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Research Article

Fuzzy optimization model for land use change

L. Jahanshahloo[1]*, E. Haghi[2]

(1) *Department of Art and Architecture, Science and Research Branch, Islamic Azad University, Tehran, Iran.*

(2) *Department of Mathematics, Robat Karim Branch, Islamic Azad University, Tehran, Iran.*

Abstract

There are some important questions in Land use change literature, for instance "How much land to allocate to each of a number of land use type in order to maximization of (household or individual) rent -paying ability, minimization of environmental impacts or maximization of population income". In this paper, we want to investigate them and propose mathematical models to find an answer for these questions. Since Most of the parameters in this process are linguistics and fuzzy logic is a powerful tools to handle them, a fuzzy linear programming model is used in model building. To this end, Fuzzy Linear Programming Problem(FLP) with fuzzy related system of constraint and fuzzy coefficient vector in the objective function, that is a full fuzzy system of simultaneous equations with fuzzy objective function is discussed. The related production operations in the objective function and in the constrains are performed in the basis of standard production between fuzzy numbers. The constraint which can be take into account depend on the case but representative objective include: Lower and upper limits on land use, availability of Labour and so on.

Keywords: Land use change, Fuzzy number, Triangular fuzzy number, Ranking
MSC2000 Classification: 34B15, 34B16.

1 Introduction

Land is the stage on which all human activity is being conducted and the source of the materials needed for this conduct. Human use of land resources gives rise to "land use" which varies with the purposes it serves, whether they be food production, provision of shelter, recreation, extraction and processing of materials, and so on, as well as the bio-physical characteristics of land itself. Hence, land use is being shaped under the influence of two broad sets of forces - human needs and environmental features and processes. Neither one of these forces stays still; they are in a constant state of flux as change is the quintessence of life. Changes in the uses of land occurring at various spatial levels and within various time periods are the material expressions, among others, of environmental and human dynamics and of their interactions which are mediated by land. These changes have at times beneficial, at times detrimental impacts and effects, the latter being the chief causes of concern as they impinge variously on human well-being and welfare [14]. The application of mathematical programming and optimization techniques to urban and regional analysis, spurred by the post-1950s developments in solution techniques and computer technology, has an impressive record and continues to attract significant research contributions as well as to offer significant decision support in various circumstances, notably in planning. As their name denotes, optimization models are exclusively oriented towards producing solutions which optimize certain objectives defined by (interested) users/decision makers. In other words, they are fit to provide support in decision situations where the question is to choose a solution to a decision problem which satisfies one or more objectives and takes into account various constraints. Hence, they are prescriptive models although they are used also as evaluation tools. They have found important

*Corresponding author. Email address: L.Jahanshahloo@srbiau.ac.ir, Tel:+989123788758

applications in the analysis of land use - especially land use planning applications - and, recently, they appear to be useful tools in the search for land use solutions which contribute to sustainable development and use of environmental and human resources. Linear programming is one of the most widely used techniques in model building since the mid-1950s as it is more manageable, understandable and computationally easier than other optimization techniques. Its use in the analysis of land use is marked perhaps by the widely known Herbert-Stevens Linear Programming Model designed for the Penn-Jersey Transportation Study [15]. Other similar models were built in the same period such as the Southern Wisconsin Regional Plan Model [16] and Britton Harris' Optimizing Model - a modification of the Herbert-Stevens model [17]. More applications ensued in the following decades up to the present [18, 19, 20, 21, 22]. In the following, the basic structure of a LP model is presented first and, then, details of its applications drawing on the published literature are offered. In conventional approach, parameters of linear programming models must be well defined and precise. However, in real world environment, this is not a realistic assumption. Usually, the value of many parameters of a linear programming model is estimated by experts. Clearly, it can not be assumed the knowledge of experts is so precise. Since Bellman and Zadeh [1] proposed the concept of decision making in fuzzy environments, a number of researchers have exhibited their interest to solve the fuzzy linear programming problems. Existing methods can be divided into two groups, depending on the fuzziness of decision parameters and decision variables. In the first group, the researchers assumed the decision parameters are fuzzy numbers while the decision variables are crisp ones, see [2, 3, 5, 6, 7, 8, 9, 10, 11, 12]. This means that in an uncertain environment, a crisp decision is made to meet some decision criteria. Tanaka *et al.* [13] can be considered as the pioneers for the second group of fuzzy linear programming problems with fuzzy decision variables and crisp decision parameters. Recently, Buckley and Feuring [4] introduced a general class of fuzzy linear programming called fully fuzzified linear programming (FFLP) problems, where all decision parameters and variables are fuzzy numbers. In fact, fully fuzzified linear programming problem is a generalized version of both classes mentioned above. As pointed out by Buckley and Feuring [4], searching for the optimal solutions of FFLP is a very difficult task. They employed a directed search technique of evolutionary algorithm type. The another kind of mathematical programming is Integer Linear Programming (ILP).This kind of problem is applied in material production field and feasibility studies of factories.Since ILP is used in real problem ,in the most of cases it should be studied in uncertain environment. Fuzzy logic is a good foundation for handling these problems. Fuzzy Integer Linear programming (FILP) has been studies by some authors [38] and solved with relating algorithms. Based on the nature of the problem , we need to study the fuzzy integer programming with fuzzy R.H.S. We reduce the FILP in two ILP problems and prove the related theorems in details. The paper is organized as follows: In Section 2 we present the basic definitions. In Section 3 a ranking function is brought and it is applied for solving full fuzzy optimization land use change models.

