An application of a semi-analytical method on linear fuzzy Volterra integral equations

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Abstract
Recently, fuzzy integral equations have attracted some interest. In this paper, we focus on linear fuzzy Volterra integral equation of the second kind (FVIE-2) and propose a new method for numerical solving it. In fact, using parametric form of fuzzy numbers we convert a linear fuzzy Volterra integral equation of the second kind to a linear system of Volterra integral equations of the second kind in crisp case. We use variational iteration method (VIM) and find the approximate solution of this system and hence obtain an approximation for fuzzy solution of the linear fuzzy Volterra integral equation of the second kind. Finally, using the proposed method, we give some illustrative examples.

Keywords: Fuzzy functions; Fuzzy integral equations; Fuzzy numbers; System of linear Volterra integral equations of the second kind; Variational iteration method

1 Introduction
The topics of fuzzy integral equations (FIEs) which growing interest for some time, in particular in relation to fuzzy control, have been rapidly developed in recent years. Prior to discussing fuzzy integral equations and their solving, it is necessary to present an appropriate brief introduction to preliminary topics such as fuzzy numbers and fuzzy calculus.

The concept of fuzzy sets, was originally introduced by Zadeh [26], led to the definition of fuzzy numbers and its implementation in fuzzy control [3] and approximate reasoning problems [27]. The basic arithmetic structure for fuzzy numbers was later developed by Mizumoto and Tanaka [16], Nahmias [17], Dubois and Prade [4], all of them observed fuzzy numbers as a collection of $\alpha$-levels, $0 \leq \alpha \leq 1$.

Goetschel and Voxman [8] suggested a new approach. They represented fuzzy number as a parameterized triple (see Section 2) and then embedded the set of fuzzy numbers into a topological vector space. This enabled them to design the basics of a fuzzy calculus. The subject of embedding fuzzy numbers in either a topological or a Banach space was investigated also by Puri and Ralescu [21, 22], Kaleva [12] and Ouyang [20]. Solving integral equations requires appropriate and applicable definitions of fuzzy function and fuzzy integral of a fuzzy function. The fuzzy mapping function was introduced by Chang and Zadeh [3]. Later, Dubois and Prade [5] presented an elementary fuzzy calculus

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based on the extension principle [26]. The concept of integration of fuzzy functions was first introduced by Dubois and Prade [5]. Alternative approaches were later suggested by Goetschel and Voxman [8], Kaleva [12], Nanda [18] and others. While Goetschel and Voxman [8] preferred a Riemann integral type approach, Kaleva [12] chose to define the integral of fuzzy function, using the Lebesgue-type concept for integration. In this work we concentrate on solving fuzzy Volterra integral equation of the second kind (FVIE-2). In Section 2, we briefly present some basic notations of fuzzy number, fuzzy function and fuzzy integral. FVIE-2 and parametric form of the FVIE-2 are discussed in Section 3. We observe parametric form of the FVIE-2 is a system of Volterra integral equations of the second kind in crisp case. In Section 4, we state the basic concepts of the variational iteration method [7, 13, 14, 15]. In Section 5, we apply variational iteration method on the linear system of Volterra integral equations of the second kind. In Section 6, we use variational iteration method for solving the linear system of Volterra integral equations of the second kind produced by the FVIE-2. We shall give some examples to illustrate our method in Section 7. We conclude in Section 8.

2 Preliminaries

In this section, we review the fundamental notations of fuzzy set theory to be used throughout this paper.

**Definition 2.1.** A fuzzy number is a fuzzy set \( u : \mathbb{R} \rightarrow [0, 1] \) which satisfies

i. \( u \) is upper semicontinuous.

ii. \( u(x) = 0 \) outside some interval \( [c, d] \).

iii. There are real numbers \( a, b : c \leq a \leq b \leq d \) for which

1. \( u(x) \) is monotonic increasing on \( [c, a] \).
2. \( u(x) \) is monotonic decreasing on \( [b, d] \).
3. \( u(x) = 1, a \leq x \leq b \).

The set of all fuzzy numbers (as given by Definition 2.1) is denoted by \( E^1 \). An alternative definition or parametric form of a fuzzy number which yields the same \( E^1 \) is given by Kaleva [12].

**Definition 2.2.** A fuzzy number \( u \) is a pair \((u, \overline{u})\) of functions \( u(r), \overline{u}(r) \) : \( 0 \leq r \leq 1 \) which satisfying the following requirements:

i. \( u(r) \) is a bounded monotonic increasing left continuous function,

ii. \( \overline{u}(r) \) is a bounded monotonic decreasing left continuous function,

iii. \( u(r) \leq \overline{u}(r), 0 \leq r \leq 1 \).

For arbitrary \( u = (u, \overline{u}), v = (\underline{v}, \overline{v}) \) and \( k > 0 \) we define addition \((u + v)\) and multiplication by \( k \) as

\[
(u + v)(r) = u(r) + v(r), \quad (u + v)(r) = \overline{u}(r) + \overline{v}(r), \quad (ku)(r) = ku(r), \quad (ku)(r) = k\overline{u}(r).
\]

(2.1)\n
(2.2)

The collection of all the fuzzy numbers with addition and multiplication as defined by Eqs. (2.1) and (2.2) is denoted by \( E^1 \) and is a convex cone. It can be shown that Eqs. (2.1) and (2.2) are equivalent to the addition and multiplication as defined by using the \( \alpha \)-cut approach [8] and the extension principles [19].

