A new approach for the sequence spaces of fuzzy level sets with the partial metric

Uğur Kadak\textsuperscript{1,2*}, Muharrem Özlik\textsuperscript{1}

(1) Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey
(2) Department of Mathematics, Faculty of Science, Bozok University, Yozgat, Turkey

Abstract
In this paper, we investigate the classical sets of sequences of fuzzy numbers by using partial metric which is based on a partial ordering. Some elementary notions and concepts for partial metric and fuzzy level sets are given. In addition, several necessary and sufficient conditions for partial completeness are established by means of fuzzy level sets. Finally, we give some illustrative examples and present some results between fuzzy and partial metric spaces.

Keywords: Sequence space, partial metric space, fuzzy level sets, complete metric space, fuzzy completeness

1 Introduction

Introduced in 1992, a partial metric space is a generalization of the notion of metric space defined in 1906 by Maurice Frechet such that the distance of a point from itself is not necessarily zero. This notion has a wide array of applications not only in many branches of mathematics, but also in the field of computer domain and semantics. Motivated by the needs of computer science for non Hausdorff Scott topology, one show that much of the essential structure of metric spaces, such as Banach’s contraction mapping theorem, can be generalised to allow for the possibility of non zero self-distances $d(x, x)$. The discipline of mathematics has traditionally taken zero self distance for granted because, before computer science, there was little reason to consider the computability of a metric distance $d(x, y)$. More precisely, mathematics has understandably assumed that each metric distance is a totally defined structure. To assert that $d(x, x) = 0$ for each $x \in X$ is in computational terms a useful means to specify that $x$ is totally computed. On the other hand; the effectiveness of level sets comes from not only their required memory capacity for fuzzy sets, but also their two valued nature. This nature contributes to an effective derivation of the fuzzy-inference algorithm based on the families of the level sets. Besides this, the definition of fuzzy sets by level sets offers advantages over membership functions, especially when the fuzzy sets are in universes of discourse with many elements. This definition considerably reduces the required memory capacity for the fuzzy sets and the processing time for fuzzy inference. In this study, the relations between fuzzy level sets and partial metric are introduced and discussed think about distance function.

*Corresponding author. Email address: ugurkadak@gmail.com, Tel:+905306403218.
The final section is devoted to the completeness of the sets concerning the fuzzy level sets, we give some lemmas about the different types of partial metric.

Section 2 is devoted to the partial distance functions of fuzzy numbers with the level sets. Prior to stating and proving follows:

The main purpose of the present paper is to study the corresponding sets \( \ell_\infty(H) \), \( c(H) \), \( c_0(H) \) and \( \ell_p(H) \) consisting of the bounded, convergent, null and p-summable sequences of fuzzy numbers with the partial metric \( H^p \), as follows:

\[
\ell_\infty(H) := \left\{ u = (u_k) \in \omega(F) : \sup_{k \in \mathbb{N}} H^p(u_k, \bar{0}) < \infty, \quad \bar{0} \in E^1 \right\},
\]
\[
c(H) := \left\{ u = (u_k) \in \omega(F) : \lim_{k \to \infty} H^p(u_k, u) = 0 \quad \text{for some} \quad u \in E^1 \right\},
\]
\[
c_0(H) := \left\{ u = (u_k) \in \omega(F) : \lim_{k \to \infty} H^p(u_k, \bar{0}) = 0 \right\},
\]
\[
\ell_p(H) := \left\{ u = (u_k) \in \omega(F) : \sum_{k=0}^{\infty} H^p(u_k, \bar{0})^p < \infty \right\}, \quad (1 \leq p < \infty),
\]

where the distance function \( H^p \) denote the partial metric of fuzzy numbers with the level sets defined by

\[
H(u, v) = \sup_{\lambda \in [0, 1]} p([u^\lambda], [v^\lambda]) = \sup_{\lambda \in [0, 1]} \left\{ \max\{u^\lambda_+, v^\lambda_+\} - \min\{u^\lambda_-, v^\lambda_-\} \right\} = \max\{u^+_0, v^+_0\} - \min\{u^-_0, v^-_0\}
\]

and

\[
H^p(u, v) = 2H(u, v) - H(u, u) - H(v, v),
\]

for any \( u, v \in E^1 \) with the partial ordering \( \sqsubseteq_H \). One can show that \( \ell_\infty(H) \), \( c(H) \) and \( c_0(H) \) are complete metric spaces with the partial metric \( H_\infty \) defined by

\[
H_\infty(u, v) := \sup_{k \in \mathbb{N}} \{ H^p(u_k, v_k) \}.
\]

where \( u = (u_k) \) and \( v = (v_k) \) are the elements of the sets \( c(H) \), \( c_0(H) \) or \( \ell_\infty(H) \). Also, the space \( \ell_p(H) \) is complete metric space with the partial metric \( H_p \) defined by

\[
H_p(u, v) := \left\{ \sum_{k=0}^{\infty} H^p(u_k, v_k)^p \right\}^{1/p}, \quad (1 \leq p < \infty),
\]

where \( u = (u_k) \) and \( v = (v_k) \) are the points of \( \ell_p(H) \).

The main purpose of the present paper is to study the corresponding sets \( \ell_\infty(H) \), \( c(H) \), \( c_0(H) \) and \( \ell_p(H) \) of sequences of fuzzy numbers by using partial metric to the classical sequence spaces. The rest of this paper is organized, as follows:

Section 2 is devoted to the partial distance functions of fuzzy numbers with the level sets. Prior to stating and proving the main results concerning the fuzzy level sets, we give some lemmas about the different types of partial metric. The final section is devoted to the completeness of the sets \( \ell_\infty(H) \), \( c(H) \), \( c_0(H) \) and \( \ell_p(H) \) by taking into account the partially ordering and some related examples.

2 Preliminaries, Background and Notation

Nonzero self-distance is thus motivated by experience from computer science, and seen to be plausible for the example of finite and infinite sequences. The question we now ask is whether nonzero self-distance can be introduced to any metric space. That is, is there a generalization of the metric space axioms to introduce nonzero self-distance such that familiar metric and topological properties are retained? The following is suggested.
Proposition 2.1. [14] (Nonzero self-distance) Let $\mathcal{S}^\omega$ be the set of all infinite sequences $x = (x_0, x_1, x_2, \ldots)$ over a set $X$. For all such sequences $x$ and $y$ let $d^i_\omega(x, y) = 2^{-k}$, where $k$ is the largest number (possibly $\infty$) such that $x_i = y_i$ for each $i < k$. Thus $d^i_\omega(x, y)$ is defined to be $1$ over $2$ to the power of the length of the longest initial sequence common to both $x$ and $y$. It can be shown that $(\mathcal{S}^\omega, d^\omega)$ is a metric space.