2 Preliminaries

In this section we are going to review the fundamental notations.

Definition 2.1. Land: *Land is a delineable area of the earth's terrestrial surface, encompassing all attributes of the biosphere immediately above or below this surface, including those of the near-surface climate, the soil and terrain forms, the surface hydrology (including shallow lakes, rivers, marshes, and swamps), the near-surface sedimentary layers and associated groundwater reserve, the plant and animal populations, the human settlement pattern and physical results of past and present human activity (terracing, water storage or drainage structures, roads, buildings, etc.), [23].*

In a short term land defines as an area of the Earth's surface, [24].

Definition 2.2. Land use: *Land use denotes the human employment of land, [25].*

Definition 2.3. Land cover: *Land cover describes the physical state of the land surface: as in cropland, mountains, or forests, [26] .*

Definition 2.4. Land use and land cover change: *Land use and land cover change means (quantitative) changes in the areal extent (increases or decreases) of a given type of land use or land cover, [27, 28].*

We represent an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements:

- $\underline{u}$(r) is a bounded left continuous nondecreasing function over [0,1].
- $\overline{u}$(r) is a bounded left continuous nonincreasing function over [0,1].
- $\underline{u}(r)$ and $\overline{u}(r)$ are right continuous in 0
- $\underline{u}(r) \leq \overline{u}(r)$,$0 \leq r \leq 1$.

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$.

Definition 2.5. *The fuzzy number*

$$T = (c - w + wr, c + w - wr) := (c; w) \qquad , 0 \leq r \leq 1$$

which $c, w \in \Re$, $c = Core(T)$ *and* $w \geq 0$ *is called Symmetric Triangular Fuzzy Number* ($\widehat{STFN}$).
Let $\widehat{S.T}$ *be the set of all* $\widehat{STFN}$.

Definition 2.6. *The integer fuzzy number*

$$T = (c - w + wr, c + w - wr) := (c; w) \qquad , 0 \leq r \leq 1$$

which $c, w \in Z$, $c = Core(T)$ *and* $w \geq 0$ *is called Integer Symmetric Triangular Fuzzy Number* ($\widehat{ISTFN}$).
Let $\widehat{I.S.T}$ *be the set of all* $\widehat{ISTFN}$.

Let $T = (c_1; w_1), U = (c_2; w_2) \in \widehat{I.S.T}$ and $k \in \Re$, by using extension principal we can define:

1. $T = U$ if and only if $c_1 = c_2$ and $w_1 = w_2$.
2. $T + U = (c_1 + c_2; w_1 + w_2)$.
3. $kT = (kc_1; \mid k \mid w_1)$.

Suppose $A \in \Re^{n \times n}$, $\tilde{X} = (T_1, T_2, \ldots, T_n)^T$ and $\tilde{Y} = (U_1, U_2, \ldots, U_n)^T$ are $\widehat{ISTFN}$ vectors this means that $\tilde{X}, \tilde{Y} \in \widehat{I.S.T}^n$. Now we have

1. $Core(\tilde{X} + \tilde{Y}) = Core(\tilde{X}) + Core(\tilde{Y})$.
2. $Core(A\tilde{X}) = ACore(\tilde{X})$.
3. $A(\tilde{X} + \tilde{Y}) = A\tilde{X} + A\tilde{Y}$.

Definition 2.7. *(Ordering on* $\widehat{I.S.T}$*) Let* $T = (c_1; w_1)$ *and* $U = (c_2; w_2)$ *are* $\widehat{ISTFN}$*s.*
We say $T <^\star U$ *if and only if:*

1. $c_1 < c_2$
2. *or* $c_1 = c_2$ *and* $w_1 < w_2$.

And $T \leq^\star U$ *if and only if* $T <^\star U$ *or* $T = U$. *It's clear that by this definition* $\widehat{ISTFN}$*s have the triple axiom.This means that for any* $T, U \in \widehat{I.S.T}$ *we have only one of this*

$$T <^\star U \quad , \quad T = U \quad , \quad T^\star > U$$

Lemma 2.1. *If* T, U, V, W *be* $\widehat{ISTFN}$*s and* $k \in \Re$ *then*

1. $T \leq^\star U$ *if and only if* $T + V \leq^\star U + V$.
2. $T <^\star U$ *if and only if* $T + V <^\star U + V$.
3. *If* $T \leq^\star U$ *and* $V \leq^\star W$ *then* $T + V \leq^\star U + W$.
4. *If* $T <^\star U$ *and* $V \leq^\star W$ *then* $T + V <^\star U + W$.
5. *If* $T <^\star U$ *and* $U <^\star V$ *then* $T <^\star V$.
6. *If* $T \leq^\star U$ *and* $U <^\star V$ *then* $T <^\star V$.
7. *If* $T <^\star U$ *and* $U \leq^\star V$ *then* $T <^\star V$.
8. *If* $T \leq^\star U$ *and* $U \leq^\star V$ *then* $T \leq^\star V$.
9. *If* $T <^\star U$ *then* $kT <^\star kU$ *for* $k \geq 0$.