We will next define the fuzzy function notation and a metric \( D \) in \( E^1 \) [8].
Definition 2.3. For arbitrary fuzzy numbers \( u = (u, \overline{u}) \) and \( v = (v, \overline{v}) \) the quantity

\[
D(u, v) = \sup_{0 \leq i \leq 1} \{ \max \{ |u(r) - v(r)|, |\overline{u}(r) - \overline{v}(r)| \} \} \tag{2.3}
\]

is the distance between \( u \) and \( v \).

This metric is equivalent to the one used by Puri and Ralescu [21] and Kaleva [12]. It is shown [23] that \((E^1, D)\) is a complete metric space.

Definition 2.4. A function \( f : \mathbb{R}^1 \rightarrow E^1 \) is called a fuzzy function. If for arbitrary fixed \( t_0 \in \mathbb{R}^1 \) and \( \varepsilon > 0 \), a \( \xi > 0 \) exists, \( f \) is said to be continuous.

\[
| t - t_0 | < \xi \implies D[f(t), f(t_0)] < \varepsilon \tag{2.4}
\]

Definition 2.5. Let \( f : [a, b] \rightarrow E^1 \). For each partition \( p = \{ t_0, t_1, \ldots, t_n \} \) of \([a, b]\) and for arbitrary \( \xi_i : t_{i-1} \leq \xi_i \leq t_i, 1 \leq i \leq n \) let

\[
R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}). \tag{2.5}
\]

The definite integral of \( f(t) \) over \([a, b]\) is

\[
\int_a^b f(t)dt = \lim_{|p| \to 0} R_p, \quad \max_{1 \leq i \leq n} \{|t_i - t_{i-1}|\} \to 0 \tag{2.6}
\]

provided that this limit exists in the metric \( D \).

If the fuzzy function \( f(t) \) is continuous in the metric \( D \), its definite integral exists [8]. Furthermore,

\[
\left( \int_a^b f(t; r)dt \right) = \int_a^b \hat{f}(t; r)dt, \quad \left( \int_a^b \overline{f}(t; r)dt \right) = \int_a^b \overline{f}(t; r)dt. \tag{2.7}
\]

It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach [12]. However, if \( f(t) \) is continuous, both approaches yield the same value. Moreover, the representation of the fuzzy integral using Eqs. (2.5) and (2.6) is more convenient for numerical calculations. More details about the properties of the fuzzy integral are given in [8, 12].

3 Fuzzy Volterra integral equation

The Volterra integral equation of the second kind is

\[
F(t) = f(t) + \beta \int_a^t K(s, t)F(s)ds, \quad a \leq t \leq b, \tag{3.8}
\]

where \( \beta > 0 \), \( K(s, t) \) is an arbitrary kernel function over \( \Delta = \{(s, t) : a \leq s \leq t \leq b\} \) and \( f(t) \) is a function of \( t : a \leq t \leq b \). If \( f(t) \) is a crisp function then the solutions of Eq. (3.8) are crisp as well. We consider the fuzzy Volterra integral equation as following form

\[
F(t; r) = f(t; r) + \beta \int_a^t K(s, t)F(s; r)ds, \quad a \leq t \leq b, \tag{3.9}
\]
where $\beta > 0$, $K(s,t)$ is an arbitrary kernel function over $\triangle = \{(s,t) : a \leq s \leq t \leq b\}$ and $f(t;r)$ is a fuzzy function on interval $[a,b]$, i.e., $f(t;r)$ is a fuzzy number for each $a \leq t \leq b$. If $f(t;r)$ is a fuzzy function this equation, i.e., Eq. (3.9), may only posses fuzzy solution, that this solution is a fuzzy function on interval $[a,b]$. Sufficient conditions for the existence of a unique solution to the fuzzy Volterra integral equation of the second kind (FVIE-2), i.e. to Eq. (3.9) are given in [25].

Now, we introduce parametric form of a FVIE-2 with respect to Definition 2.2. Let $(f(t;r), \mathcal{F}(t;r))$ and $(\mathcal{F}(t;r), \mathcal{F}(t;r))$, $0 \leq r \leq 1$ and $t \in [a,b]$ are parametric form of $f(t)$ and $F(t)$, respectively. Then, parametric form of FVIE-2 is as follows:

$$
\begin{align*}
\mathcal{F}(t;r) &= f(t;r) + \beta \int_{a}^{r} K(s,t)F(s;r)ds, \\
\mathcal{F}(t;r) &= \mathcal{F}(t;r) + \beta \int_{a}^{r} K(s,t)F(s;r)ds,
\end{align*}
$$

(3.10)

where

$$
K(s,t)F(s;r) = \begin{cases}
K(s,t)F(s;r), & K(s,t) \geq 0, \\
K(s,t)F(s;r), & K(s,t) < 0,
\end{cases}
$$

(3.11)

and

$$
\tilde{K}(s,t)F(s;r) = \begin{cases}
K(s,t)\tilde{F}(s;r), & K(s,t) \geq 0, \\
K(s,t)\tilde{F}(s;r), & K(s,t) < 0.
\end{cases}
$$

(3.12)

for each $0 \leq r \leq 1$ and $a \leq t \leq b$. We can see that (3.10) is a system of linear Volterra integral equations in crisp case for each $0 \leq r \leq 1$ and $a \leq t \leq b$.