To be interested in an infinite sequence $x$ they would want to know how to compute it, that is, how to write a computer program to print out the values $x_0$, then $x_1$, then $x_2$, and so on. As $x$ is an infinite sequence, its values cannot be printed out in any finite amount of time, and so computer scientists are interested in how the sequence $x$ is formed from its parts, the finite sequences $(x_0), (x_0, x_1), (x_0, x_1, x_2)$ and so on. After each value $x_k$ is printed, the finite sequence $x = (x_0, x_1, x_2, \ldots, x_k)$ represents that part of the infinite sequence produced so far. Each finite sequence is thus thought of in computer science as being a partially computed version of the infinite sequence $x$, which is totally computed. Suppose now that the above definition of $d^\omega$ is extended to $\mathcal{S}^*$, the set of all finite sequences over $S$. If $x$ is a finite sequence then $d^\omega_k(x, x) = 2^{-k}$ for some number $k < \infty$, which is not zero, since $x_j = x_j$ can only hold if $x_j$ is defined.

Definition 2.1. [10] Let $X$ be a non-empty set and $p$ be a function from $X \times X$ to the set $\mathbb{R}^+$ of non-negative real numbers. Then the pair $(X, p)$ is called a partial metric space and $p$ is a partial metric for $X$, if the following partial metric axioms are satisfied for all $x, y, z \in X$:

(P1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,

(P2) $0 \leq p(x, x) \leq p(x, y)$,

(P3) $p(x, y) = p(y, x),

(P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Each partial metric space thus gives rise to a metric space with the additional notion of nonzero self-distance introduced. Also, a partial metric space is a generalization of a metric space; indeed, if an axiom $p(x, x) = 0$ is imposed, then the above axioms reduce to their metric counterparts. Thus, a metric space can be defined to be a partial metric space in which each self-distance is zero.

It is clear that $p(x, y) = 0$ implies $x = y$ from (P1) and (P2). But, $x = y$ does not imply $p(x, y) = 0$, in general. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$.

Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ which has as a base the family open $p$-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Remark 2.1. [9] Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(\ldots)$ need not be continuous in the sense that $x_n \rightarrow x$ and $y_n \rightarrow y$ implies $p(x_n, y_n) \rightarrow p(x, y)$. For example, if $X = [0, +\infty)$ and $p(x, y) = \max\{x, y\}$ for $x, y \in X$, then for $\{x_n\} = \{1\}$, $p(x_n, x) = x = p(x, x)$ for each $x \geq 1$ and so, e.g., $x_n \rightarrow 2$ and $x_n \rightarrow 3$ when $n \rightarrow \infty$.

Proposition 2.2. [15] If $p$ is a partial metric on $X$, then the function $p^\prime$ defined by

$$p^\prime : X \times X \rightarrow \mathbb{R}^+$$

$$(x, y) \mapsto p^\prime(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a usual metric on $X$. For example, in $(\mathbb{R}^+, p)$ where $p$ is the usual partial metric on $\mathbb{R}^+$, we obtain the usual distance in $\mathbb{R}^-$ since for any $x, y \in \mathbb{R}^-$, $p^\prime(x, y) = 2p(x, y) - p(x, x) - p(y, y) = x + y - 2\min\{x, y\} = |x - y|$.

Definition 2.2. [14] A partial order on $X$ is a binary relation $\sqsubseteq$ on $X$ such that

(i) $x \sqsubseteq x$ (reflexivity)

(ii) If $x \sqsubseteq y$ and $y \sqsubseteq x$ then $x = y$ (antisymmetry)

(iii) If $x \sqsubseteq y$ and $y \sqsubseteq z$ then $x \sqsubseteq z$ (transitivity).

A partially ordered set (or poset) is a pair $(X, \sqsubseteq)$ such that $\sqsubseteq$ is a partial order on $X$. For each partial metric space $(X, p)$ let $\sqsubseteq_p$ be the binary relation over $X$ such that $x \sqsubseteq_p y$ (to be read, $x$ is part of $y$) if and only if $p(x, x) = p(x, y)$. Then it can be shown that $(X, \sqsubseteq_p)$ is a poset.
Definition 2.3. [9] Let \((X, \leq)\) be a partially ordered set. Then:
(a) elements \(x, y \in X\) are called comparable if \(x \leq y\) or \(y \leq x\) holds;
(b) a subset \(\mathcal{H}\) of \(X\) is said to be well ordered if every two elements of \(\mathcal{H}\) are comparable;
(c) a mapping \(f : X \to X\) is called nondecreasing with respect to \(\leq\) if \(x \leq y\) implies \(f(x) \leq f(y)\).

For the partial metric space \(\max\{a, b\} \not\in p\) over the nonnegative reals, \(\max\) is the usual \(\geq\) ordering. For intervals, \([a, b] \subseteq \rho\) if and only if \([c, d]\) is a subset of \([a, b]\). Thus the notion of a partial metric extends that of a metric by introducing nonzero self-distance, which can then be used to define the relation is part of, which, for example, can be applied to model the output from a computer program.

Definition 2.4. (cf. [11, 15, 16, 1]) Let \((X, p)\) be a partial metric space and \((x_n)\) be a sequence in \((X, p)\). Then, we say that
(a) A sequence \((x_n)\) converges to a point \(x \in X\) if and only if \(p(x, x_n) = \lim_{n \to \infty} p(x_n, x)\).
(b) A sequence \((x_n)\) is a Cauchy sequence if there exists (and is finite) \(\lim_{n,m \to \infty} p(x_n, x_m)\).
(c) A partial metric space \((X, p)\) is said to be complete if every Cauchy sequence \((x_n)\) in \(X\) converges, respect to the topology \(\tau_p\), to a point \(x \in X\) such that \(p(x, x_n) = \lim_{n \to \infty} p(x_n, x)\). It is easy to see that, every closed subset of a partial metric space is complete.
(d) A mapping \(f : X \to X\) is called to be continuous at \(x_0 \in X\) if for every \(\epsilon > 0\), there exists \(\delta > 0\) such that \(f(B_p(x_0, \delta)) \subseteq B_p(f(x_0), \epsilon)\).
(e) A sequence \((x_n)\) in a partial metric space \((X, p)\) converges to a point \(x \in X\), for any \(\epsilon > 0\) such that \(x \in B_p(x, \epsilon)\), there exists \(n_0 \geq 1\) so that for any \(n \geq n_0\), \(x_n \in B_p(x, \epsilon)\).

