First Order Linear Homogeneous Ordinary Differential Equation in Fuzzy Environment Based On Laplace Transform

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Abstract

In this paper the First Order Linear Ordinary Differential Equations (FOLODE) are described in fuzzy environment. Here coefficients and /or initial condition of FOLODE are taken as Generalized Triangular Fuzzy Numbers (GTFNs). The solution procedure of the FOLODE is developed by Laplace transform. It is illustrated by numerical examples. Finally imprecise bank account problem and concentration of drug in blood problem are described.

Keywords: Fuzzy Differential Equation, Generalized Triangular fuzzy number, 1st Order differential equation, Laplace transform.

1 Introduction

The concept of Fuzzy number and fuzzy arithmetic were first introduced by L. A. Zadeh [10] and Dubois & Parade [5]. Fuzzy differential equation (FDE) were First formulated by O. Kaleva [15]. To Model dynamical system under possibilistic uncertainty in a natural way, the usage of fuzzy differential equation has been proving resultant. In many applications, First order linear fuzzy differential equation are one of the simplest fuzzy differential equation. Buckley & Feuring [9] and Buckley et al [8] gave a very general formulation of first order initial value problem.

In many papers initial condition of a FDE was taken as different type of fuzzy numbers. Buckley et al [8] used triangular fuzzy number, Duraisamy & Usha [4] used Trapezoidal fuzzy number, Bede et al [3] used LR type fuzzy number. FDE has also used in many models such as HIV model (Hassan et al [7]), decay model (Diniz et al[6]), predator-prey model (Ahmad & Baets[12]), population model (Barros et al[11]), civil engineering (Oberguggenberger & Pittschmann [13] ) and hydraulic (Bencsik et al,[2]) models, Growth model [22], Bacteria culture model [23].

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Laplace transform is a very useful tool to solve differential equation. Laplace transforms give the solution of a differential equations satisfying the initial condition directly without use the general solution of the differential equation. Fuzzy Laplace Transform (FLT) was first introduced by Allahviranloo & Ahmadi [19]. Here first order fuzzy differential equation with fuzzy initial condition is solved by FLT. Tolouti & Ahmadi [18] applied the FLT in 2nd order FDE. FLT also used to solve many areas of differential equation. Salahshour et al [16] used FLT in Fuzzy fractional differential equation. Salahshour & Haghi used FLT in Fuzzy Heat Equation. Ahmad et al [14] used FLT in Fuzzy Duffing’s Equation.

In this paper, we have considered FOLODE and have described its solution procedure in section-3 by Fuzzy Laplace Transform. Here all fuzzy numbers are taken as GTFNs. The method was discussed by different examples. In section-4, we have also described two models (bank account and concentration of drug in blood problem) in fuzzy environment and illustrated numerically.

2 Preliminary Concept

Definition 2.1.

Fuzzy Set: Let X be a universal set. The fuzzy set $\tilde{A} \subseteq X$ is defined by the set of tuples as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : \mu_{\tilde{A}}: X \rightarrow [0,1]\}$. The membership function $\mu_{\tilde{A}}(x)$ of a fuzzy set is a function with mapping $\mu_{\tilde{A}}: X \rightarrow [0,1]$. So every element x in X has membership degree $\mu_{\tilde{A}}(x)$ in [0,1] which is a real number. As closer the value of $\mu_{\tilde{A}}(x)$ is to 1, so much x belongs to $\tilde{A}$. $\mu_{\tilde{A}}(x_1) > \mu_{\tilde{A}}(x_2)$ implies relevance of $x_1$ in $\tilde{A}$ is greater than the relevance of $x_2$ in $\tilde{A}$. If $\mu_{\tilde{A}}(x_0) = 1$, then we say $x_0$ exactly belongs to $\tilde{A}$, if $\mu_{\tilde{A}}(x_1) = 0$ we say $x_1$ does not belong to $\tilde{A}$, and if $\mu_{\tilde{A}}(x_2) = a$ where $0 < a < 1$. We say the membership value of $x_2$ in $\tilde{A}$ is a. When $\mu_{\tilde{A}}(x)$ is always equal to 1 or 0 we get a crisp (classical) subset of X. Here the term “crisp” means not fuzzy. A crisp set is a classical set. A crisp number is a real number.

Definition 2.2.