In the next section, we state the basic concepts of the variational iteration method.

4 Variational iteration method

The Variational iteration method (VIM) is proposed by He [9, 10] as a modification of a general Lagrange multiplier method [11]. This method has been shown to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions [1, 2, 24]. To illustrate its basic idea of the technique, we consider following general nonlinear system:

$$
L[u(t)] + N[u(t)] = g(t),
$$

(4.13)

where $L$ is linear operator, $N$ is a nonlinear operator, and $g(t)$ is a given continuous function. The basic character of the method is to construct a correction functional for system (4.13), which reads

$$
u_{n+1}(t) = u_{n}(t) + \int_{0}^{t} \lambda(\tau)\{Lu_{n}(\tau) + Nu_{n}(\tau) - g(\tau)\}d\tau,
$$

(4.14)

where $\lambda(\tau)$ is a general Lagrangian multiplier [9, 10, 11] which can be identified optimally via variational theory, the subscript $n$ denotes the nth-order approximation and $\tilde{u}_{n}$ is considered as a restricted variation [6], i.e. $\delta \tilde{u}_{n} = 0$.

For example, when $L \equiv \frac{d}{dt}$, we can construct the following correction functional

$$
u_{n+1}(t) = u_{n}(t) + \int_{0}^{t} \lambda(\tau)\{u'_{n}(\tau) + Nu_{n}(\tau) - g(\tau)\}d\tau,
$$

(4.15)

calculating variation with respect to $u_{n}$, noticing that $\delta u_{n}(0) = 0$, yields

$$
\delta u_{n+1}(t) = \delta u_{n}(t) + \delta \int_{0}^{t} \lambda(\tau)\{u'_{n}(\tau) + Nu_{n}(\tau) - g(\tau)\}d\tau
$$

$$
= \delta u_{n}(t) + \delta \int_{0}^{t} \lambda(\tau)\{u'_{n}(\tau)\}d\tau
$$

$$
= (1 + \lambda(\tau))\delta u_{n}(\tau)|_{\tau=t} - \int_{0}^{t} \lambda'(\tau)\delta u_{n}(\tau)d\tau = 0.
$$

(4.16)
Therefore, we have the following stationary conditions:
\[ \lambda'(\tau) = 0, \hspace{2cm} (4.17) \]
\[ 1 + \lambda(\tau) = 0 |_{\tau=t}. \hspace{2cm} (4.18) \]
So, the Lagrange multiplier, can be readily identified as
\[ \lambda(\tau) = -1. \hspace{2cm} (4.19) \]
Substituting this value of the Lagrange multiplier into functional (4.15) gives the iteration formula:
\[ u_{n+1}(t) = u_n(t) - \int_0^t \{ u_n'(\tau) + Nu_n(\tau) - g(\tau) \} d\tau. \hspace{2cm} (4.20) \]
Iteration formula (4.20) will give several approximations, and the exact solution is obtained at the limit of the resulting successive approximations.

5 VIM for linear system of Volterra integral equations

In this section, we consider the variational iteration method for linear system of Volterra integral equations of the form
\[ U(t) = F(t) + \int_a^t K(s,t)U(s)ds, \hspace{2cm} (5.21) \]
where
\[ U(t) = (u_1(t), u_2(t), \ldots, u_m(t))^T, \]
\[ F(t) = (f_1(t), f_2(t), \ldots, f_m(t))^T, \]
\[ K(s,t) = [k_{ij}(s,t)], \hspace{0.5cm} i = 1, 2, \ldots, m, \hspace{0.5cm} j = 1, 2, \ldots, m. \]
Consider the \(i\)th equation of (5.21):
\[ u_i(t) = f_i(t) + \int_a^t \sum_{j=1}^m k_{ij}(s,t)u_j(s)ds. \hspace{2cm} (5.22) \]
Differentiating both sides of Eq. (5.22) with respect to \(t\) yields
\[ u_i'(t) = f_i'(t) + \sum_{j=1}^m k_{ij}(t,t)u_j(t) + \int_a^t \sum_{j=1}^m \frac{\partial k_{ij}}{\partial t}(s,t)u_j(s)ds. \hspace{2cm} (5.23) \]
In view of the variational iteration method, we construct a correction functional in the following form
\[ u_{in+1}(t) = u_{in}(t) \]
\[ + \int_0^t \lambda_i(\tau)\{ u'_i(\tau) - \int_a^t \sum_{j=1}^m k_{ij}(\tau, \tau)\tilde{u}_j(\tau) - \int_a^t \sum_{j=1}^m \frac{\partial k_{ij}}{\partial \tau}(s, \tau)\tilde{u}_j(s)ds \} d\tau, \hspace{2cm} (5.24) \]
where \(\tilde{u}_j\) is considered as restricted variational, i.e. \(\delta \tilde{u}_j = 0\).
To find optimal value of \(\lambda_i(\tau)\), we have
\[ \delta u_{in+1}(t) = \delta u_{in}(t) + \delta \int_0^t \lambda_i(\tau)\{ u'_i(\tau) \} d\tau \]
\[ = (1 + \lambda_i(\tau))\delta u_{in}(\tau)|_{\tau=t} - \int_0^t \lambda_i(\tau)\delta u_{in}(\tau)d\tau = 0. \hspace{2cm} (5.25) \]
The stationary conditions can be obtained as follows:

\[ \lambda_i'(\tau) = 0, \]

\[ 1 + \lambda_i(\tau) = 0 \mid_{\tau = t}. \]