Lemma 2.1. [15] Let \((X, p)\) be a partial metric space. Then,
(i) \((x_n)\) is a Cauchy sequence in \((X, p)\) if and only if it is a Cauchy sequence in the metric space \((X, p^*)\).

(ii) A partial metric space \((X, p)\) is complete if and only if the metric space \((X, p^*)\) is complete. Furthermore, \(\lim_{n \to \infty} p^*(x_n, x) = 0\) if and only if \(p(x, x_n) = \lim_{n \to \infty} p(x_n, x) = \lim_{n,m \to \infty} p(x_n, x_m)\).

In the partial metric space \((\mathbb{R}^+, p)\), the limit of the sequence \((-1/n)\) is 0 since one has \(\lim_{n \to \infty} p^*(-1/n, 0)\) where \(p^*\) is the usual metric induced by \(p\) on \(\mathbb{R}^+\).

Lemma 2.2. [9] Let \((X, p)\) be a partial metric space, \(f : X \to X\) be a given mapping. Suppose that \(f\) is continuous at \(x_0 \in X\) and for each sequence \(x_n \in X, x_n \to x_0\) in \(\tau_p\) then \(f(x_n) \to f(x_0)\) in \(\tau_p\) holds.

Definition 2.5. [1] Suppose that \((X_1, p_1)\) and \((X_2, p_2)\) are partial metric spaces with induced metrics \(p_1^*\) and \(p_2^*\) respectively. Then the function \(f: (X_1, p_1) \to (X_2, p_2)\) is said to be continuous if both \(f: (X_1, \tau_{p_1}) \to (X_2, \tau_{p_2})\) and \(f : (X_1, p_1^*) \to (X_2, p_2^*)\) are respectively continuous in the sense of topological spaces and metric spaces continuity.

Definition 2.6. [9] Let \(X\) be a nonempty set. Then \((X, p, \leq)\) is called an ordered (partial) metric space if:
(i) \((X, p)\) is a (partial) metric space,
(ii) \((X, \leq)\) is a partially ordered set.

Definition 2.7. [14] A sequence \(x = (x_n)\) of points in a partial metric space \((X, p)\) is Cauchy if there exists \(a \geq 0\) such that for each \(\epsilon > 0\) there exists \(k\) such that for all \(n, m > k\), \(|p(x_n, x_m)| = 0 < \epsilon\). In other words, \(x\) is Cauchy if the numbers \(p(x_n, x_m)\) converge to some \(a\) as \(n\) and \(m\) approach infinity, that is, \(\lim_{n,m \to \infty} p(x_n, x_m) = a\). Note that then \(\lim_{n \to \infty} p(x_n, x_m) = a\), and so if \((X, p)\) is a metric space then \(a = 0\).

Definition 2.8. [14] A sequence \(x = (x_n)\) of points in a partial metric space \((X, p)\) converges to \(y\) in \(X\) if \(p(y) = \lim_{n \to \infty} p(x_n, x_n) = \lim_{n \to \infty} p(x_n, y)\).

Thus if a sequence converges to a point then the self-distances converge to the selfdistance of that point.

Lemma 2.3. [13] Assume that \(x_n \to z\) as \(n \to \infty\) a partial metric space \((X, p)\) such that \(p(z, z) = 0\). Then \(\lim_{n \to \infty} p(x_n, y) = p(z, y)\) for every \(y \in X\).
2.1 The Level sets of fuzzy numbers

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $u : \mathbb{R} \to [0, 1]$ which satisfies the following four conditions:

(i) $u$ is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(ii) $u$ is fuzzy convex, i.e., $u[\lambda x + (1 - \lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0, 1]$.

(iii) $u$ is upper semi-continuous.

(iv) The set $[u]_0 = \{x \in \mathbb{R} : u(x) > 0\}$ is compact, (cf. Zadeh [4]), where $\{x \in \mathbb{R} : u(x) > 0\}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by $E^1$ and call it as the space of fuzzy numbers. $\lambda$-level set $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \lambda\}, & 0 < \lambda \leq 1, \\ \{x \in \mathbb{R} : u(x) > \lambda\}, & \lambda = 0. \end{cases}$$

The set $[u]_\lambda$ is closed, bounded and non-empty interval for each $\lambda \in [0, 1]$ which is defined by $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$. $\mathbb{R}$ can be embedded in $E^1$, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\tau$ defined by

$$\tau(x) = \begin{cases} 1, & x = r, \\ 0, & x \neq r. \end{cases}$$

Theorem 2.1. [7] Let $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ for $u \in E^1$ and for each $\lambda \in [0, 1]$. Then the following statements hold:

(i) $u^-$ is a bounded and non-decreasing left continuous function on $[0, 1]$.

(ii) $u^+$ is a bounded and non-increasing left continuous function on $[0, 1]$.

(iii) The functions $u^-$ and $u^+$ are right continuous at the point $\lambda = 0$.

(iv) $u^-(1) \leq u^+(1)$.

Conversely, if the pair of functions $u^-$ and $u^+$ satisfies the conditions (i)-(iv), then there exists a unique $u \in E^1$ such that $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ for each $\lambda \in [0, 1]$. The fuzzy number $u$ corresponding to the pair of functions $u^-$ and $u^+$ is defined by $u : \mathbb{R} \to [0, 1]$, $u(x) = \sup\{\lambda : u^-(\lambda) \leq x \leq u^+(\lambda)\}$.

Now we give the definitions of triangular fuzzy numbers with the $\lambda$-level set.