Proof. Proof is clear. □

Lemma 2.2. *Let* $w_1, w_2 \geq 0$, $w_1 \leq w_2$ *if and only if for any* $0 \leq r \leq 1$, $w_1 - w_1 r \leq w_2 - w_2 r$.

Proof. Proof is clear. □

Lemma 2.3. *Consider two* $\widehat{ISTFN}$*s* $T = (-w_1 + w_1 r, w_1 - w_1 r)$ *and* $U = (-w_2 + w_2 r, w_2 - w_2 r)$ *that their cores are zero then*

$$T \leq^\star U \ if \ and \ only \ if \ w_1 - w_1 r \leq w_2 - w_2 r \quad , for \ any \ 0 \leq r \leq 1.$$

Proof. Proof is clear. □

Remark 2.1. *[29] The $m \times n$ fuzzy linear system of equations (FSLE)*

$$\begin{cases} a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \ldots + a_{1n}\tilde{x}_n = \tilde{y}_1 \\ a_{21}\tilde{x}_1 + a_{21}\tilde{x}_2 + \ldots + a_{1n}\tilde{x}_n = \tilde{y}_2 \\ \dot{} \\ a_{m1}\tilde{x}_1 + a_{m2}\tilde{x}_2 + \ldots + a_{mn}\tilde{x}_n = \tilde{y}_m \end{cases} .$$

where the coefficient matrix $A = (a_{ij}), 1 \le i \le m$ and $1 \le j \le n$ is a crisp $n \times n$ matrix and $\tilde{y}_i$ is an arbitrary fuzzy number, $1 \le i \le n$ is called a FSLEs. The i-th equations is as follow

$$\underline{a_{i1}\tilde{x}}_1 + \ldots + \underline{a_{ii}\tilde{x}}_i + \ldots + \underline{a_{in}\tilde{x}}_n = \underline{\tilde{y}}_i(r)$$

$$\overline{a_{i1}\tilde{x}}_1 + \ldots + \overline{a_{ii}\tilde{x}}_i + \ldots + \overline{a_{in}\tilde{x}}_n = \overline{\tilde{y}}_i(r)$$

for $1 \le i \le m, 0 \le r \le 1$.
From above we have two crisp $m \times n$ linear systems for all i that there can be extended to a $2m \times 2n$ crisp linear system as follows:

$$SX = Y \longrightarrow \begin{bmatrix} S_1 \ge 0 & S_2 \le 0 \\ S_2 \le 0 & S_1 \ge 0 \end{bmatrix} \begin{bmatrix} \underline{X} \\ \overline{X} \end{bmatrix} = \begin{bmatrix} \underline{Y} \\ \overline{Y} \end{bmatrix}.$$

Thus FSLE is extended to above crisp system where $A = S_1 + S_2$. And it can be written as follows:

$$\begin{cases} S_1\underline{X} + S_2\overline{X} = \underline{Y} \\ S_2\underline{X} + S_1\overline{X} = \overline{Y} \end{cases}$$

3 Fuzzy Integer Linear Programming

In this section we are going to reduce FILP (3.1) to two crisp ILP.

$$\begin{cases} Min\ C\ \tilde{X} \\ \quad s.t. \quad A\tilde{X} = \tilde{b} \\ \quad \tilde{X} \ge^{\star} 0, \tilde{X} \in \widehat{I.S.T}^n \end{cases} \tag{3.1}$$

Which $A \in \Re^{m\times n}$, $C \in \Re^n$ and $\tilde{b}$ is an arbitrary fuzzy number vector.

Theorem 3.1. *If (3.1) has a feasible solution then $\tilde{b}$ is a vector of $\widehat{STFN}$s. This means that $\tilde{b} \in \widehat{S.T}^m$.*

Proof. Note that every combination of $\widehat{STFN}$s is a $\widehat{STFN}$. We have $\sum_{j=1}^{n} a_j\tilde{x}_j = \tilde{b}$. So $\tilde{b}$ is a vector of $\widehat{STFN}$s this means that $\tilde{b} \in \widehat{S.T}^m$ and proof is completed. □

By theorem 3.1 we only consider the following FILP as follows:

$$\begin{cases} Min\ C\ \tilde{X} \\ \quad s.t. \quad A\tilde{X} = T \\ \quad \tilde{X} \ge^{\star} 0, \tilde{X} \in \widehat{I.S.T}^n \end{cases} \tag{3.2}$$

Which $A \in \Re^{m\times n}$, $C \in \Re^n$ and $T \in \widehat{S.T}^m$.