(5.26)

(5.27)

Therefore, the Lagrange multiplier can be identified as

\[ \lambda_i(\tau) = -1. \]

(5.28)

Substituting this value of the Lagrange multiplier into functional (5.23) gives the iteration formula:

\[ u^{n+1}(t) = u^n(t) \]

\[ - \int_0^t \{ u'_n(\tau) - f'_1(\tau) - \sum_{j=1}^{m} k_{ij}(\tau, \tau)u_{j,n}(\tau) - \int_{\tau}^{t} \sum_{j=1}^{m} \frac{\partial k_{ij}}{\partial \tau}(s, \tau)u_{j,n}(s)ds \} d\tau. \]

Therefore, we can write the following iteration formulas:

\[
\begin{cases}
  u^{n+1}_a(t) = u^n_a(t) - \int_0^t \{ u'_a(\tau) - f'_1(\tau) - \sum_{j=1}^{m} k_{aj}(\tau, \tau)u_{j,n}(\tau) - \int_{\tau}^{t} \sum_{j=1}^{m} \frac{\partial k_{aj}}{\partial \tau}(s, \tau)u_{j,n}(s)ds \} d\tau, \\
  u^{n+1}_b(t) = u^n_b(t) - \int_0^t \{ u'_b(\tau) - f'_2(\tau) - \sum_{j=1}^{m} k_{bj}(\tau, \tau)u_{j,n}(\tau) - \int_{\tau}^{t} \sum_{j=1}^{m} \frac{\partial k_{bj}}{\partial \tau}(s, \tau)u_{j,n}(s)ds \} d\tau, \\
  \vdots \\
  u^{n+1}_m(t) = u^n_m(t) - \int_0^t \{ u'_m(\tau) - f'_m(\tau) - \sum_{j=1}^{m} k_{mj}(\tau, \tau)u_{j,n}(\tau) - \int_{\tau}^{t} \sum_{j=1}^{m} \frac{\partial k_{mj}}{\partial \tau}(s, \tau)u_{j,n}(s)ds \} d\tau.
\end{cases}
\]

(5.29)

(5.30)

Iteration formulas (5.30) will give several approximations, and the exact solution is obtained at the limit of the resulting successive approximations.

6 VIM for system produced by the FVIE-2

In this section, we apply variational iteration method on the system of Volterra integral equations of the second kind produced by the FVIE-2, i.e. system of Volterra integral equations (3.10), and obtain iteration formulas for it. Prior to applying VIM for system (3.10), we suppose that the kernel \( K(s, t) \) is nonnegative for \( a \leq s \leq c \) and negative for \( c \leq s \leq t \). Therefore, we rewrite system (3.10) in the following form

\[
\begin{align*}
F(t;r) &= f(t;r) + \beta \int_{a}^{c} K(s,t)F(s;r)ds + \beta \int_{a}^{c} K(s,t)F(s;r)ds, \\
F(t;r) &= f(t;r) + \beta \int_{a}^{c} K(s,t)F(s;r)ds + \beta \int_{a}^{c} K(s,t)F(s;r)ds.
\end{align*}
\]

(6.31)

Eq. (6.31) is a system of linear Volterra integral equations in crisp case for each \( 0 \leq r \leq 1 \) and \( a \leq t \leq b \), therefore using VIM and Eq. (5.30), we can obtain the following iteration formulas:

\[
\begin{align*}
F_{n+1}(t;r) &= F_{n}(t;r) - \int_{0}^{t} \{ F'_n(\tau;r) - f'(\tau;r) - \beta \int_{a}^{c} \frac{\partial K}{\partial \tau}(s,\tau)F_{n}(s;r)ds \} d\tau, \\
F_{n+1}(t;r) &= F_{n}(t;r) - \int_{0}^{t} \{ F'_n(\tau;r) - f'(\tau;r) - \beta \int_{a}^{c} \frac{\partial K}{\partial \tau}(s,\tau)F_{n}(s;r)ds \} d\tau.
\end{align*}
\]

(6.32)

Using the current iteration formulas, we can find a solution of system (6.31) and hence obtain a fuzzy solution of the linear fuzzy Volterra integral equation of the second kind.
7 Test problems

In this section, we apply variational iteration method for three fuzzy Volterra integral equations. We compare results with exact solutions (see Tables 1, 2 and 3) so, approximate solutions and exact solutions are compared in Figs. 1, 2 and 3 for a fixed t.

