Some results on t-nearest points in fuzzy normed linear spaces

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Abstract
In this paper we consider t-best nearest point and find algorithms for find t-best nearest points in fuzzy normed linear space.

Keywords: t-Nearest point, Fuzzy normed space, Distance, Convergent algorithm, t-norm, Fuzzy normed spaces.

1 Introduction
The concept of fuzzy sets was introduced by Zadeh [9]. It was developed extensively by many authors and used in various fields. To use this concept in topology and analysis, several researchers have defined fuzzy metric spaces in various ways. (see [1]-[9]).

2 Preliminaries and notations
In this paper we consider the set of all t-nearest point and t-nearest points on fuzzy normed spaces and prove several theorems pertaining to this set.

Definition 2.1. ([6]) A binary operation \(\ast: [0, 1] \times [0, 1] \rightarrow [0, 1]\) is said to be a continuous t-norm if \(([0, 1], \ast)\) is a topological monoid with unit 1 such that \(a \ast b \leq c \ast d\) whenever \(a \leq c \) and \(b \leq d\) (\(a, b, c, d \in [0, 1]\)).

Definition 2.2. ([6]) The 3-tuple \((X, N, \ast)\) is said to be a fuzzy normed space if \(X\) is a vector space, \(\ast\) is a continuous t-norm and \(N\) is a fuzzy set on \(X\times[0, \infty)\) satisfying the following conditions for every \(x, y \in X\) and \(t, s > 0\),

(i) \(N(x, t) > 0\),

(ii) \(N(x, t) = 1 \iff x = 0\),

(iii) \(N(\alpha x, t) = N(x, t/|\alpha|),\) for all \(\alpha \neq 0\),

(iv) \(N(x, t) \ast N(y, s) = N(x + y, t + s)\),

(v) \(N(x, \cdot) : (0, \infty) \rightarrow [0, 1]\) is continuous,

(vi) \(\lim_{t \to \infty} N(x, t) = 1\).

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Lemma 2.1. (\cite{6}) Let $N$ be a fuzzy norm. Then:

(i) $N(x,t)$ is nondecreasing with respect to $t$ for each $x \in X$,

(ii) $N(x-y,t) = N(y-x,t)$.

As was shown in \cite{6}, every fuzzy normed space induces a fuzzy metric space on it and is therefore a topological space.

Definition 2.3. (\cite{6}) Let $(X, N, \ast)$ be a fuzzy normed space. The open ball $B(x, r, t)$ and the closed ball $B[x, r, t]$ with the center $x \in X$ and radius $0 < r < 1$, $t > 0$ are defined as follows:

$$B[x, r, t] = \{y \in X : N(x-y, t) > 1-r\}.$$ 

Lemma 2.2. (\cite{6}) If $(X, N, \ast)$ is a fuzzy normed space. Then:

(i) the function $(x, y) \to x + y$ is continuous,

(ii) the function $(a, x) \to ax$ is continuous.

Definition 2.4. (\cite{7}) Let $A$ be a nonempty subset of a fuzzy normed space $(X, N, \ast)$. For $x \in X$, $t > 0$, let

$$d(x, A, t) = \sup\{N(y-x, t) : y \in A\}.$$ 

An element $y_0 \in A$ is said to be a $t$-nearest point of $x$ from $A$ if

$$N(y_0-x, t) = d(A, x, t).$$

Definition 2.5. (\cite{7}) Let $A$ be a nonempty set of a fuzzy normed space $(X, N, \ast)$. For $x \in X$, $t > 0$, we shall denote the set of all elements of $t$-nearest point of $x$ from $A$ by $P^t_A(x)$; i.e.,

$$P^t_A(x) = \{y \in A : d(A, x, t) = N(y-x, t)\}.$$ 

3 t-nearest point

Theorem 3.1. Let $(X, M, \ast)$ be a fuzzy normed space and $M$ is a nonempty subset of $X$. If $x \in X \setminus M$, then there exist a minimizing sequence $\{g_k\}_{k \geq 1}$ in $M$ $N(x-g_k, t) \to d(x, M, t)$, a decreasing sequence $\{\alpha_k\}_{k \geq 1}$ and an increasing sequence $\{\beta_k\}_{k \geq 1}$ of positive real numbers such that $\beta_k < \alpha_k$, $\beta_k \leq N(x-g_k, t) \leq d(x, M, t) \leq \alpha_k$ and $\alpha_k - \beta_k = 1/2^{k-1}(\alpha_1 - \beta_1)$ for all $k \in N$.