Definition 2.9. [3, Definition, p. 137] (Triangular fuzzy number) The membership function $\mu_{(u)}$ of a triangular fuzzy number $u$ represented by $(u_1, u_2, u_3)$ is interpreted, as follows:

$$\mu_{(u)}(x) = \begin{cases} \frac{x - u_1}{u_2 - u_1}, & u_1 \leq x \leq u_2, \\ \frac{u_3 - x}{u_3 - u_2}, & u_2 \leq x \leq u_3, \\ 0, & x < u_1, \ x > u_3. \end{cases}$$

Then, the result $[u]_\lambda := [u^-(\lambda), u^+(\lambda)] = [(u_2 - u_1)\lambda + u_1, (u_2 - u_3)\lambda + u_3]$ holds for each $\lambda \in [0, 1]$.

Let $u, v, w \in E^1$ and $\alpha \in \mathbb{R}$. Then the operations addition, scalar multiplication and product defined on $E^1$ by $u + v = w \iff [w]_\lambda = [u]_\lambda + [v]_\lambda$ for all $\lambda \in [0, 1]$ then $w^-(\lambda) = u^-(\lambda) + v^-(\lambda)$ and $w^+(\lambda) = u^+(\lambda) + v^+(\lambda)$ for all $\lambda \in [0, 1]$. Let $W$ be the set of all closed bounded intervals $A$ of real numbers with endpoints $\bar{A}$ and $\underline{A}$, i.e., $A := [\underline{A}, \bar{A}]$. Define the relation $d$ on $W$ by $d(A, B) := \max\{|\underline{A} - \underline{B}|, |\bar{A} - \bar{B}|\}$. Then it can easily be observed that $d$ is a metric on $W$ (cf. Diamond and Kloeden [6]) and $(W, d)$ is a complete metric space, (cf. Nanda [8]). Now, we can define the metric $D$ on $E^1$ by means of the Hausdorff metric $d$ as

$$D(u, v) := \sup_{\lambda \in [0, 1]} d([u]_\lambda, [v]_\lambda) := \sup_{\lambda \in [0, 1]} \max\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}.$$
Proposition 2.3. [2] Let $u,v,w,z \in E^1$ and $\alpha \in \mathbb{R}$. Then, the following statements hold:

(i) $(E^1,D)$ is a complete metric space, (cf. Puri and Ralescu [17]).

(ii) $D(\alpha u,\alpha v) = |\alpha|D(u,v)$.

(iii) $D(u+v,w+v) = D(u,w)$.

(iv) $D(u+v,w+z) \leq D(u,w) + D(v,z)$.

(v) $|D(u,\overline{u}) - D(v,\overline{v})| \leq D(u,v) \leq D(u,\overline{u}) + D(v,\overline{v})$.

Definition 2.10. [5] $u \in E^1$ is said to be a non-negative fuzzy number if and only if $u(x) = 0$ for all $x < 0$. It is immediate that $u \geq 0$ if $u$ is a non-negative fuzzy number. By $E^+_1$, we denote the set of non-negative fuzzy numbers. Similarly, $u \in E^1$ is said to be a non-positive fuzzy number if and only if $u(x) = 0$ for all $x > 0$. It is immediate that $u \leq 0$ if $u$ is a non-positive fuzzy number. By $E^+_1$, we denote the set of non-positive fuzzy numbers. By $E^1_+$, we denote the set of either non-negative fuzzy numbers or non-negative fuzzy numbers.

Then, it is trivial that the following statements hold:

(i) $u \geq 0 \Leftrightarrow u^-(\lambda) \geq 0$ and $u^+(\lambda) \geq 0$ for all $\lambda \in [0,1]$.

(ii) $u \leq 0 \Leftrightarrow u^-(\lambda) \leq 0$ and $u^+(\lambda) \leq 0$ for all $\lambda \in [0,1]$.

(iii) $u \not\sim 0 \Leftrightarrow u^-(\lambda) < 0$ and $u^+(\lambda) > 0$ for some $\lambda \in [0,1]$.

Definition 2.11. [5] The following statements hold:

(a) A sequence $u = (u_n)$ of fuzzy numbers is a function $u$ from the set $\mathbb{N}$ into the set $E^1$. The fuzzy number $u_k$ denotes the value of the function at $k \in \mathbb{N}$ and is called as the general term of the sequence.

(b) A sequence $(u_n) \in \omega(F)$ is called convergent to $u \in E^1$, if and only if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $D(u_n,u) < \varepsilon$ for all $n \geq n_0$.

(c) A sequence $(u_n) \in \omega(F)$ is called bounded if and only if the set of its terms is a bounded set. That is to say that a sequence $(u_n) \in \omega(F)$ is said to be bounded if and only if there exist two fuzzy numbers $m$ and $M$ such that $m \leq u_n \leq M$ for all $n \in \mathbb{N}$. This means that $m^-(\lambda) \leq u_n^-(\lambda) \leq M^-(\lambda)$ and $m^+(\lambda) \leq u_n^+(\lambda) \leq M^+(\lambda)$ for all $\lambda \in [0,1]$.

The boundedness of the sequence $(u_n) \in \omega(F)$ is equivalent to the fact that

$$\sup_{n \in \mathbb{N}}D(u_n,\overline{u}) = \sup_{n \in \mathbb{N}}\sup_{\lambda \in [0,1]}\max\{|u_n^-(\lambda)|,|u_n^+(\lambda)|\} < \infty.$$ 

If the sequence $(u_k) \in \omega(F)$ is bounded then the sequences of functions $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are uniformly bounded in $[0,1]$.

Definition 2.12. [20] Let $(u_n)$ be a sequence of fuzzy-valued functions defined on a set $A \subseteq E^1$. We say that $(u_n)$ converges pointwise on $A$ if for each $x \in A$ the sequence $(u_n(x))$ of fuzzy-valued functions converges. If the sequence $(u_n)$ converges pointwise on a set $A$, then we can define $u : A \rightarrow E^1$ by

$$\lim_{n \rightarrow \infty} u_n(x) = u(x) \text{ for each } x \in A.$$

Thus, in terms of $\varepsilon - N$ notation, $(u_n)$ converges to a fuzzy number $u$ on $A \subseteq E^1$ iff for each $x \in A$ and for an arbitrary $\varepsilon > 0$, there exists an integer $N = N(\varepsilon,x)$ such that $D(u_n(x),u(x)) < \varepsilon$ whenever $n > N$. The integer $N$ in the definition of pointwise convergence may, in general, depend on both $\varepsilon > 0$ and $x \in A$. If, however, one integer can be found that works for all points in $A$, then the convergence is said to be uniform. That is, a sequence of fuzzy-valued functions $(u_n)$ converges uniformly to $u$ on a set $A$ if for each $\varepsilon > 0$, there exists an integer $N_0 = N(\varepsilon)$ such that

$$D(u_n(x),u(x)) < \varepsilon \text{ whenever } n > N_0 \text{ and for all } x \in A.$$
Obviously the sequence \((u_n) \in \omega(F)\) of fuzzy-valued functions converges to a fuzzy number \(u\) if and only if \(\{u_n(\lambda)\}\) and \(\{u^+(\lambda)\}\) converge uniformly to \(u^-\) and \(u^+\) on \([0, 1]\), respectively. Often, we say that \(u\) is the uniform limit of the sequence \((u_n)\) on \(A\) and write \(u_n \to u\) uniformly on \(A\).