Example 7.1. Consider the fuzzy Volterra integral equation with

\[ F(t;r) = \frac{1}{30} \sin(t/2)[16 + 22r + 26r^2 - 4r^3] \]
\[ + \frac{1}{30} \sin^2(t/2)[-12 + 3r + 3r^3] + \frac{1}{30} \sin(t/2) \sin(3t/2)[4 - r - r^3], \tag{7.33} \]
\[ \overline{F}(t;r) = \frac{1}{30} \sin(t/2)[104 - 22r + 4r^2 - 26r^3] \]
\[ + \frac{1}{30} \sin^2(t/2)[-3r - 3r^2] + \frac{1}{30} \sin(t/2) \sin(3t/2)[r + r^3], \tag{7.34} \]

and kernel

\[ K(s,t) = 0.1 \sin(s) \sin(t/2), \quad 0 \leq t \leq 2\pi, \quad 0 \leq s \leq t, \tag{7.35} \]

and \( a = 0, b = 2\pi, \beta = 1 \) and \( F(0;r) = \overline{F}(0;r) = 0 \). The exact solution in this case is given by

\[ F(t;r) = (r + r^2) \sin(t/2), \tag{7.36} \]
\[ \overline{F}(t;r) = (4 - r - r^3) \sin(t/2). \tag{7.37} \]

In this example, \( K(s,t) \geq 0 \) for each \( 0 \leq s \leq \pi \) and \( K(s,t) \leq 0 \) for each \( \pi \leq s \leq t \). Considering the initial approximations \( F_0(t;r) = 0 \) and \( \overline{F}_0(t;r) = 0 \) in Eq. (6.32) and using MATLAB software we obtain

\[ F_1(t;r) = \frac{1}{15} \sin(t/2)[11r + 13r^2 + 8 - 2r^3] + \frac{1}{15} \cos(t)[4 - r - r^3] \]
\[ - \frac{1}{60} \cos(2t)[4 - r - r^3] - \frac{1}{20}[4 - r - r^3], \]

\[ \overline{F}_1(t;r) = \frac{1}{15} \sin(t/2)[52 - 11r - 13r^3 + 2r^2] + \frac{1}{15} \cos(t)[r + r^3] \]
\[ - \frac{1}{60} \cos(2t)[r + r^3] - \frac{1}{20}[r + r^3], \]

\[ F_2(t;r) = \frac{1}{7200} \sin(t/2)[6781r + 6973r^2 + 768 - 192r^3] \]
\[ + \frac{7}{2400} \sin(3t/2)[r + r^2] - \frac{7}{1200} \sin(5t/2)[r + r^2] \]
\[ + \frac{1}{1200} \sin(7t/2)[r + r^2] + \frac{2}{225} \cos(t)[4 - 2r - r^2 - r^3] \]
\[ - \frac{1}{450} \cos(2t)[4 - 2r - r^2 - r^3] - \frac{1}{150}[4 - 2r - r^2 - r^3], \]

\[ \overline{F}_2(t;r) = \frac{1}{7200} \sin(t/2)[28892 - 6781r - 6973r^3 + 192r^2] \]
\[ + \frac{7}{2400} \sin(3t/2)[4 - r - r^3] - \frac{7}{1200} \sin(5t/2)[4 - r - r^3] \]
\[ + \frac{1}{1200} \sin(7t/2)[4 - r - r^3] - \frac{2}{225} \cos(t)[4 - 2r - r^2 - r^3] \]
\[ + \frac{1}{450} \cos(2t)[4 - 2r - r^2 - r^3] + \frac{1}{150}[4 - 2r - r^2 - r^3], \]
\[
F_3(t,r) = \frac{1}{162000} \sin \left( \frac{t}{2} \right) [16012r + 161063r^2 + 3748 - 937r^3] \\
- \frac{7}{18000} \sin \left( \frac{3t}{2} \right) [4 - 2r - r^2 - r^3] \\
+ \frac{7}{270000} \sin \left( \frac{5t}{2} \right) [2 - 10r - 5r^2 - 5r^3] \\
- \frac{1}{270000} \sin \left( \frac{7t}{2} \right) [2 - 10r - 5r^2 - 5r^3] \\
+ \frac{1}{216000} \cos \left( t \right) [1676 - 803r - 384r^2 - 419r^3] \\
- \frac{1}{270000} \cos \left( 2t \right) [58 - 265r - 120r^2 - 145r^3] \\
+ \frac{1}{28800} \cos \left( 3t \right) [4 - r - r^2] \\
- \frac{1}{129600} \cos \left( 4t \right) [4 - r - r^3] \\
+ \frac{1}{1296000} \cos \left( 5t \right) [4 - r - r^3] \\
- \frac{1}{72000} [412 - 199r - 96r^2 - 103r^3],
\]

\[
F_3(t,r) = \frac{1}{162000} \sin \left( \frac{t}{2} \right) [644252 - 160126r - 161063r^3 + 937r^2] \\
+ \frac{7}{18000} \sin \left( \frac{3t}{2} \right) [4 - 2r - r^2 - r^3] \\
- \frac{7}{270000} \sin \left( \frac{5t}{2} \right) [2 - 10r - 5r^2 - 5r^3] \\
+ \frac{1}{270000} \sin \left( \frac{7t}{2} \right) [2 - 10r - 5r^2 - 5r^3] \\
- \frac{1}{216000} \cos \left( t \right) [1536 - 803r - 419r^2 - 384r^3] \\
+ \frac{1}{54000} \cos \left( 2t \right) [96 - 53r - 29r^2 - 24r^3] \\
+ \frac{1}{28800} \cos \left( 3t \right) [r + r^2] \\
- \frac{1}{129600} \cos \left( 4t \right) [r + r^2] \\
+ \frac{1}{1296000} \cos \left( 5t \right) [r + r^2] \\
+ \frac{1}{72000} [384 - 199r - 103r^2 - 96r^3],
\]
Table 1.a The numerical results for $F_3(p; r)$, $F_7(p; r)$ and $F_{10}(p; r)$ in comparison with $F(p; r)$

