quad (1-\lambda)\wedge\{t>0 : N(T^{2n}u-T^{2n-2}u,t)\geq\alpha\}$$

$$\leq \quad (1-\lambda)\wedge\{t>0 : N(T^{2n}u-T^{2n-1}u,t)\geq\alpha\}$$

$$+ \quad (1-\lambda)\wedge\{t>0 : N(T^{2n-1}u-T^{2n-2}u,t)\geq\alpha\}, \tag{2.3}$$

since

$$f(\wedge\{t>0 : N(T^{n}u-Tv,t)\geq\alpha\}) \quad = \quad f(\wedge\{t>0 : N(T^{2n}u-T^{2n-1}u,t)\geq\alpha\}$$

$$\leq f(\lambda\wedge\{t>0 : N(T^{2n-1}u-T^{2n-1}u,t)\geq\alpha\}+(1-\lambda)\wedge\{t>0 : N(T^{2n}u-T^{2n-2}u,t)\geq\alpha\})$$

$$- \quad g(\wedge\{t>0 : N(T^{2n-1}u-T^{2n-1}u,t)\geq\alpha\},\wedge\{t>0 : N(T^{2n}u-T^{2n-2}u,t)\geq\alpha\})$$

$$= \quad f((1-\lambda)\wedge\{t>0 : N(T^{2n}u-T^{2n-2}u,t)\geq\alpha\})-g(0,\wedge\{t>0 : N(T^{2n}u-T^{2n-2}u,t)\geq\alpha\})$$

$$\leq f((1-\lambda)\wedge\{t>0 : N(T^{2n}u-T^{2n-2}u,t)\geq\alpha\}) \tag{2.4}$$

Hence,

$$\wedge\{t>0:N(T^{2n}u-T^{2n-1}u,t)\geq\alpha\} \leq \frac{1-\lambda}{\lambda}\wedge\{t>0:N(T^{2n-1}u-T^{2n-2}u,t)\geq\alpha\}$$

Thus,

$$\wedge\{t>0:N(T^{n}u-T^{n+1}u,t)\geq\alpha\} \leq \frac{1-\lambda}{\lambda}\wedge\{t>0:N(T^{n-1}u-T^{n}u,t)\geq\alpha\}$$

Since $\lambda\in(\frac{1}{2},1]$, we deduce that $lim_{n\to\infty}\wedge\{t>0:N(T^{n}u-T^{n+1}u,t)\geq\alpha\}$ exists. Set $lim_{n\to\infty}\wedge\{t>0:N(T^{n}u-T^{n+1}u,t)\geq\alpha\}=l\geq 0$. If $l>0$, then (2.3) implies that

$$l\leq(1-\lambda)lim_{n\to\infty}\wedge\{t>0:N(T^{2n}u-T^{2n-2}u,t)\geq\alpha\}\leq 2(1-\lambda)l.$$

Therefore, we have $\lambda\leq\frac{1}{2}$, which is a contradiction. So,

$$lim_{n\to\infty}\wedge\{t>0:N(T^{n}u-T^{n+1}u,t)\geq\alpha\}=0.$$

This means that, for any $\varepsilon>0$, there exists a natural number N_0 such that for any natural number $n\geq N_0$,

$$\wedge\{t>0:N(T^{n}u-T^{n+1}u,t)\geq\alpha\}<\varepsilon.$$

Furthermore, for any natural number $m>n>0$, we have

$$\begin{aligned}\wedge\{t>0:N(T^{n+N_0}u-T^{m+N_0}u,t)\geq\alpha\} &\leq \sum_{i=n}^{m-1}\wedge\{t>0:N(T^{N_0+i}u-T^{N_0+i+1}u,t)\geq\alpha\}\\ &\leq \sum_{i=n}^{m-1}(\frac{1-\lambda}{\lambda})^{i}\wedge\{t>0:N(T^{N_0}u-T^{N_0+1}u,t)\geq\alpha\}\\ &< \frac{(\frac{1-\lambda}{\lambda})^{n}}{1-(\frac{1-\lambda}{\lambda})}\varepsilon.\end{aligned}$$

So, $lim_{n\to\infty}\wedge\{t>0:N(T^{n+N_0}u-T^{m+N_0}u,t)\geq\alpha\}=0$, since $\frac{1-\lambda}{\lambda}\in[0,1)$. Therefore, $\{T^{n}u\}$ is a Cauchy sequence. □

Lemma 2.2. *Let (U,N) be a fuzzy norm space, $T:A\cup B\to A\cup B$ be a generalized eventual cyclic gross contractive mapping and $\lambda\in[0,\frac{1}{2}]$. Then $\{T^{n}u\}$ is a Cauchy sequence for every $x\in A\cup B$.*