$\alpha$-Level or $\alpha$-cut of a fuzzy set: Let X be an universal set. Let $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\} (\subseteq X)$ be a fuzzy set. $\alpha$-cut of the fuzzy set $\tilde{A}$ is a crisp set. It is denoted by $A_{\alpha}$. It is defined as $A_{\alpha} = \{x : \mu_{\tilde{A}}(x) \geq \alpha \ \forall x \in X\}$

Note: $A_{\alpha}$ is a crisp set with its characteristic function $\chi_{A_{\alpha}}(x)$ defined as

$\chi_{A_{\alpha}}(x) = 1 \mu_{\tilde{A}}(x) \geq \alpha \ \forall x \in X$

$= 0$ otherwise.

Definition 2.3.

Generalized Fuzzy number (GFN): Generalized Fuzzy number $\tilde{A}$ as $\tilde{A} = (a_1, a_2, a_3, a_4 : \omega)$ where $0 < \omega \leq 1$, and $a_1, a_2, a_3, a_4$ ($a_1 < a_2 < a_3 < a_4$) are real numbers. The generalized fuzzy number $\tilde{A}$ is a fuzzy subset of real line R, whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

1) $\mu_{\tilde{A}}(x): R \rightarrow [0, 1]$
2) $\mu_{\tilde{A}}(x) = 0$ for $x \leq a_1$
3) $\mu_{\tilde{A}}(x)$ is strictly increasing function for $a_1 \leq x \leq a_2$
4) $\mu_{\tilde{A}}(x) = \omega$ for $a_2 \leq x \leq a_3$
5) $\mu_{\tilde{A}}(x)$ is strictly decreasing function for $a_3 \leq x \leq a_4$
6) $\mu_{\tilde{A}}(x) = 0$ for $a_4 \leq x$
Definition 2.4.

**Generalized triangular fuzzy number (GTFN):** A Generalized Fuzzy number is called a Generalized Triangular Fuzzy Number if it is defined by $\tilde{A} = (a_1, a_2, a_3; \omega)$ its membership function is given by

$$
\mu_{\tilde{A}}(x) = \begin{cases} 
0, & x \leq a_1 \\
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\
\frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3 \\
0, & x \geq a_3 
\end{cases}
$$

or, $\mu_{\tilde{A}}(x) = \max \left( \min \left( \frac{x-a_1}{a_2-a_1}, \omega, \frac{a_3-x}{a_3-a_2} \right), 0 \right)$

Definition 2.5.

**TFN:** In the previous definition if $\omega = 1$ then $\tilde{A}$ is called a triangular fuzzy number (TFN). Then $\tilde{A} = (a_1, a_2, a_3)$ and its membership function is given by

$$
\mu_{\tilde{A}}(x) = \begin{cases} 
0, & x \leq a_1 \\
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\
1, & x = a_2 \\
\frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3 \\
0, & x \geq a_3 
\end{cases}
$$

or, $\mu_{\tilde{A}}(x) = \max \left( \min \left( \frac{x-a_1}{a_2-a_1}, 1, \frac{a_3-x}{a_3-a_2} \right), 0 \right)$
Definition 2.6.
A generalized fuzzy number is completely determined by an interval \([X_1(\alpha), X_2(\alpha)]\). Here two functions \(X_1(\alpha), X_2(\alpha)\) for \(0 \leq \alpha \leq \omega, \ \omega \in (0,1)\) satisfy the following axioms:

1. \(X_1(\alpha)\) is a bounded monotonic increasing left continuous function over \([0, \omega]\) i.e. \(\frac{\partial}{\partial \alpha} [X_1(\alpha)] > 0\).

2. \(X_2(\alpha)\) is a bounded monotonic decreasing left continuous function over \([0, \omega]\) i.e. \(\frac{\partial}{\partial \alpha} [X_2(t, \alpha)] < 0\).

3. \(X_1(\alpha) \leq X_2(\alpha) \forall \alpha \in [0, \omega]\) where \(0 < \omega \leq 1\).

Definition 2.7.
Fuzzy ordinary differential equation (FODE):
Consider the 1st Order Linear Homogeneous Ordinary Differential Equation (ODE)
\[
\frac{dx}{dt} = kx
\]
with initial condition \(x(t_0) = x_0\).
The above ODE is called FODE if any one of the following three cases holds:

(i) Only \(x_0\) is a generalized fuzzy number (Type-I).
(ii) Only \(k\) is a generalized fuzzy number (Type-II).
(iii) Both \(k\) and \(x_0\) are generalized fuzzy numbers (Type-III).