We emphasize that uniform convergence on a set implies (pointwise) convergence on that set. But the converse is not true, as we shall soon see in a number of examples. Thus, uniform convergence is a stronger form of convergence. Finally, we remark that it is apparent that if a sequence of fuzzy-valued functions converges uniformly on a set \(A\), then it converges on every compact subset of \(A\).

**Definition 2.13.** [18] A sequence \(\{u_n(x)\}\) of fuzzy valued functions converges uniformly to \(u(x)\) on a set \(I\) if for each 
\(\varepsilon > 0\) there exists a number \(n_0(\varepsilon)\) such that \(D(u_n(x), u(x)) < \varepsilon\) for all \(x \in I\) and \(n > n_0(\varepsilon)\).

### 3 Completeness of the sequence spaces of fuzzy level sets with the partial metric

**Proposition 3.1.** [1] Let \(x, y \in X\) and define the partial distance function \(p\) by

\[
p: X \times X \longrightarrow \mathbb{R}^+ \quad (x, y) \longmapsto p(x, y) = \max\{x, y\},
\]

and

\[
p: X \times X \longrightarrow \mathbb{R}^+ \quad (x, y) \longmapsto p(x, y) = -\min\{x, y\},
\]

for \(X = \mathbb{R}^+\) and \(X = \mathbb{R}^+\), respectively. Then, \((\mathbb{R}^+, p)\) is complete partial metric space where the self-distance for any point \(x \in \mathbb{R}^+\) is its value itself. The pair \((\mathbb{R}^+, p)\) is complete partial metric space for which \(p\) is called the usual partial metric on \(\mathbb{R}^+\), and where the self-distance for any point \(x \in \mathbb{R}^+\) is its absolute value.

The open balls are of the form \(B_p(x, \varepsilon) = \{y \in \mathbb{R}^+: \max\{x, y\} < \varepsilon\}\) for all \(x \in \mathbb{R}^+\) and \(\varepsilon > 0\) with \(x \leq -\varepsilon\) otherwise, if \(x > \varepsilon\) then \(B_p(x, \varepsilon) = \emptyset\). Suppose that \(y \in B_p(x, \varepsilon)\), then \(\max\{x, y\} < \varepsilon\) which implies that \(y < \varepsilon\).

Similarly, the open balls are of the form \(B_p(x, \varepsilon) = \{y \in \mathbb{R}^-: -\min\{x, y\} < \varepsilon\}\) for all \(x \in \mathbb{R}^-\) and \(\varepsilon > 0\) with \(x \geq -\varepsilon\) otherwise, if \(x < -\varepsilon\) then \(B_p(x, \varepsilon) = \emptyset\). Suppose that \(y \in B_p(x, \varepsilon)\), then \(-\min\{x, y\} < \varepsilon\) which implies that \(\min\{x, y\} > -\varepsilon\), hence \(y > -\varepsilon\).

**Lemma 3.1.** Let us consider the set \(I = \{u^-\lambda, u^+\lambda: u^-\lambda, u^+\lambda \in \mathbb{R}\} \subset E^1\) of the level sets and the distance function \(H: I \times I \longrightarrow \mathbb{R}^+\) by means of the Hausdorff partial metric \(p\) with respect to the usual ordering on \(\mathbb{R}\) defined by

\[
H(u, v) = \sup_{\lambda \in [0, 1]} p([u^-\lambda, v^-\lambda]) = \sup_{\lambda \in [0, 1]} (\max\{u^-\lambda, v^-\lambda\} - \min\{u^-\lambda, v^-\lambda\}) = \max\{u^-0, v^-0\} - \min\{u^-0, v^-0\}
\]

for any \(u, v \in E^1\). Then the pair \((E^1, H)\) is a fuzzy partial metric space.

The partial ordering relation on \(E^1\) is defined by \(u \leq v \Leftrightarrow [u^-\lambda] \subseteq [v^-\lambda] \Leftrightarrow u^-\lambda \leq v^-\lambda\) and \(u^+\lambda \leq v^+\lambda\) for all \(\lambda \in [0, 1]\). Here, \([u^-\lambda] \subseteq [v^-\lambda]\) if, and only if, \([v^-\lambda] \subseteq [u^-\lambda]\). Indeed, \(p([u^-\lambda], [v^-\lambda]) = p([u^+\lambda], [v^+\lambda])\) implies \(u^-0 = u^-0 = \max\{u^-0, v^-0\} - \min\{u^-0, v^-0\}\). Suppose that \(v^-0 > u^-0\), then \(u^-0 = u^-0 - \min\{u^-0, v^-0\}\) and \(\min\{u^-0, v^-0\} = v^-0\). Consequently, \([v^-\lambda] \subseteq [u^-\lambda]\) if, and only if, \(p([u^-\lambda], [v^-\lambda]) = p([u^+\lambda], [v^+\lambda]).

**Theorem 3.1.** The spaces \((E^1, H)\) and \((E^1, H^*)\) are complete.

**Definition 3.1.** [12] Let \((X, p)\) be a partial metric space. For any \(x \in X\) and \(\varepsilon > 0\), respectively the open and closed ball for the partial metric \(p\) by setting

\[
B_p(x) = \{y \in X: p(x, y) < \varepsilon\}, \quad \overline{B}_p(x) = \{y \in X: p(x, y) \leq \varepsilon\}
\]

**Remark 3.1.** [12] Contrary to the metric space case, some open balls may be empty. As an example, in a partial metric space \((X, p)\), the open balls \(B_p(x, \varepsilon)\) are empty for any \(x \in X\).