Remark 3.1. *The (3.1) is reduce to (3.2).*

Consider two following problems:

$$\begin{cases} Min\ C\,\tilde{X} \\ s.t. \quad A\tilde{X} = Core(T) \\ \tilde{X}^{\star} \geq 0, \tilde{X} \in \widehat{I.S.T}^{n} \end{cases} \tag{3.3}$$

Which $A \in \Re^{m\times n}$, $C \in \Re^{n}$ and $T \in \widehat{S.T}^{m}$ and i-th entry of $Core(T)$ is core of i-th entry of T.

and

$$\begin{cases} Min\ C\,\tilde{X} \\ s.t. \quad A\tilde{X} = H(T) = (-b+br, b-br) \\ \tilde{X}^{\star} \geq 0, \tilde{X} \in \widehat{I.S.T}^{n} \end{cases} \tag{3.4}$$

Which $A \in \Re^{m\times n}$, $C \in \Re^{n}$ and $T \in \widehat{S.T}^{m}$ and $H(T) = T - Core(T)$. It means that , if (c;w) is the solution of (3.4) then w is an integer number. We note that

$$Core(H(T)) = Core(T) - Core(Core(T)) = Core(T) - Core(T) = 0.$$

Theorem 3.2. *$\tilde{X}^2$ is a feasible solution of (3.2) if and only if there exist two feasible solutions $\tilde{X}^3$ and $\tilde{X}^4$ of problems (4.8) and (3.4) that*

$$\tilde{X}^2 = \tilde{X}^3 + \tilde{X}^4.$$

Also $Core(\tilde{X}^2)$ and $\tilde{X}^2 - Core(\tilde{X}^2) = H(\tilde{X}^2)$ are feasible solutions of problems (4.8) and (3.4).

Proof. Let $\tilde{X}^2$ is a feasible solution of problem (3.2) put:

$$\tilde{X}^3 := Core(\tilde{X}^2).$$

$$\tilde{X}^4 := \tilde{X}^2 - Core(\tilde{X}^2) = \tilde{X}^2 - \tilde{X}^3.$$

it is clear that $X^2 = X^3 + X^4$. Now $\tilde{X}^3 = Core(X^2) \geq 0$ so $\tilde{X}^{3\,\star} \geq 0$ and since $\tilde{X}^3$ is crisp vector so $\tilde{X}^3 \in \widehat{I.S.T}^{n}$. Now

$$A\tilde{X}^3 = ACore(\tilde{X}^2) = Core(A\tilde{X}^2) = Core(T).$$

In other hand $Core(\tilde{X}^4) = 0 \geq 0$ so $\tilde{X}^4 \geq^{\star} 0$ and $\tilde{X}^4 = \tilde{X}^2 - \tilde{X}^3$ so $\tilde{X}^4 \in \widehat{I.S.T}^{n}$ vector. Now

$$A\tilde{X}^4 = A(\tilde{X}^2 - \tilde{X}^3) = A\tilde{X}^2 - A\tilde{X}^3 = T - Core(T) = H(T).$$

So we have two feasible solution $\tilde{X}^3$, $\tilde{X}^4$ of problems (4.8) and (3.4) which $\tilde{X}^2 = \tilde{X}^3 + \tilde{X}^4$.
Conversly: If we have two feasible solution $\tilde{X}^3$, $\tilde{X}^4$ of problems (4.8) and (3.4) so by $\tilde{X}^2 := \tilde{X}^3 + \tilde{X}^4$ we have $Core(\tilde{X}^2) = Core(\tilde{X}^3) + Core(\tilde{X}^4) \geq 0$ so $\tilde{X}^2 \geq^{\star} 0$ and $\tilde{X}^2$ is a combination of two $\widehat{ISTFN}$ vector so $\tilde{X}^2 \in \widehat{I.S.T}^{n}$. Now

$$A\tilde{X}^2 = A(\tilde{X}^3 + \tilde{X}^4) = A\tilde{X}^3 + A\tilde{X}^4 = Core(T) + H(T) = T.$$

So $\tilde{X}^2$ is a feasible solution of problem (3.2) and proof is completed. □

Corollary 3.1. *The problem (3.2) is infeasible if and only if (4.8) or problem (3.4) are infeasible.*

Theorem 3.3. *If $\tilde{X}^{*^3}$, $\tilde{X}^{*^4}$ are optimal solutions of problems (4.8) and (3.4) then*

$$\tilde{X}^{*^2} = \tilde{X}^{*^3} + \tilde{X}^{*^4}$$

is an optimal solution of (3.2).

Proof. Suppose $\tilde{X}^{*^3}$, $\tilde{X}^{*^4}$ are optimal solutions of problems (4.8) and (3.4) let $\tilde{X}^{*^2} = \tilde{X}^{*^3} + \tilde{X}^{*^4}$ And $\tilde{X}^2$ is a feasible solution of problem (3.2) by theorem 3.2, $\tilde{X}^3 = Core\tilde{X}^2$ and $\tilde{X}^4 = H(\tilde{X}^2)$ are feasible solutions of problems (4.8) and (3.4) so

$$C\tilde{X}^{*^3} \leq^{\star} C\tilde{X}^3$$

$$C\tilde{X}^{*^4} \leq^{\star} C\tilde{X}^4$$

Thus by lemma 2.1 we have

$$C\tilde{X}^{*^3} + C\tilde{X}^{*^4} \leq^\star C\tilde{X}^3 + C\tilde{X}^4$$

So

$$C\tilde{X}^{*^2} = C(\tilde{X}^{*^3} + \tilde{X}^{*^4}) = C\tilde{X}^{*^3} + C\tilde{X}^{*^4} \leq^\star C\tilde{X}^3 + C\tilde{X}^4 = C(\tilde{X}^3 + \tilde{X}^4) = C\tilde{X}^2.$$

Hence $\tilde{X}^{*^2}$ is an optimal solution of problem (3.2) and proof is completed. □