Definition 2.8.
Strong and Weak solution of FODE:
Consider the 1st order linear homogeneous fuzzy ordinary differential equation
\[
\frac{dx}{dt} = kx
\]
with \(x(t_0) = x_0\). Here \(k\) or (and) \(x_0\) be generalized fuzzy number(s).
Let the solution of the above FODE be \(\tilde{x}(t)\) and its \(\alpha\)-cut be \(x(t, \alpha) = [x_1(t, \alpha), x_2(t, \alpha)]\).
If \(x_1(t, \alpha) \leq x_2(t, \alpha) \forall \alpha \in [0, \omega]\) where \(0 < \omega \leq 1\) then \(\tilde{x}(t)\) is called strong solution otherwise \(\tilde{x}(t)\) is called weak solution and in that case the \(\alpha\)-cut of the solution is given by \(x(t, \alpha) = [\min\{x_1(t, \alpha), x_2(t, \alpha)\}, \max\{x_1(t, \alpha), x_2(t, \alpha)\}]\).

Definition 2.9.
[20] Let \(f : (a, b) \to E\) and \(x_0 \in (a, b)\). We say that \(f\) is strongly generalized differential at \(x_0\) (Bede-Gal differential) if there exists an element \(f'(x_0) \in E\), such that

(i) for all \(h > 0\) sufficiently small, \(\exists f(x_0 + h) - h f(x_0), \exists f(x_0) - h f(x_0 - h)\) and the limits(in the metric \(D\))
\[
\lim_{h \to 0} \frac{f(x_0 + h) - h f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) - h f(x_0 - h)}{h} = f'(x_0)
\]
Or

(ii) for all \(h > 0\) sufficiently small, \(\exists f(x_0) - h f(x_0 + h), \exists f(x_0 - h) - h f(x_0)\) and the limits(in the metric \(D\))
\[
\lim_{h \to 0} \frac{f(x_0) - h f(x_0 + h)}{-h} = \lim_{h \to 0} \frac{f(x_0 - h) - h f(x_0)}{-h} = f'(x_0)
\]
Or

(iii) for all \(h > 0\) sufficiently small, \(\exists f(x_0 + h) - h f(x_0), \exists f(x_0 - h) - h f(x_0)\) and the limits(in the metric \(D\))
\[
\lim_{h \to 0} \frac{f(x_0 + h) - h f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0 - h) - h f(x_0)}{-h} = f'(x_0)
\]
Or
(iv) for all $h > 0$ sufficiently small, $\exists f(x_0)^{-h}f(x_0 + h), \exists f(x_0)^{-h}f(x_0 - h)$ and the limits (in the metric $D$)

$$
limit_{h \to 0} \frac{f(x_0)^{-h}f(x_0 + h)}{-h} = limit_{h \to 0} \frac{f(x_0)^{-h}f(x_0 - h)}{-h} = f'(x_0)
$$

($h$ and $-h$ at denominators mean $\frac{1}{h}$ and $-\frac{1}{h}$, respectively).

**Definition 2.10.**

[21] Let $f: R \to E$ be a function and denote $f(t) = (f(t, r), \overline{f}(t, r))$, for each $r \in [0, 1]$. Then

1. If $f$ is (i)-differentiable, then $f(t, r)$ and $\overline{f}(t, r)$ are differentiable function and $f'(t) = (f'(t, r), \overline{f}'(t, r))$.
2. If $f$ is (ii)-differentiable, then $f(t, r)$ and $\overline{f}(t, r)$ are differentiable function and $f'(t) = (\overline{f}'(t, r), f'(t, r))$.

**Definition 2.11.**

Let $f: [0, T] \to \mathbb{R}_F$. The integral of $f$ in $[0, T]$, (denoted by $\int_{[0,T]} f(t)dt$ or $\int_0^T f(t) dt$) is defined levelwise as the set if integrals of the (real) measurable selections for $[f]'$, for each $r \in [0, 1]$. We say that $f$ is integrable over $[0, T]$ if $\int_{[0,T]} f(t)dt \in \mathbb{R}_F$ and we have

$$
\left[ \int_0^T f(t) dt \right]' = \left[ \int_0^T f'(t) dt , \int_0^T \overline{f}'(t) dt \right]
$$

for each $r \in [0, 1]$.

3 Laplace Transform Method for Solving 1st Order Fuzzy Differential Equation

3.1 Solution Procedure of 1st Order Linear Homogeneous FODE of Type-I

Consider the initial value problem

$$
\frac{dx}{dt} = kx \tag{1}
$$

with fuzzy Initial Condition (IC) $\overline{x}(0) = \overline{y}_0 = (y_1, y_2, y_3; \omega)$.