Now, we can give the relationship of the fuzzy partial metrics \(H^+_\lambda\) and fuzzy Hausdorff metric \(D\) where the fuzzy distance

\[
D(u, v) := \sup_{\lambda \in [0, 1]} d([u^-\lambda, v^-\lambda]) := \sup_{\lambda \in [0, 1]} \max\{|u^-\lambda - v^-\lambda|, |u^+\lambda - v^+\lambda|\}.
\]
If we take $\lambda = 0$, and for all $u, v \in E^r_+$ then

$$D(u, v) := \max \{ |u_0^- - v_0^-|, |u_0^+ - v_0^+| \} := H^r_\pm(u, v).$$

(3.1)

**Proposition 3.2.** Define $H_\omega$ on the space $\gamma(H)$ by

$$H_\omega : \gamma(H) \times \gamma(H) \rightarrow \mathbb{R}^+ \quad \quad (u, v) \mapsto H_\omega(u, v) = \sup_{k \in \mathbb{N}} \{ H^r(u_k, v_k) \}$$

where $u = (u_k), v = (v_k) \in \gamma(H)$ where $\gamma(H)$ denotes any of the spaces $\ell_\omega(H), c(H)$ or $c_0(H)$. Then, $(\gamma(H), H_\omega)$ is complete metric space with the partial metric.

**Proof.** Since the proof is similar for the spaces $c(H)$ and $c_0(H)$, we prove the theorem only for the space $\ell_\omega(H)$. Let $u = (u_k), v = (v_k)$ and $w = (w_k) \in \ell_\omega(H)$. Then,

(i) By using the axiom (P1) in Definition 2.1, it is trivial that

$$u = v \Leftrightarrow H_\omega(u, v) = \sup_{k \in \mathbb{N}} \{ H^r(u_k, v_k) \} = \sup_{k \in \mathbb{N}} \{ 2H(u_k, v_k) - H(u_k, u_k) - H(v_k, v_k) \} = 0$$

(ii) By using the axiom (P2) in Definition 2.1, it follows that

$$H_\omega(u, v) = \sup_{k \in \mathbb{N}} \{ H^r(u_k, v_k) \} \leq \sup_{k \in \mathbb{N}} \{ H^r(u_k, v_k) \} = H_\omega(u, v)$$

(iii) By using the axiom (P3) in Definition 2.1, it is clear that

$$H_\omega(u, v) = \sup_{k \in \mathbb{N}} \{ H^r(u_k, v_k) \} = \sup_{k \in \mathbb{N}} \{ H^r(v_k, u_k) \} = H_\omega(v, u).$$

(iv) By using the axiom (P4) in Definition 2.1 with the inequalities $H(u_k, w_k) \leq H(u_k, v_k) + H(v_k, w_k) - H(v_k, v_k)$ and $H^r(u_k, u_k) = 0$, we have

$$H_\omega(u, w) \leq \sup_{k \in \mathbb{N}} \{ H^r(u_k, w_k) \} = \sup_{k \in \mathbb{N}} \{ 2H(u_k, w_k) - H(u_k, u_k) - H(w_k, w_k) \}$$

$$\leq \sup_{k \in \mathbb{N}} \{ 2[H(u_k, u_k) + H(v_k, w_k)] - H(u_k, u_k) - H(w_k, w_k) \}$$

$$= \sup_{k \in \mathbb{N}} \{ 2[H(u_k, v_k) - H(u_k, u_k) - H(v_k, v_k)] + 2[H(v_k, w_k) - H(v_k, v_k)] \}$$

$$\leq \sup_{k \in \mathbb{N}} \{ H^r(u_k, v_k) \} + \sup_{k \in \mathbb{N}} \{ H^r(v_k, w_k) \} + \sup_{k \in \mathbb{N}} \{ H^r(v_k, v_k) \}$$

$$= \sup_{k \in \mathbb{N}} \{ H^r(u_k, v_k) \} + H_\omega(v, w) - H_\omega(v, v).$$

Therefore, one can conclude that $(\ell_\omega(H), H_\omega)$ is a partial metric space on $\ell_\omega(H)$. It remains to prove the completeness of the space $\ell_\omega(H)$. Let $(u_m)$ be any Cauchy sequence on $\ell_\omega(H)$ where $u_m = \{ u^{(m)}_1, u^{(m)}_2, \ldots \}$. Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ for all $m, r > N$ such that

$$H_\omega(u_m, u_r) = \sup_{k \in \mathbb{N}} \{ H^r(u_k^{(m)}, u_k^{(r)}) \} < \varepsilon.$$
A fortiori, for every fixed $k \in \mathbb{N}$ and for $m, r > N$

$$
\{H^t\left(u_k^{(m)} , u_k^{(r)}\right) : k \in \mathbb{N}\} < \varepsilon. \tag{3.2}
$$

Hence for every fixed $k \in \mathbb{N}$, by using the completeness of $\{E^t, H^t\}$ in Theorem 3.1, we say the sequence $(u_k^{(m)}) = \{u_k^{(1)}, u_k^{(2)}, \ldots\}$ is a Cauchy sequence and is uniformly convergent. Now, we suppose that $\lim_{m \to \infty} u_k^{(m)} = u_k$ and $u = (u_1, u_2, \ldots)$. We must show that

$$
\lim_{m \to \infty} H_\varepsilon(u_m, u) = 0 \quad \text{and} \quad u \in \ell_\varepsilon(H). \tag{3.3}
$$

The constant $N \in \mathbb{N}$ for all $m > N$, taking the limit for $r \to \infty$ in (3.2), we obtain

$$
H^t\left(u_k^{(m)} , u_k\right) < \varepsilon \tag{3.4}
$$

for all $k \in \mathbb{N}$. Since $(u_k^{(m)}) \in \ell_\varepsilon(H)$, there exists a fuzzy number $M > \overline{0}$ such that $H^t(u_k^{(m)} , \overline{0}) \leq M$ for all $k \in \mathbb{N}$. Thus, (3.4) gives together with the triangle inequality of partial metric for $m > N$ that

$$
H^t(u_k, \overline{0}) \leq H^t(u_k, u_k^{(m)}) + H^t(u_k^{(m)} , \overline{0}) - H^t(u_k^{(m)} , u_k^{(m)}) \leq \varepsilon + M. \tag{3.5}
$$