Theorem 3.4. *If $\tilde{X}^{*^2}$ is an optimal solution of problem (3.2) then there exist two optimal solutions of problems (4.8) and (3.4) like $\tilde{X}^{*^3}$, $\tilde{X}^{*^4}$ which*

$$\tilde{X}^{*^2} = \tilde{X}^{*^3} + \tilde{X}^{*^4}.$$

Proof. Suppose $\tilde{X}^{*^2}$ is an optimal solution of problem (3.2) let $\tilde{X}^{*^3} = Core(\tilde{X}^{*^2})$ and $\tilde{X}^{*^4} = H(\tilde{X}^{*^2})$ by theorem 3.2 these are feasible solutions of problems (4.8) and (3.4). Now if $\tilde{X}^3$ and $\tilde{X}^4$ are feasible solutions of problems (4.8) and (3.4) we know that $\tilde{X}^2 = \tilde{X}^3 + \tilde{X}^4$ is a feasible solution of problem (3.2). Since $\tilde{X}^{*^2}$ is an optimal solution of problem (3.2) so by definition **??**, $C\tilde{X}^{*^3} = Core(C\tilde{X}^{*^2}) \leq Core(C\tilde{X}^2) = Core(C\tilde{X}^3)$ hence $C\tilde{X}^{*^3} \leq^\star C\tilde{X}^3$ thus $\tilde{X}^{*^3}$ is an optimal solution of problem (4.8). In other hand if $C\tilde{X}^4 <^\star C\tilde{X}^{*^4}$ then $\tilde{Y}^{*^2} = C\tilde{X}^{*^3} + C\tilde{X}^4 <^\star C\tilde{X}^{*^3} + C\tilde{X}^{*^4} = C\tilde{X}^{*^2}$ and it's a feasible solution of problem (3.2) but this is a contradiction so $C\tilde{X}^{*^4} \leq^\star C\tilde{X}^4$ hence $\tilde{X}^{*^4}$ is an optimal solution of problem (3.4) and proof is completed. □

Corollary 3.2. *If problem (3.2) be feasible then the problem (3.2) has unbounded optimal solution if and only if the problem (4.8) or problem (3.4) have unbounded optimal solution.*

Remark 3.2. *The problem (3.2) is reduce to problems (4.8) and (3.4).*

Now we are going to reduce problem (4.8) to problem (3.5) Consider the following problem:

$$\begin{cases} Min\ C\,X \\ \quad s.t. \quad AX = Core(T) \qquad\qquad X \geq 0, X \in Z^n \end{cases} \tag{3.5}$$

which is a crisp linear programming.

Theorem 3.5. *A crisp vector $X \in Z^n$ is a feasible solution of problem (4.8) if and only if it's a feasible solution of (3.5).*

Proof. We note that a crisp vector $X \in Z^n$ is a $\widehat{ISTFN}$ vector now we have X is a feasible solution of problem (4.8) if and only if $AX = Core(T)$ and $X \geq^\star 0$ if and only if $AX = Core(T)$ and $X \geq 0$ if and only if X is a feasible solution of problem (3.5) and proof is complete. □

Theorem 3.6. *If problem (4.8) has an optimal solution then it has a crisp optimal solution.*

Proof. Let $\tilde{X}^{*^3}$ is an optimal solution of (4.8) and $X^* = Core(\tilde{X}^{*^3})$ then

$$AX^* = A(Core(\tilde{X}^{*^3})) = Core(A\tilde{X}^{*^3}) = Core(Core(T)) = Core(T)$$

Also $Core(X^*) = Core((\tilde{X}^*)^3) \geq 0$ so $X^* \geq^\star 0$ and X^* is a crisp vector, and it is a $\widehat{ISTFN}$ vector therefore X^* is a feasible solution of problem (4.8) we know that

$$CX^* + CH(\tilde{X}^{*^3}) = C(X^* + H(\tilde{X}^{*^3})) = C(Core(\tilde{X}^{*^3}) + H(\tilde{X}^{*^3})) = C\tilde{X}^{*^3}$$

By definition 2.7 since $Core(H(X^{*^3})) = 0$ and 0 is the least $\widehat{ISTFN}$ around 0 we have $C.H(\tilde{X}^{*^3}) \geq^\star 0$. Hence by lemma 2.1,

$$CX^* \leq^\star C\tilde{X}^{*^3}$$

therefore X^* is an optimal solution of problem (4.8) and proof is completed. □

Theorem 3.7. *An optimal solution of problem (3.5) is an optimal solution of problem (4.8).*

Proof. Suppose X^* is an optimal solution of problem (3.5). by theorem 3.5, X^* is a feasible solution of problem (4.8). Let $\tilde{X}^3$ is a feasible solution of problem (4.8) and $X^3 = Core(\tilde{X}^3)$. Since X^3 is a feasible solution of (3.5) hence $CX^* \leq CX^3$ since X^* is crisp so $CX^\star \leq^\star CX^3$ thus

$$CX^* \leq^\star CX^3 + CH(\tilde{X}^3) = C(Core(\tilde{X}^3)) + CH(\tilde{X}^3) = C(Core(\tilde{X}^3) + H(\tilde{X}^3)) = C\tilde{X}^3$$

therefore X^* is an optimal solution of problem (4.8) and proof is complete. □

Remark 3.3. *The problem (4.8) is reduce to problem (3.5).*

Theorem 3.8. *If problem (3.4) has an optimal solution then there is a $\tilde{X}^\circ$ which $Core(\tilde{X}^\circ) = 0$ and it is an optimal solution of (3.4).*