Let $\overline{x}(t)$ be a solution of FODE (1).

Let $x(t, \alpha) = [x_1(t, \alpha), x_2(t, \alpha)]$ be the $\alpha$-cut of $\overline{x}(t)$.

and $[\overline{y}_0]_\alpha = [\overline{x}(0, \alpha), \overline{x}(0, \alpha)] = \left[ y_1 + \frac{a_1\tau_0}{\omega}, y_3 - \frac{a_2\tau_0}{\omega} \right] \forall \alpha \in [0, \omega], 0 < \omega \leq 1$

where $l_{y_0} = y_2 - y_1$ and $r_{y_0} = y_3 - y_2$

Here we solve the given problem for $k > 0$ and $k < 0$ respectively.

**Case 1: when $k > 0$**

Taking $\alpha$-cut of (1) we get

$$
\frac{dx}{dt} = kx_1(t, \alpha) \tag{2}
$$

and

$$
\frac{dx}{dt} = kx_2(t, \alpha) \tag{3}
$$

Taking Laplace Transform both sides of (2) we get

$$
l\left\{ \frac{dx_1}{dt} \right\} = l\{kx_1(t, \alpha)\}
$$
Or, \( sl\{x_1(t, \alpha)\} - x_1(0, \alpha) = kl\{x_1(t, \alpha)\} \)
\[ \text{Or, } l\{x_1(t, \alpha)\} = \frac{\gamma_1 + \frac{a\gamma_0}{\omega}}{s-k} \] (4)

Taking inverse Laplace Transform of (4) we get
\[ x_1(t, \alpha) = l^{-1}\left(\frac{\gamma_1 + \frac{a\gamma_0}{\omega}}{s-k}\right) = \left(\gamma_1 + \frac{a\gamma_0}{\omega}\right)e^{kt} \] (5)

similarly from equation (3) we get
\[ l\left(\frac{dx_2(t, \alpha)}{dt}\right) = l\{kx_2(t, \alpha)\} \]
Or, \( sl\{x_2(t, \alpha)\} - x_2(0, \alpha) = kl\{x_2(t, \alpha)\} \)
\[ \text{Or, } l\{x_2(t, \alpha)\} = \frac{\gamma_3 - \frac{ar\gamma_0}{\omega}}{s-k} \] (6)

Taking inverse Laplace Transform of (6) we get
\[ x_2(t, \alpha) = l^{-1}\left(\frac{\gamma_3 - \frac{ar\gamma_0}{\omega}}{s-k}\right) = \left(\gamma_3 - \frac{ar\gamma_0}{\omega}\right)e^{kt} \] (7)

Here \( \frac{\partial}{\partial \alpha} [x_1(t, \alpha)] = \frac{iv_0}{\omega}e^{kt} > 0 \), \( \frac{\partial}{\partial \alpha} [x_2(t, \alpha)] = -\frac{r\gamma_0}{\omega}e^{kt} < 0 \)
and \( x_1(t, \omega) = \gamma_2e^{kt} = x_2(t, \omega) \).

So the solution is a generalized fuzzy number with \( \alpha \)-cut
\[ x(t, \alpha) = \left[\left(\gamma_1 + \frac{a\gamma_0}{\omega}\right), \left(\gamma_3 - \frac{ar\gamma_0}{\omega}\right)\right]e^{kt}. \] It is a strong solution.

**Case 2: when \( k < 0 \)**

Let \( k = -m \) where \( m \) is a positive real number.

Then the FODE (1) becomes
\[ \frac{dx_1(t, \alpha)}{dt} = -mx_2(t, \alpha) \] (8)

and
\[ \frac{dx_2(t, \alpha)}{dt} = -mx_1(t, \alpha) \] (9)

Taking Laplace Transform both sides of (8) we get
\[ l\left(\frac{dx_1(t, \alpha)}{dt}\right) = l\{-mx_2(t, \alpha)\} \]
Or, \( sl\{x_1(t, \alpha)\} - x_1(0, \alpha) = -ml\{x_2(t, \alpha)\} \)
\[ \text{Or, } sl\{x_1(t, \alpha)\} + ml\{x_2(t, \alpha)\} = \left(\gamma_1 + \frac{a\gamma_0}{\omega}\right) \] (10)