It is clear that (3.5) holds for every $k \in \mathbb{N}$ whose right-hand side does not involve $k$. This leads us to the consequence that $u = (u_1, u_2, \ldots)$ is bounded sequence of fuzzy numbers hence $u \in \ell_\varepsilon(H)$. Also, from (3.4) we obtain for $m > N$ that

$$
H_\varepsilon(u_m, u) = \sup_{k \in \mathbb{N}} \left\{H^t\left(u_k^{(m)} , u_k\right)\right\} \leq \varepsilon.
$$

This shows that (3.3) holds and $\lim_{m \to \infty} H_\varepsilon(u_m, u) = 0$. Since $(u_k)$ is an arbitrary Cauchy sequence, $(\ell_\varepsilon(H))$ is complete. \hfill \Box

**Example 3.1.** Consider the membership functions $u_k(t)$ and $v_k(t)$ defined by the triangular fuzzy numbers as

$$
u_k(t) = \begin{cases} 
(k+1)t - 1 & , \frac{1}{k+1} t \leq \frac{2}{k+1}, \\
3 - (k+1)t & , \frac{2}{k+1} t < \frac{3}{k+1}, \\
0 & , \text{otherwise}
\end{cases} \quad \text{and} \quad v_k(t) = \begin{cases} 
(k+1)(t - 1) & , 1 \leq t \leq 1 + \frac{1}{k+1}, \\
2 - (k+1)(t - 1) & , 1 + \frac{1}{k+1} < t \leq 1 + \frac{2}{k+1}, \\
0 & , \text{otherwise}
\end{cases}
$$

for all $k \in \mathbb{N}$.

It is trivial that $u_k^{-}(\lambda) = \frac{k+1}{k+1}$ and $u_k^{+}(\lambda) = \frac{3-\lambda}{k+1}$ for all $\lambda \in [0, 1]$. Therefore we see that $(u_k)_0^{-} = 1/(k+1)$ and $(u_k)_0^{+} = 1/(k+1)$. Then, it is concluded that

$$
sup_k \left\{H^t(u_k, \overline{0})\right\} = sup_k \left\{2H(u_k, \overline{0}) - H(u_k, u_k) - H(\overline{0}, \overline{0})\right\} = sup_k \left\{(u_k)^{-} + (u_k)^{+}\right\} = sup_k \left\{4/(k+1)\right\} < \infty
$$

and $(u_k) \in \ell_\varepsilon(H)$. Similarly, $v_k^{-}(\lambda) = 1 + \frac{k+1}{k+1}$ and $v_k^{+}(\lambda) = 1 + \frac{3-\lambda}{k+1}$ for all $\lambda \in [0, 1]$. It is clear that $(v_k)_0^{-} = 1$ and $(v_k)_0^{+} = 1 + \frac{2}{k+1}$. We observe that

$$
sup_k \left\{H^t(v_k, \overline{0})\right\} = sup_k \left\{2H(v_k, \overline{0}) - H(v_k, v_k) - H(\overline{0}, \overline{0})\right\} = sup_k \left\{2 + \frac{2}{k+1}\right\} < \infty
$$

Then, $(v_k) \in \ell_\varepsilon(H)$. Now we can calculate the partial distance between the sequences of fuzzy numbers $u = (u_k)$ and $v = (v_k) \in \ell_\varepsilon(H)$ that

$$
H_\varepsilon(u, v) = \sup_{k \in \mathbb{N}} \left\{H^t(u_k, v_k)\right\} = \sup_{k \in \mathbb{N}} \left\{2H(u_k, v_k) - H(u_k, u_k) - H(v_k, v_k)\right\}
$$

$$
= \sup_{k \in \mathbb{N}} \left\{2\max\{u_k^{-}(0), v_k^{-}(0)\} - \min\{u_k^{+}(0), v_k^{+}(0)\}\right\} - \sup_{k \in \mathbb{N}} \left\{u_k^{-}(0) - u_k^{+}(0)\right\} - \sup_{k \in \mathbb{N}} \left\{v_k^{-}(0) - v_k^{+}(0)\right\}
$$

$$
= \sup_{k \in \mathbb{N}} \left\{2\left(1 + \frac{1}{k+1}\right) - \left(\frac{2}{k+1}\right) - \left(\frac{2}{k+1}\right)\right\}
$$

$$
= \sup_{k \in \mathbb{N}} \left\{\left(2 - \frac{2}{k+1}\right)\right\} = 2
$$
Proposition 3.3. Define the distance function $H_p$ by

$$H_p : \ell_p(H) \times \ell_p(H) \longrightarrow \mathbb{R}^+$$

$$(u, v) \longmapsto H_p(u, v) = \left\{ \sum_{k=0}^{\infty} H^s(u_k, v_k)^p \right\}^{1/p}, \ (1 \leq p < \infty)$$

where $u = (u_k), \ v = (v_k) \in \ell_p(H)$. Then, $(\ell_p(H), H_p)$ is complete metric space with the partial metric.

Proof. It is obvious that $H_p$ satisfies the axioms (P1), (P2) and (P3). To prove (P4), we use Minkowski’s inequality on real and $H_p(v, v) = 0$. Let $u = (u_k), \ v = (v_k)$ and $w = (w_k) \in \ell_p(H)$. Then,

$$H_p(u, w) = \left\{ \sum_{k=0}^{\infty} (H^s(u_k, w_k))^p \right\}^{1/p} = \left\{ \sum_{k=0}^{\infty} [2H(u_k, v_k) - H(u_k, u_k) - H(w_k, w_k)]^p \right\}^{1/p} \leq \left\{ \sum_{k=0}^{\infty} [2H(u_k, v_k) - H(v_k, v_k)]^p \right\}^{1/p} + \left\{ \sum_{k=0}^{\infty} [2H(v_k, w_k) - H(w_k, w_k)]^p \right\}^{1/p} = H_p(u, v) + H_p(v, w) - H_p(v, v).$$

Therefore, one can conclude that $(\ell_p(H), H_p)$ is a partial metric space and complete. Since the proof is analogous for the cases $p = 1$ and $p = \infty$ we omit their detailed proof and we consider only case $1 < p < \infty$. It remains to prove the completeness of the space $(\ell_p(H), H_p)$.