| $r$  | $|F - F_3|$  | $|F - F_7|$  | $|F - F_{10}|$ |
|-----|-------------|-------------|-------------|
| 0   | 1.9490e-03  | 2.7045e-06  | 1.9250e-08  |
| 0.1 | 1.8447e-03  | 2.5618e-06  | 1.8234e-08  |
| 0.2 | 1.7275e-03  | 2.4016e-06  | 1.7094e-08  |
| 0.3 | 1.5945e-03  | 2.2197e-06  | 1.5799e-08  |
| 0.4 | 1.4427e-03  | 2.0121e-06  | 1.4322e-08  |
| 0.5 | 1.2691e-03  | 1.7748e-06  | 1.2633e-08  |
| 0.6 | 1.0710e-03  | 1.5037e-06  | 1.0703e-08  |
| 0.7 | 8.4529e-04  | 1.1947e-06  | 1.5035e-09  |
| 0.8 | 5.8912e-04  | 8.4379e-07  | 8.0059e-09  |
| 0.9 | 2.9955e-04  | 4.4691e-07  | 6.1810e-09  |
| 1.0 | 2.6373e-05  | 4.9549e-12  | 0           |

Table 1.b The numerical results for $F_3(p; r)$, $F_7(p; r)$ and $F_{10}(p; r)$ in comparison with $F(p; r)$

| $r$  | $|F - F_3|$  | $|F - F_7|$  | $|F - F_{10}|$ |
|-----|-------------|-------------|-------------|
| 0   | 2.0016e-03  | 2.7045e-06  | 1.9250e-08  |
| 0.1 | 1.8975e-03  | 2.5618e-06  | 1.8234e-08  |
| 0.2 | 1.7806e-03  | 2.4016e-06  | 1.7094e-08  |
| 0.3 | 1.6480e-03  | 2.2197e-06  | 1.5799e-08  |
| 0.4 | 1.4966e-03  | 2.0121e-06  | 1.4322e-08  |
| 0.5 | 1.3235e-03  | 1.7748e-06  | 1.2633e-08  |
| 0.6 | 1.1256e-03  | 1.5037e-06  | 1.0703e-08  |
| 0.7 | 8.9990e-04  | 1.1947e-06  | 1.5035e-09  |
| 0.8 | 6.4348e-04  | 8.4380e-07  | 8.0059e-09  |
| 0.9 | 3.5329e-04  | 4.4692e-07  | 6.1810e-09  |
| 1.0 | 2.6373e-05  | 4.9549e-12  | 0           |

Example 7.2. Consider the fuzzy Volterra integral equation with

$$\mathcal{L}(t; r) = rt - t^2\left[\frac{2}{3}rt^3 - \frac{4}{3}t^3 + \frac{1}{2}rt^2 + \frac{1}{12}r - \frac{1}{12}\right],$$

(7.38)

$$\mathcal{J}(t; r) = (2-r)t + t^2\left[\frac{2}{3}rt^3 - \frac{1}{2}rt^2 + \frac{1}{12}r - \frac{1}{12}\right],$$

(7.39)

and kernel

$$K(s, t) = t^2(1-2s), \quad 0 \leq t \leq 1, \quad 0 \leq s \leq t,$$

(7.40)

and $a = 0$, $b = 1$, $\beta = 1$ and $E(0; r) = F(0; r) = 0$. The exact solution in this case is given by

$$E(t; r) = rt,$$

(7.41)

$$F(t; r) = (2-r)t.$$  

(7.42)

In this example, $K(s, t) \geq 0$ for each $0 \leq s \leq \frac{1}{2}$ and $K(s, t) \leq 0$ for each $\frac{1}{2} \leq s \leq t$. Considering the initial approximations $F_0(t; r) = 0$ and $F_{10}(t; r) = 0$ in Eq. (6.32) and using MATLAB software we obtain

$$E_1(t; r) = rt - t^2\left[\frac{2}{3}rt^3 - \frac{4}{3}t^3 + \frac{1}{2}rt^2 + \frac{1}{12}r - \frac{1}{12}\right],$$

$$F_1(t; r) = (2-r)t + t^2\left[\frac{2}{3}rt^3 - \frac{1}{2}rt^2 + \frac{1}{12}r - \frac{1}{12}\right],$$