Remark 3.1. Suppose $y_1$ is an arbitrary point in $M$ and $\alpha_1 = N(x - y_1, t)$ such that $B[x, \alpha_1, t] \cap M \neq \emptyset$. Choose $g_1 \in B[x, \alpha_1, t] \cap M$. We can choose positive real number $\beta_1$ such that $0 < \beta_1 < \alpha_1$ and $B[x, \beta_1, t] \cap M = \emptyset$. Assume that $\gamma_1 = (\alpha_1 + \beta_1)/2$, then we can consider two cases:

1. If $B[x, \gamma_1, t] \cap M \neq \emptyset$, choose $g_2 \in B[x, \gamma_1, t] \cap M$, put $\alpha_2 = \gamma_1$ and $\beta_2 = \beta_1$. In this case $\beta_2 = \beta_1 < N(x - g_2, t) \leq \alpha_2 = \gamma_1$.

2. If $B[x, \gamma_1, t] \cap M = \emptyset$, put $\alpha_2 = \alpha_1$, $\beta_2 = \gamma_1$ and $g_2 := g_1$. Also in this case $\beta_2 = \gamma_1 < N(x - g_2, t) \leq \alpha_2 = \alpha_1$.

Continuing this process $k$-times, we can find positive real numbers $\alpha_1, \beta_1$, $\alpha_2, \beta_2, \ldots, \alpha_k, \beta_k$ such that for $1 \leq i \leq k$, $0 < \beta_i < \alpha_i$. We have $g_k \in B[x, \alpha_k, t] \cap M$ and $B[x, \beta_k, t] \cap M = \emptyset$, also $\beta_i < N(x - g_k, t) \leq \alpha_i$ for all $1 \leq i \leq k$. Now put $\gamma_k = (\alpha_k + \beta_k)/2$, and consider the following two cases:

1. If $B[x, \gamma_k, t] \cap M \neq \emptyset$, put $\alpha_{k+1} = \gamma_k$, $\beta_{k+1} = \beta_k$ and choose $g_{k+1} \in B[x, \gamma_k, t] \cap M$.

2. If $B[x, \gamma_k, t] \cap M = \emptyset$, put $\alpha_{k+1} = \alpha_k$, $\beta_{k+1} = \gamma_k$ and $g_{k+1} = g_k$.
In each of above cases we have for all \( k \geq 1 \), \( \beta_k < \beta_{k+1} < \alpha_{k+1} \leq \alpha_k \) we have \( g_k \in B(x, \alpha_k, t) \cap M \) and \( B(x, \beta_k) \cap M = \emptyset \). Moreover, we have

\[
|\alpha_{k+1} - \beta_{k+1}| = 1/2|\alpha_k - \beta_k|.
\]

(3.1)

It follows that for all \( k \geq 1 \),

\[
(\alpha_k - \beta_k) = 1/2^{(k-1)}(\alpha_1 - \beta_1)
\]

(3.2)

Now since \( d(x, M, t) \leq \alpha_k \), then

\[
\beta_k < N(x - g_k, t) \leq d(x, M, t) \leq \alpha_k.
\]

(3.3)

From (3.2) and (3.3) we have

\[
\lim_{k \to \infty} N(x - g_k, t) = \lim_{k \to \infty} \alpha_k = \lim_{k \to \infty} \beta_k = d(x, M, t).
\]

Algorithm

Suppose \( (X, N, +) \) is a fuzzy normed space, \( M \) is a nonempty subset of \( X \) and \( x \in X \setminus M \), let \( \varepsilon \geq 0 \) be given. Then

1. Put \( k = 1 \), choose \( 0 < \beta_k < \alpha_k \) such that \( B(x, \alpha_k, t) \cap M \neq \emptyset \) and \( B(x, \beta_k, t) \cap M = \emptyset \).
2. If \( |\alpha_k - \beta_k| < \varepsilon \), put \( d(x, M, t) := \alpha_k \), and stop, else
3. Put \( \gamma_k := (\alpha_k + \beta_k)/2 \).
4. If \( B(x, \gamma_k, t) \cap M = \emptyset \), put \( \alpha_{k+1} = \alpha_k \), \( \beta_{k+1} = \gamma_k \) and \( k := k + 1 \), then go to step 2.
5. Else, put \( \alpha_{k+1} := \gamma_k \), \( \beta_{k+1} := \beta_k \), \( k := k + 1 \), and then go to step 3.