Let $(u_m)$ be any Cauchy sequence on $\ell_p(H)$ where $u_m = \{u_1^{(m)}, u_2^{(m)}, \ldots\}$. Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$H_p(u_m, u_r) = \left\{ \sum_{k=0}^{\infty} H^s(u_k^{(m)}, u_k^{(r)})^p \right\}^{1/p} < \varepsilon \quad (3.6)$$

for all $m, r > N$. We obtain for each fixed $k \in \mathbb{N}$ from (3.6) that

$$H^s(u_k^{(m)}, u_k^{(r)}) < \varepsilon. \quad (3.7)$$

For all $m, r > N$ and the constant $k \in \mathbb{N}$, by using the completeness $(E^1, H^s)$ in Theorem 3.1, we say the sequence $(u_k^{(m)})$ is a Cauchy sequence of fuzzy numbers and is uniformly convergent.

Now, we suppose that $\lim_{m \to \infty} u_k^{(m)} = u_k$ and $u = (u_1, u_2, \ldots)$. We must show that

$$\lim_{m \to \infty} H_p(u_m, u) = 0 \quad \text{and} \quad u \in \ell_p(H).$$

We have from (3.7) for each $j \in \mathbb{N}$ and $m, r > N$ that

$$\sum_{k=0}^{j} H^s(u_k^{(m)}, u_k^{(r)})^p < \varepsilon^p \quad (3.8)$$

Take any $m > N$. Let us pass to limit firstly $r \to \infty$ and next $j \to \infty$ in (3.8) to obtain $H_p(u_m, u) < \varepsilon$ for all $m > N$ it follows that $\lim_{m \to \infty} H_p(u_m, u) = 0$. By using Minkowski’s inequality for each $j \in \mathbb{N}$ that

$$\left\{ \sum_{k=0}^{j} H^s(u_k, 0)^p \right\}^{1/p} \leq \left\{ \sum_{k=0}^{j} H^s(u_k, u_k^{(m)})^p \right\}^{1/p} + \left\{ \sum_{k=0}^{j} H^s(u_k^{(m)}, 0)^p \right\}^{1/p} - \left\{ \sum_{k=0}^{j} H^s(u_k^{(m)}, u_k^{(m)})^p \right\}^{1/p} < \infty$$

which implies that $u \in \ell_p(H)$. Since $(u_m)$ is an arbitrary Cauchy sequence of fuzzy numbers, the space $(\ell_p(H), H_p)$ is complete. This step concludes the proof. □
Example 3.2. Consider the membership functions \( u_k(t) \) and \( v_k(t) \) defined by the trapezoidal fuzzy numbers as

\[
u_k(t) = \begin{cases} k(k+1)t - 1 &, \quad \frac{1}{2k(k+1)} \leq t \leq \frac{2}{k(k+1)}, \\ 3 - k(k+1)t &, \quad \frac{2}{k(k+1)} < t \leq \frac{3}{k(k+1)}, \\ 0 & \text{otherwise}, \end{cases}
\]

and

\[
v_k(t) = \begin{cases} 6k(k+1)t - 3 &, \quad \frac{1}{2k(k+1)} \leq t \leq \frac{2}{k(k+1)}, \\ 1 &, \quad \frac{2}{k(k+1)} < t \leq \frac{3}{k(k+1)}, \\ 4 - 4k(k+1)t &, \quad \frac{3}{k(k+1)} < t \leq \frac{4}{k(k+1)}, \\ 0 & \text{otherwise} \end{cases}
\]

for all \( k \in \mathbb{N} \).

It is obvious that \( u_k^- (\lambda) = \frac{\lambda + 1}{k(k+1)} \) and \( u_k^+ (\lambda) = \frac{3 - \lambda}{k(k+1)} \), \( v_k^- (\lambda) = \frac{\lambda + 3}{6k(k+1)} \) and \( v_k^+ (\lambda) = \frac{4 - \lambda}{4k(k+1)} \) for all \( \lambda \in [0, 1] \). Then, taking \( p = 1 \)

\[
\sum_{k=0}^{\infty} H^p(u_k, 0)^p = \sum_{k=0}^{\infty} \left\{ u_k^-(0) - u_k^+(0) \right\} = \sum_{k=0}^{\infty} \left\{ \frac{3}{k(k+1)} - \frac{1}{k(k+1)} \right\} = \sum_{k=0}^{\infty} \left\{ \frac{2}{k(k+1)} \right\} = 2
\]

and \( u \in \ell_p(H) \) and similarly one can conclude that \( v \in \ell_p(H) \). Now we can calculate the partial distance between the sequences of fuzzy numbers \( u = (u_k) \) and \( v = (v_k) \in \ell_p(H) \) that

\[
H_p(u, v) = \left( \sum_{k=0}^{\infty} H^p(u_k, v_k)^p \right)^{1/p}
\]

\[
= \sum_{k=0}^{\infty} \left\{ 2 \left( \max \left\{ u_k^+(0), v_k^+(0) \right\} - \min \left\{ u_k^-(0), v_k^-(0) \right\} \right) - \left( u_k^+(0) - u_k^-(0) \right) - \left( v_k^+(0) - v_k^-(0) \right) \right\}
\]

\[
= \sum_{k=0}^{\infty} \left\{ \frac{5}{k(k+1)} - \frac{3}{k(k+1)} - \frac{1}{2k(k+1)} \right\} = \sum_{k=0}^{\infty} \left\{ \frac{5}{2k(k+1)} \right\} = 5/2.
\]

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4 Conclusion

Many authors have extensively developed fuzzy level sets theory and their applications. Prior to introducing Kadak and Basar defined the power of a fuzzy number and introduced some sets of fuzzy-valued sequences with the level sets, in [19, 21]. In addition to, Kadak [20], determine the sets of fuzzy-valued function includes the sets of bounded and continuous functions and its related applications. Indeed, some useful results have been obtained by using level sets for defining different metric types of fuzzy numbers like as partial metric. We should record from now on that the main results given in the last section of the present paper will base on examining some fuzzy-valued function spaces with respect to the partial metric. The potential applications of the obtained results include the generalization of sequence spaces based on partial metric. One of the purposes of this work is to extend the fuzzy level set calculus to the classical calculus.
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