$$F_{10}(t; r) = 0.$$
\begin{align*}
F_2(t;r) &= rt - t^2 \left[ \frac{4}{21} r t^7 - \frac{5}{18} r t^6 + \frac{1}{24} r t^5 - \frac{1}{24} t^4 - \frac{1}{36} r t^3 \\
&\quad + \frac{1}{36} r^3 + \frac{1}{840} r - \frac{1}{840} t^3 \right],\\
F_3(t;r) &= (2 - r) t + t^2 \left[ \frac{4}{21} r t^7 - \frac{8}{21} r t^6 + \frac{5}{6} r t^5 + \frac{1}{10} r t^4 - \frac{1}{5} r t^3 \right] \\
&\quad + \frac{1}{24} r t^4 - \frac{1}{24} t^4 - \frac{1}{36} r t^3 + \frac{1}{36} r^3 + \frac{1}{840} r - \frac{1}{840} t^3,\\
F_4(t;r) &= rt - t^2 \left[ \frac{8}{231} r t^{11} - \frac{16}{231} t^{11} - \frac{47}{630} r t^{10} - \frac{47}{315} t^{10} + \frac{43}{216} r t^9 - \frac{43}{405} t^9 \right] \\
&\quad - \left( \frac{1}{480} r t^8 + \frac{7}{480} t^8 - \frac{1}{72} r t^7 + \frac{1}{72} t^7 + \frac{1}{216} r t^6 + \frac{1}{216} t^6 + \frac{1}{1680} r t^4 \right) \\
&\quad - \left( \frac{1}{1680} r t^4 - \frac{1}{2520} r t^3 + \frac{1}{2520} t^3 + \frac{1}{2291} r t^2 + \frac{1}{2291} t^2 + \frac{1}{127733760} f - \frac{1}{127733760} \right],\
F_5(t;r) &= (2 - r) t + t^2 \left[ \frac{8}{231} r t^{11} - \frac{47}{630} r t^{10} + \frac{43}{810} r t^9 - \frac{1}{480} r t^8 - \frac{1}{96} t^8 \right] \\
&\quad - \left( \frac{1}{72} r t^7 + \frac{1}{72} t^7 + \frac{1}{216} r t^6 + \frac{1}{216} t^6 + \frac{1}{1680} r t^4 \right) \\
&\quad - \left( \frac{1}{1680} r t^4 - \frac{1}{2520} r t^3 + \frac{1}{2520} t^3 + \frac{1}{2291} r t^2 + \frac{1}{2291} t^2 + \frac{1}{127733760} f - \frac{1}{127733760} \right],
\end{align*}

Table 2.a The numerical results for $F_3(\frac{1}{2}; r)$, $F_7(\frac{1}{2}; r)$ and $F_{10}(\frac{1}{2}; r)$ in comparison with $F(\frac{1}{2}; r)$

| $r$   | $|F - F_3| - 7.2920 \times 10^{-8}$ | $|F - F_7| - 1.7190 \times 10^{-15}$ | $|F - F_{10}| - 5.8342 \times 10^{-21}$ |
|-------|-------------------------------------|-------------------------------------|----------------------------------------|
| 0     | 3.2920e-08                         | 1.7190e-15                         | 5.8342e-21                            |
| 0.1   | 2.9550e-08                         | 1.5543e-15                         | 0                                      |
| 0.2   | 2.6180e-08                         | 1.3878e-15                         | 0                                      |
| 0.3   | 2.2809e-08                         | 1.2212e-15                         | 0                                      |
| 0.4   | 1.9439e-08                         | 1.0547e-16                         | 0                                      |
| 0.5   | 1.6069e-08                         | 8.8818e-16                         | 0                                      |
| 0.6   | 1.2699e-08                         | 7.2164e-16                         | 0                                      |
| 0.7   | 9.3283e-09                         | 5.5511e-16                         | 0                                      |
| 0.8   | 5.9580e-09                         | 3.8858e-16                         | 0                                      |
| 0.9   | 2.5877e-09                         | 2.2204e-16                         | 0                                      |
| 1.0   | 7.8256e-10                         | 0                                  | 0                                      |
In this example, and a $= \frac{\pi}{2}$, $\beta = 1$ and $F(0; r) = F(0; r) = 0$. The exact solution in this case is given by

$$E(t; r) = t^3 (r^5 + 2r),$$

$$F(t; r) = t^3 (6 - 3r^3).$$

In this example, $K(s, t) \geq 0$ for each $0 \leq s \leq t$. Considering the initial approximations $E_0(t; r) = 0$ and $F_0(t; r) = 0$ in Eq. (6.32) and using MATLAB software we obtain

$$E_1(t; r) = 2t(r^5 + 2r)[3 - 3\cos(t) - t^2],$$

$$F_1(t; r) = 6t(2 - r^3)[3 - 3\cos(t) - t^2],$$

$$E_2(t; r) = \frac{1}{2}t(r^5 + 2r)[48 - 3\cos(t)(t^2 + 16) - 3t\sin(t) - 16r^2],$$

$$F_2(t; r) = \frac{3}{2}t(2 - r^3)[48 - 3\cos(t)(t^2 + 16) - 3t\sin(t) - 16r^2],$$

$$E_3(t; r) = \frac{1}{16}t(r^5 + 2r)[1248 + 416r^2 + 3\cos(t)(r^4 + 33r^2 + 416) + \sin(t)(10r^3 + 93r)],$$

$$F_3(t; r) = \frac{3}{16}t(2 - r^3)[1248 + 416r^2 + 3\cos(t)(r^4 + 33r^2 + 416) + \sin(t)(10r^3 + 93r)].$$