Proposition 3.1. The above algorithm shows that \( d(x, M, t) \) is a number in the interval \([\beta_1, \alpha_1]\). Because if \( \alpha_{k+1} = d(x, M, t) \) and change to formula \( \gamma_k := (\alpha_k + \beta_k)/2 \), \( d(x, M, t) \) is the number in the interval \([\beta_1, \alpha_1]\). In attention to calculate \( \gamma_k \), we have

\[
|\gamma_1 - d(x, M, t)| < (\alpha_1 - \beta_1)/2.
\]

Also, it can be seen that \( \gamma_2 \in [\gamma_1, \beta_1] \), then

\[
|\gamma_2 - d(x, M, t)| < (\alpha_1 - \beta_1)/2^2.
\]

After the the \( n \)-times iteration, then

\[
0 < |\gamma_n - d(x, M, t)| < (\alpha_1 - \beta_1)/2^n.
\]

Therefore

\[
\lim_{n \to \infty} |\gamma_n - d(x, M, t)| = 0,
\]

and

\[
\lim_{n \to \infty} \gamma_n = d(x, M, t).
\]

The upper relationship shows convergent algorithm to \( d(x, M, t) \). In this algorithm distance the point to the set is approximate, then since calculate approximate of the unknown, have mean calculation error. With considering \((\alpha_1 - \beta_1)/2^n \) instead error bounded, we wants algorithm stop when erroring smaller than of \( \varepsilon \). In this case, the minimum number of times, for that algorithm error smaller than of following equation

\[
|\gamma_n - d(x, M, t)| < (\alpha_1 - \beta_1)/2^n \leq \varepsilon
\]

\[
(\alpha_1 - \beta_1)/2^n \leq \varepsilon \Rightarrow 2^n \leq (\alpha_1 - \beta_1)\varepsilon
\]

If \( n \) is stratify in this relation such that \( d(x, M, t) \in [\beta_1, \alpha_1] \), then

\[
n \geq \log_2(\alpha_1 - \beta_1)/\varepsilon.
\]
Algorithm
Let \((X, N, *)\) be a fuzzy normed space and \(M\) is a nonempty subset of \(X\) and \(x \in X \setminus M\). This algorithm approximates a nearest point for \(x\) arbitrarily good.

Let \(\epsilon > 0\) be given. Then

1. Put \(k = 1\), choose \(0 < \beta_k < \alpha_k\) such that \(g_k \in B[x, \alpha_k, t] \cap M\) and \(B[x, \beta_k, t] \cap \emptyset = \emptyset\).
2. If \((\text{diam}[x, \alpha_k, t] \cap M) < \epsilon\), put \(g_0 := g_k\) and stop.
3. Put \(\gamma - k = (\alpha - k + \beta - k)/2\).
4. If \(g_0 \in B[x, \gamma_k, t] \cap M\), then put \(g_k := g_0\), \(\alpha_k + 1 := \gamma_k\), \(\beta_k + 1 := \beta_k\) and \(k := k + 1\) then go to step 2.
5. Else put \(\alpha_{k+1} := \alpha_k\), \(\beta_{k+1} := \gamma_k\) and \(k := k + 1\) then go to step 3.

Also in algorithm find nearest point since nearest point is the number between \(\alpha_1, \beta_1\), similar recently algorithm we can obtained the minimum number of times, for nearest point with arbitrary \(\epsilon\), then

\[
n \geq \ln(\alpha_1 - \beta_1)/\epsilon.
\]

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References

http://dx.doi.org/10.1155/AAA.2005.259


http://dx.doi.org/10.1016/0165-0114(92)90338-5

http://dx.doi.org/10.1016/1016-0114(94)90338-5

http://dx.doi.org/10.1016/S0165-0114(96)00207-2

http://dx.doi.org/10.1016/j.na.2005.09.026


http://dx.doi.org/10.1016/S0016-9958(65)90241-X