Proof. For every $x\in A\cup B$, let

$$n\geq n_0,\ \ y=T^{2n}x. \tag{2.5}$$

Then, (2.1) and the monotone increasing property of f imply that

$$\wedge\{t>0:N(T^{2n}u-T^{2n+1}u,t)\geq\alpha\} \leq \lambda\wedge\{t>0:N(T^{2n-1}u-T^{2n+1}u,t)\geq\alpha\}$$

$$\leq \lambda \wedge \{t>0 : N(T^{2n-1}u - T^{2n}u,t) \geq \alpha\} + \lambda \wedge \{t>0 : N(T^{2n}u - T^{2n+1}u,t) \geq \alpha\}, \tag{2.6}$$

since

$$f(\wedge\{t>0 : N(T^{n}u - Tv,t) \geq \alpha\}) \quad = \quad f(\wedge\{t>0 : N(T^{2n}u - T^{2n+1}u,t) \geq \alpha\}$$

$$\leq f(\lambda \wedge \{t>0 : N(T^{2n-1}u - T^{2n+1}u,t) \geq \alpha\} + (1-\lambda) \wedge \{t>0 : N(T^{2n}u - T^{2n}u,t) \geq \alpha\})$$

$$-g(\wedge\{t>0 : N(T^{2n-1}u - T^{2n+1}u,t) \geq \alpha\}, \wedge\{t>0 : N(T^{2n}u - T^{2n}u,t) \geq \alpha\})$$

$$= \quad f(\lambda \wedge \{t>0 : N(T^{2n-1}u - T^{2n+1}u,t) \geq \alpha\}) - g(\wedge\{t>0 : N(T^{2n-1}u - T^{2n}u,t) \geq \alpha\}, 0)$$

$$\leq f(\lambda \wedge \{t>0 : N(T^{2n-1}u - T^{2n+1}u,t) \geq \alpha\}) \tag{2.7}$$

Thus,

$$\wedge\{t>0 : N(T^{2n}u - T^{2n+1}u,t) \geq \alpha\} \quad \leq \quad \frac{\lambda}{1-\lambda} \wedge \{t>0 : N(T^{2n-1}u - T^{2n}u,t) \geq \alpha\}$$

So,

$$\wedge\{t>0 : N(T^{n}u - T^{n+1}u,t) \geq \alpha\} \quad \leq \quad \frac{\lambda}{1-\lambda} \wedge \{t>0 : N(T^{n-1}u - T^{n}u,t) \geq \alpha\}$$

Since $\lambda \in [0, \frac{1}{2}]$, we implies that for all $n \geq n_0$, the nonnegative sequence $\{\wedge\{t>0 : N(T^{n}u - T^{n+1}u,t) \geq \alpha\}\}$ is decreasing. Let $lim_{n\to\infty} \wedge \{t>0 : N(T^{n}u - T^{n+1}u,t) \geq \alpha\} = l \geq 0$. So, we have

$$l \leq \lambda lim_{n\to\infty} \wedge \{t>0 : N(T^{2n-1}u - T^{2n+1}u,t) \geq \alpha\} \leq 2l\lambda.$$

If $l > 0$, Then we have $\lambda \geq \frac{1}{2}$, which is a contradiction. Furthermore, $\lambda = \frac{1}{2}$ is also impossible by [4]. So,

$$lim_{n\to\infty} \wedge \{t>0 : N(T^{n}u - T^{n+1}u,t) \geq \alpha\} = 0.$$

This means that, for any $\varepsilon > 0$, there exists a natural number N_0 such that for any natural number $n \geq N_0$,

$$\wedge\{t>0 : N(T^{n}u - T^{n+1}u,t) \geq \alpha\} < \varepsilon.$$

Furthermore, for $\lambda \in [0, \frac{1}{2})$ and any natural number $m > n > 0$, we have

$$\begin{aligned} \wedge\{t>0 : N(T^{n+N_0}u - T^{m+N_0}u,t) \geq \alpha\} \quad &\leq \quad \sum_{i=n}^{m-1} \wedge\{t>0 : N(T^{N_0+i}u - \\ &\qquad T^{N_0+i+1}u,t) \geq \alpha\} \\ &\leq \quad \sum_{i=n}^{m-1} (\frac{\lambda}{1-\lambda})^{i} \wedge \{t>0 : N(T^{N_0}u - \\ &\qquad T^{N_0+1}u,t) \geq \alpha\} \\ &< \quad \frac{(\frac{\lambda}{1-\lambda})^{n}}{1-(\frac{\lambda}{1-\lambda})} \varepsilon. \end{aligned}$$

So, $lim_{n\to\infty} \wedge \{t>0 : N(T^{n+N_0}u - T^{m+N_0}u,t) \geq \alpha\} = 0$, since $\frac{\lambda}{1-\lambda} \in [0,1)$. Therefore, $\{T^{n}u\}$ is a Cauchy sequence. For $\lambda = \frac{1}{2}$, by [4], we know that $\{T^{n}u\}$ also is a Cauchy sequence.