### Table 2.b

The numerical results for $F_3(\frac{1}{2}; r)$, $F_7(\frac{1}{2}; r)$ and $F_{10}(\frac{1}{2}; r)$ in comparison with $\vec{F}(\frac{1}{2}; r)$

| r  | $|F - F_3| |$ | $|F - F_7| |$ | $|F - F_{10}| |$ |
|----|---------|---------|---------|
| 0  | 3.4485e-08 | 1.7764e-15 | 0 |
| 0.1| 3.1115e-08 | 1.5543e-15 | 0 |
| 0.2| 2.7745e-08 | 1.4433e-15 | 0 |
| 0.3| 2.4375e-08 | 1.2212e-15 | 0 |
| 0.4| 2.1004e-08 | 1.1102e-15 | 0 |
| 0.5| 1.7634e-08 | 8.8818e-16 | 0 |
| 0.6| 1.4264e-08 | 6.6613e-16 | 0 |
| 0.7| 1.0893e-08 | 5.5111e-16 | 0 |
| 0.8| 7.5231e-09 | 4.4009e-16 | 0 |
| 0.9| 4.1528e-09 | 3.3307e-16 | 0 |
| 1.0| 7.8256e-10 | 1.1102e-16 | 0 |

Example 7.3. Consider the fuzzy Volterra integral equation with

$$f(t; r) = 2t(r^5 + 2r)[3 - 3\cos(t) - t^2],$$

$$\vec{f}(t; r) = 6t(2 - r^3)[3 - 3\cos(t) - t^2],$$

and kernel

$$K(s, t) = t\cos(s - t), \quad 0 \leq t \leq \frac{\pi}{2}, \quad 0 \leq s \leq t,$$

and $a = 0$, $b = \frac{\pi}{2}$, $\beta = 1$ and $E(0; r) = \vec{F}(0; r) = 0$. The exact solution in this case is given by

$$E(t; r) = t^3 (r^5 + 2r),$$

$$F(t; r) = t^3 (6 - 3r^3).$$
Table 3.a The numerical results for $F_{3}(\frac{\pi}{4};r)$, $F_{7}(\frac{\pi}{4};r)$ and $F_{10}(\frac{\pi}{4};r)$ in comparison with $F(\frac{\pi}{4};r)$

| r     | $|F - F_{3}|$ | $|F - F_{7}|$ | $|F - F_{10}|$ |
|-------|-------------|-------------|-------------|
| 0     | 0           | 0           | 0           |
| 0.1   | 7.1048e-06  | 2.1245e-11  | 2.2866e-13  |
| 0.2   | 1.4220e-05  | 4.2522e-11  | 4.5761e-13  |
| 0.3   | 2.1400e-05  | 6.3991e-11  | 6.8873e-13  |
| 0.4   | 2.8782e-05  | 8.6065e-11  | 9.2626e-13  |
| 0.5   | 3.6632e-05  | 1.0954e-10  | 1.1789e-12  |
| 0.6   | 4.5389e-05  | 1.3572e-10  | 1.4606e-12  |
| 0.7   | 5.5701e-05  | 1.6656e-10  | 1.7927e-12  |
| 0.8   | 6.8475e-05  | 2.0476e-10  | 2.2038e-12  |
| 0.9   | 8.4916e-05  | 2.5392e-10  | 2.7329e-12  |
| 1.0   | 1.0657e-04  | 3.1866e-10  | 3.4297e-12  |

Table 3.b The numerical results for $F_{3}(\frac{\pi}{4};r)$, $F_{7}(\frac{\pi}{4};r)$ and $F_{10}(\frac{\pi}{4};r)$ in comparison with $F(\frac{\pi}{4};r)$

| r     | $|F - F_{3}|$ | $|F - F_{7}|$ | $|F - F_{10}|$ |
|-------|-------------|-------------|-------------|
| 0     | 2.1313e-04  | 6.3733e-10  | 6.8594e-12  |
| 0.1   | 2.1303e-04  | 6.3701e-10  | 6.8563e-12  |
| 0.2   | 2.1228e-04  | 6.3478e-10  | 6.8319e-12  |
| 0.3   | 2.1026e-04  | 6.2872e-10  | 6.7670e-12  |
| 0.4   | 2.0631e-04  | 6.1693e-10  | 6.6400e-12  |
| 0.5   | 1.9981e-04  | 5.9749e-10  | 6.4313e-12  |
| 0.6   | 1.9011e-04  | 5.6850e-10  | 6.1187e-12  |
| 0.7   | 1.7658e-04  | 5.2803e-10  | 5.6830e-12  |
| 0.8   | 1.5857e-04  | 4.7417e-10  | 5.1039e-12  |
| 0.9   | 1.3545e-04  | 4.0502e-10  | 4.3592e-12  |
| 1.0   | 1.0657e-04  | 3.1866e-10  | 3.4297e-12  |

Fig. 1. Exact and approximate solutions for Example 7.1 with three iterations ($t = \pi$).
8 conclusion

In this paper, we used variational iteration method (VIM) to obtain solution of the linear fuzzy Volterra integral equation of the second kind. We emphasize that this work which presents applicable computational methods, may help to narrow the existing gap between the theoretical research on FIEs.

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