Proof. Suppose $\tilde{X}^{*^4}$ is an optimal solution of problem (3.4) put $\tilde{X}^\circ = \tilde{X}^{*^4} - Core(\tilde{X}^{*^4})$ and $X^\circ = Core(\tilde{X}^{*^4})$ then it is an $\widehat{ISTFN}$, it's sufficient to show that $\tilde{X}^\circ$ is an optimal solution of problem (3.4). We have

$$A\tilde{X}^\circ + AX^\circ = A(\tilde{X}^\circ + X^\circ) = A\tilde{X}^{*^4} = H(T)$$

Since $Core(H(T)) = 0$ we have $AX^\circ = 0$ and $A\tilde{X}^\circ = H(T)$ therefore $\tilde{X}^\circ$ is a feasible solution of (3.4). To show that $\tilde{X}^\circ$ is optimal three cases discussed

1. $CX^\circ <^\star 0$: Since $AX^\circ = 0$ and $X^\circ \geq 0$, $\tilde{Y}^* = \tilde{X}^{*^4} + X^\circ$ is a feasible solution and

   $$C\tilde{X}^{**} = C(\tilde{X}^{*^4} + X^\circ) = C\tilde{X}^{*^4} + CX^\circ <^\star C\tilde{X}^{*^4}$$

   But this is a contradiction.
2. $CX^\circ >^\star 0$: In this case we have

   $$C\tilde{X}^\circ = C(\tilde{X}^{*^4} - X^\circ) = C\tilde{X}^{*^4} - CX^\circ <^\star C\tilde{X}^{*^4}$$

   this is a contradiction too.
3. $CX^\circ = 0$: In this case since

   $$C\tilde{X}^\circ = C(\tilde{X}^{*^4} - X^\circ) = C\tilde{X}^{*^4} - CX^\circ = C\tilde{X}^{*^4}$$

   therefore $\tilde{X}^\circ$ is an optimal solution of problem (3.4) and proof is completed.

□

Corollary 3.3. *If problem (3.4) has an optimal solution then there is an optimal solution $\tilde{X}^* = (-\overline{X^*}(r), \overline{X^*}(r))$ which $\underline{X}^*(1) = \overline{X}^*(1) = 0$ this means that $Core(\tilde{X}^*) = 0$.*

Consider the following parametric problem:

$$\begin{cases} Min \;\; |C|\overline{X} \\ s.t. \quad |A|\overline{X}(r) = \overline{H(T)}(r) = b - rb \\ X - Xr = \overline{X}(r) \geq 0, for\ any\ 0 \leq r \leq 1 \end{cases} \tag{3.6}$$

Which i-th entry of $|C|$ is $|c_i|$ and i,j-th entry of $|A|$ is $|a_{ij}|$, X is crisp and an integer number.

Theorem 3.9. *$\tilde{X} = (-\overline{X}(r), \overline{X}(r))$ is a feasible solution of problem (3.4) if and only if $\overline{X}(r)$ is a feasible solution of (3.6).*

Proof. $\tilde{X} = (-\overline{X}(r), \overline{X}(r))$ is a feasible solution of problem (3.4) if and only if

$$A\tilde{X} = H(T)$$

And by remark 2.1 if and only if

$$\begin{cases} S_1\underline{X} + S_2\overline{X} = \underline{H(T)} \\ S_2\underline{X} + S_1\overline{X} = \overline{\overline{H(T)}} \end{cases}$$

If and only if

$$\begin{cases} -S_1\overline{X} + S_2\overline{X} = -\overline{H(T)} \\ -S_2\overline{X} + S_1\overline{X} = \overline{H(T)} \end{cases}$$

If and only if

$$\begin{cases} -S_2\overline{X}+S_1\overline{X}=\overline{H(T)} \\ -S_2\overline{X}+S_1\overline{X}=\overline{H(T)} \end{cases}$$

If and only if

$$-S_2\overline{X}+S_1\overline{X}=\overline{H(T)}$$

If and only if

$$(S_1-S_2)\overline{X}=\overline{H(T)}=b-rb$$

If and only if

$$\mid A\mid \overline{X}=b-rb.$$

If and only if $\overline{X}$ is a feasible solution of problem (3.6). Hence $\tilde{X}^*=(-\overline{X^*}(r),\overline{X^*}(r))$ is a feasible solution of problem (3.4) if and only if $\overline{X^*}(r)$ is a feasible solution of (3.6) and proof is completed. □

Lemma 3.1. *If $C\in\Re^n$ and $\tilde{X}$ be a symmetric fuzzy number over 0 then $\overline{C\tilde{X}}=\mid C\mid \overline{\tilde{X}}$.*

Proof. Let $\tilde{X}=(-X(r),X(r))$.

$$C\tilde{X}=C(-X(r),X(r))=\sum_{i=1}^{n}c_i(-x_i(r),x_i(r))=\sum_{i=1}^{n}(-|c_i|x_i(r),|c_i|x_i(r))$$