□

Theorem 2.1. *Let (U,N) be a complete fuzzy norm space, A and B are closed, and $T : A\cup B \to A\cup B$ be a generalized eventual cyclic gross contractive mapping. Then, $A\cap B$ is nonempty and T has a unique fixed point in $A\cap B$.*

Proof. By virtue of Lemma 2.1 and Lemma 2.2, we know that for every $u \in A\cup B$, $\{T^n u\}$ is a Cauchy sequence. Since (U,N) is a complete fuzzy norm space, A and B are closed, there exists some $u_0 \in A\cup B$ such that

$$lim_{n\to\infty} T^n u = u_0$$

Therefore,

$$u_0 = lim_{n\to\infty} T^{2n+1} u \in A,$$
$$u_0 = lim_{n\to\infty} T^{2n} u \in B.$$

So, $u_0 \in A\cap B$; that is $A\cap B \neq \emptyset$. On the other hand, we obtain

$$\begin{aligned} f(\wedge\{t>0 : N(Tu_0 - u_0,t) \geq \alpha\}) \quad &\leq \quad f(\lambda \wedge \{t>0 : N(u_0 - u_0,t) \geq \alpha\} \\ &+ f((1-\lambda)\wedge\{t>0 : N(Tu_0 - u_0) \geq \alpha\} \\ &- g(\wedge\{t>0 : N(u_0 - u_0,t) \geq \alpha\}, \\ &\wedge\{t>0 : N(Tu_0 - u_0,t) \geq \alpha\}) \\ &= f((1-\lambda)\wedge\{t>0 : N(u_0 - Tu_0,t) \geq \alpha\}) \\ &- g(0, \wedge\{t>0 : N(Tu_0 - u_0,t) \geq \alpha\}), \end{aligned} \tag{2.8}$$

in view of

$$\begin{aligned} f(\wedge\{t>0 : N(Tu_0 - T^{2n}u,t) \geq \alpha\}) \quad &\leq \quad f((1-\lambda)\wedge\{t>0 : N(Tu_0 - T^{2n}u) \geq \alpha\} \\ &- \quad g(\wedge\{t>0 : N(u_0 - T^{2n}u,t) \geq \alpha\}, \\ &\wedge\{t>0 : N(Tu_0 - T^{2n}u,t) \geq \alpha\}). \end{aligned}$$

If $\wedge\{t>0 : N(Tu_0 - u_0,t) \geq \alpha\} = 0$, then u_0 is a fixed point of T. Otherwise, if

$$\wedge\{t>0 : N(Tu_0 - u_0,t) \geq \alpha\} > 0,$$

then we have $g(0, \wedge\{t>0 : N(Tu_0 - u_0,t) \geq \alpha\}) > 0$. Equation (2.8) implies that

$$f(\wedge\{t>0 : N(Tu_0 - u_0,t) \geq \alpha\}) \quad \leq \quad f((1-\lambda)\wedge\{t>0 : N(u_0 - Tu_0,t) \geq \alpha\}).$$

Hence, $\wedge\{t>0 : N(Tu_0 - u_0,t) \geq \alpha\} < (1-\lambda)\wedge\{t>0 : N(u_0 - Tu_0,t) \geq \alpha\}$, This is impossible since $(1-\lambda) \in [0,1]$. According to previous discussions, it is concluded that

$$\wedge\{t>0 : N(Tu_0 - u_0,t) \geq \alpha\} = 0,$$

and therefore, u_0 is a fixed point of T. Now, if there is $v_0 \in A\cap B$ such that $Tv_0 = v_0$, then $\wedge\{t>0 : N(u_0 - v_0,t) \geq \alpha\} = 0$; that is, the fixed point of T is unique, since

$$g(\wedge\{t>0 : N(u_0 - v_0,t) \geq \alpha\} < \wedge\{t>0 : N(u_0 - v_0,t) \geq \alpha\}) \leq 0,$$

in view of

$$\begin{aligned}
f(\wedge\{t>0:N(u_0-v_0,t)\geq\alpha\}) &\leq f(\wedge\{t>0:N(Tu_0-Tv_0,t)\geq\alpha\}\\
&\leq f(\lambda\wedge\{t>0:N(u_0-Tv_0)\geq\alpha\}\\
&+ g((1-\lambda)\wedge\{t>0:N(Tu_0-v_0,t)\geq\alpha\}\\
&- g(\wedge\{t>0:N(u_0-Tv_0,t)\geq\alpha\},\\
&\quad \wedge\{t>0:N(Tu_0-v_0,t)\geq\alpha\})\\
&\leq f(\wedge\{t>0:N(u_0-v_0,t)\geq\alpha\})\\
&- g(\wedge\{t>0:N(u_0-v_0,t)\geq\alpha\},\\
&\quad \wedge\{t>0:N(u_0-v_0,t)\geq\alpha\}).
\end{aligned}$$

The proof is completed. □

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