Hence $\overline{C\tilde{X}}=\mid C\mid \overline{\tilde{X}}$ and proof is completed. □

Theorem 3.10. *If $\tilde{X}^*=(-\overline{X^*}(r),\overline{X^*}(r))$ be an optimal solution of problem (3.4) then $\overline{X}^*(r)$ is an optimal solution of (3.6).*

Proof. By theorem 3.9, since $\tilde{X}^*=(-\overline{X^*}(r),\overline{X^*}(r))$ is a feasible solution of problem (3.4) so $\overline{X^*}(r)$ is a feasible solution of (3.6). Let $\overline{X}(r)$ is a feasible solution of then $\tilde{X}=(-\overline{X}(r),\overline{X}(r))$ is a feasible solution of the problem (3.4) so $C\tilde{X}^*\preceq^\star C\tilde{X}$ thus by lemma 2.3 and lemma 3.1, $\mid C\mid \overline{X^*}(r)\leq \mid C\mid \overline{X}(r)$ hence $\overline{X^*}(r)$ is an optimal solution of problem (3.6) and proof is completed. □

Theorem 3.11. *If $\overline{X^*}(r)$ be an optimal solution of problem (3.6) then $\tilde{X}^*=(-\overline{X^*}(r),\overline{X^*}(r))$ is an optimal solution of (3.4).*

Proof. By theorem 3.9, since $\overline{X^*}(r)$ is a feasible solution of problem (3.6) so $\tilde{X}^*=(-\overline{X^*}(r),\overline{X^*}(r))$ is a feasible solution of problem (3.4). Let $\tilde{X}$ is a feasible solution of problem (3.4) so
$H(\tilde{X})=(-\overline{X},\overline{X})$ is a feasible solution of problem (3.4) therefore $\mid C\mid \overline{X^*}(r)\leq\mid C\mid \overline{X}(r)$ now by lemma 2.3 and lemma 3.1 we have

$$C\tilde{X}^*\preceq^\star C.H(\tilde{X})\preceq^\star C\tilde{X}$$

Hence $\tilde{X}^*$ is an optimal solution of problem (3.4) and proof is complete. □

Corollary 3.4. *Problem (3.4) has an optimal solution if and only if for some optimal solution $\overline{X^*}(r)$ of (3.6), $\tilde{X}^*=(-\overline{X^*}(r),\overline{X^*}(r))$ is an optimal solution of problem (3.4).*

Remark 3.4. *The problem (3.4) is reduce to problem (3.6).*

Consider the following problem:

$$\begin{cases} Min\ \mid C\mid X \\ \quad s.t.\ \mid A\mid\ X=b \\ \quad X\geq 0 \\ ,X\in Z^n \end{cases} \tag{3.7}$$

Which is a crisp linear programming.

Theorem 3.12. *$\overline{X}(r)=X-Xr$ is a feasible solution of problem (3.6) if and only if X is a feasible solution of (3.7).*

Proof. Suppose $X - Xr$ is a feasible solution of problem (3.6). Since for any $0 \leq r \leq 1$, $X - Xr \geq 0$ so by $r = 0$ we have $X \geq 0, X \in Z^n$. In other hand

$$|A|(X - Xr) = b - br$$

Thus

$$|A|X - |A|Xr = b - br$$

Therefore by $r = 0$ we have

$$|A|X = b$$

Hence X is a feasible solution of problem (3.7).
Conversely:Let X is a feasible solution of problem (3.7). If $0 \leq r \leq 1$ then $X \geq Xr$ so $X - Xr \geq 0$. Now since X is a feasible solution of problem (3.7) we have

$$|A|X = b$$

And

$$-|A|Xr = -br$$

Therefore

$$|A|(X - Xr) = b - br$$

Hence $X - Xr$ is a feasible solution of problem (3.6) and proof is complete. □

Theorem 3.13. *If $\overline{X^*}(r) = X^* - X^*r$ be an optimal solution of problem (3.6) then $\overline{X^*}(r)$ is an optimal solution of (3.7).*

Proof. By theorem 3.12 since $X^* - X^*r$ is a feasible solution of problem (3.6) so X^* is a feasible solution of problem (3.7). Let X is a feasible solution of problem (3.7) therefore $X - Xr$ is a feasible solution of the problem (3.6) so $|C|(X^* - X^*r) \leq |C|(X - Xr)$ thus by lemma 2.2,
$|C|X^* \leq |C|X$ hence X^* is an optimal solution of the problem (3.7) and proof is complete. □

Theorem 3.14. *If X^* be an optimal solution of problem (3.7) then $\overline{X^*}(r) = X^* - X^*r$ is an optimal solution of (3.6).*

Proof. Let $Y - Yr$ is a feasible solution of problem (3.6). By lemma 3.12, Y is a feasible solution of problem (3.7), therefore $|C|X^* \leq |C|Y$ finally by lemma 2.2, $|C|(X^* - X^*r) \leq |C|(Y - Yr)$ hence $X^* - X^*r$ is an optimal solution of problem (3.6) and proof is complete. □

Corollary 3.5. *problem (3.6) has an optimal solution if and only if for some optimal solution X^* of (3.7), $\tilde{X}^* = (-\overline{X^*}(r), \overline{X^*}(r))$ is an optimal solution of problem (3.6).*

Remark 3.5. *The problem (3.6) is reduce to problem (3.7).*

Remark 3.6. *The problem (3.2) is reduce to crisp problems (3.5) and (3.7).*

Remark 3.7. *Above proofs for the case maximization analogues.*

4 Fuzzy optimization model for land use change

The models of land use change in fuzzy environment are defined variously. More broadly, models can be considered as abstractions, approximations of reality which is achieved though simplification of complex real world relations to the point that they are understandable and analytically manageable (Briassoulis 1999). There are some important questions in Land use change literature, for instance "How much land to allocate to each of a number of land use type in order to maximization of (household or individual) rent -paying ability, minimization of environmental impacts or maximization of population income". In this section, we want to investigate them and propose mathematical models to find an answer for these questions. Since Most of the parameters in this process are linguistics and fuzzy logic is a powerful tools to handle them, a FILP model is used in model building. To this end, FILP with fuzzy related system of constraint and fuzzy coefficient vector in the objective function, that is a full fuzzy system of simultaneous equations with fuzzy objective function is discussed. The related production operations in the objective function and in the constrains are performed in the basis of standard production between fuzzy numbers. The constraint which can be take into account depend on the case but representative objective include: Lower and upper limits on land use, availability of Labour and so on. The following principal categories of fuzzy optimization models are presented below:

- Fuzzy linear programming models

- Fuzzy dynamic programming models
- Fuzzy goal programming, hierarchical programming, linear and quadratic assignment, and nonlinear programming models
- Fuzzy utility maximization models
- Fuzzy multi-Objective/Multi-Criteria Decision Making models.

In this section we are going to minimizes the total cost of developing urban land in a given zone of the study area under land availability constraints. Based on the previous research [?],the most important factors are cost of developing a unit of land of a given type in the zone and total land use demand requirement for land use. In the most researches, the model's factors have been considered as crisp numbers while determining these factors as a crisp number are so difficult and reduce the flexibility of the model. In the present work, it is tried to remove this difficultly by modeling the problem in a fuzzy environment. The proposed model is as follows:

$$\begin{cases} min\ C_t = \sum_{i=1}^{N} c_i X_i \\ \quad s.t. \\ d_i X_i = \tilde{E}_i \qquad\qquad i = 1\dots n \\ \sum_{i=1}^{N} X_i \preceq \tilde{F} \\ X_n \preceq \tilde{G} X_m \end{cases} \tag{4.8}$$

where X_i is units of a given land use type in the zone. c_i is cost of developing a unit of land of a given type in the zone. $\tilde{E}_i$ is total land use demand requirement for land use i. d_i is the service ratio coefficients which provide for supporting service land requirements necessary for primary land use developments such as streets. $\tilde{F}$ is upper limit on land use of a particular type in zone m. $\tilde{G}$ is the ratio of land use type n allowed relative to land use type m with the land use types m and n in the same or in different zones.

The model emphasizes mostly the determining effect of development costs (which are affected, among others, by environmental factors such as soil conditions, etc.) on the land use distribution in an area. Romanos (1976) observes that this model, while inheriting all the problems associated with LP models, does not represent any improvement over the Herbert-Stevens models; on the contrary, it lacks the detail and ingenuity of the latter model as well as the economic nature assumptions about individual behavior.

Now, we show the ability of the proposed model and its solving method by investigating a case study.

4.1 Case study

The objective is minimizing the total cost of developing urban land in Tehran under land availability constraints.
In Tehran 's comprehensive plan for 2008 , 6 kind of land use is considered as follow :

1	Residental Area
2	Green Space Area
3	Urban Service Area
4	Millitary Area
5	Official & Commercial Area
6	Industrial Transportation & Storge Area

The street ratio coefficient in Tehran is equal for all kinds of land use and is approximately 20 percent.

The total land use demand requirement for land use k, upper limit on land use of a particular type in zone m, the ratio of land use type n allowed relative to land use type m with the land use types m and n in the same or in different zones are considered in fuzzy environment as follow: upper limit on Land use is $\tilde{F}_m = (616+r, 618-r)$

1	(179+r,181-r)
2	(104+2r, 107-r)
3	(110+2r,114-2r)
4	(5+2r,6.5+0.5r)
5	(43+2r,47-2r)
5	(40+2r,44-2r)

Total Land use Demand Requirement $\tilde{E}_i$

1	(179+r,181-r)
2	(104+2r, 107-r)
3	(110+2r,114-2r)
4	(5+2r,6.5+0.5r)
5	(43+2r,47-2r)
5	(40+2r,44-2r)

Total Land use Demand Requirement $\tilde{E}_i$

By substituting from the above tables in model (4.8) ,the number of units of a given land use type in the zone have been determined as they have been shown in the following table.

5 conclusion

In this paper, we investigated the mathematical models to find an answer for mentioned questions in abstract. Since Most of the parameters in this process are linguistics and fuzzy logic is a powerful tools to handle them, a fuzzy Integer linear programming model is used in model building. To this end, FILP with fuzzy related system of constraint ,that is a fuzzy system of simultaneous equations with fuzzy objective function is discussed. The related production operations in the objective function and in the constrains are performed in the basis of standard production between fuzzy numbers. The constraint which can be take into account depend on the case but representative objective include: Lower and upper limits on land use, availability of Labour and so on.

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