Taking Laplace Transform both sides of (9) we get
\[ l\left(\frac{dx_2(t, \alpha)}{dt}\right) = l\{-mx_1(t, \alpha)\} \]
Or, \( sl\{x_2(t, \alpha)\} - x_2(0, \alpha) = -ml\{x_1(t, \alpha)\} \)
\[ \text{Or, } ml\{x_1(t, \alpha)\} + sl\{x_2(t, \alpha)\} = \left(\gamma_3 - \frac{ar\gamma_0}{\omega}\right) \] (11)

Solving (10) and (11) we get
\[ l(x_1(t, \alpha)) = \left( y_1 + \frac{at_{\gamma_0}}{\omega} \right) \frac{s}{s^2 - m^2} - \left( y_3 - \frac{ar_{\gamma_0}}{\omega} \right) \frac{m}{s^2 - m^2} \tag{12} \]

and

\[ l(x_2(t, \alpha)) = \left( y_3 - \frac{ar_{\gamma_0}}{\omega} \right) \frac{s}{s^2 - m^2} - \left( y_1 + \frac{at_{\gamma_0}}{\omega} \right) \frac{m}{s^2 - m^2} \tag{13} \]

Taking inverse Laplace Transform of (12) we get

\[
x_1(t, \alpha) = \left( y_1 + \frac{at_{\gamma_0}}{\omega} \right) \frac{1}{l-1}\left( \frac{s}{s^2 - m^2} \right) - \left( y_3 - \frac{ar_{\gamma_0}}{\omega} \right) \frac{1}{l-1}\left( \frac{m}{s^2 - m^2} \right)
\]

\[
= \left( y_1 + \frac{at_{\gamma_0}}{\omega} \right) \cosh mt - \left( y_3 - \frac{ar_{\gamma_0}}{\omega} \right) \sinh mt
\]

\[
= \frac{1}{2} \left[ y_1 + y_3 + \frac{\alpha}{\omega} (l_{\gamma_0} - r_{\gamma_0}) \right] e^{-mt} + \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) (l_{\gamma_0} + r_{\gamma_0}) e^{mt}
\tag{14}
\]

Taking inverse Laplace Transform of (13) we get

\[
x_2(t, \alpha) = \left( y_3 - \frac{ar_{\gamma_0}}{\omega} \right) \frac{1}{l-1}\left( \frac{s}{s^2 - m^2} \right) - \left( y_1 + \frac{at_{\gamma_0}}{\omega} \right) \frac{1}{l-1}\left( \frac{m}{s^2 - m^2} \right)
\]

\[
= \left( y_3 - \frac{ar_{\gamma_0}}{\omega} \right) \cosh mt - \left( y_1 + \frac{at_{\gamma_0}}{\omega} \right) \sinh mt
\]

\[
= \frac{1}{2} \left[ y_1 + y_3 + \frac{\alpha}{\omega} \left( l_{\gamma_0} - r_{\gamma_0} \right) \right] e^{-mt} - \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) \left( l_{\gamma_0} + r_{\gamma_0} \right) e^{mt}
\tag{15}
\]

Now

\[
\frac{\partial}{\partial t} [x_1(t, \alpha)] = \frac{1}{2 \omega} \left( l_{\gamma_0} - r_{\gamma_0} \right) e^{-m(t-t_0)} + \frac{1}{2 \omega} \left( l_{\gamma_0} + r_{\gamma_0} \right) e^{m(t-t_0)}
\]

\[
\frac{\partial}{\partial t} [x_2(t, \alpha)] = \frac{1}{2 \omega} \left( l_{\gamma_0} - r_{\gamma_0} \right) e^{-m(t-t_0)} - \frac{1}{2 \omega} \left( l_{\gamma_0} + r_{\gamma_0} \right) e^{m(t-t_0)}
\]

and \( x_1(t, \omega) = x_2(t, \omega) \)

Here three cases arise.

**Case 3.1.2.1: When \( l_{\gamma_0} = r_{\gamma_0} \)**

Here \( \gamma_0 = (y_1, y_2, y_3; \omega) \) is a symmetric GTFN.

\[ \frac{\partial}{\partial t} [x_1(t, \alpha)] > 0 \quad \text{and} \quad \frac{\partial}{\partial t} [x_2(t, \alpha)] < 0 \]

Hence

\[
\left[ \frac{1}{2} \left( y_1 + y_3 + \frac{\alpha}{\omega} (l_{\gamma_0} - r_{\gamma_0}) \right) e^{-m(t-t_0)} + \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) \left( l_{\gamma_0} + r_{\gamma_0} \right) e^{m(t-t_0)} \right]
\]

\[
\left[ \frac{1}{2} \left( y_1 + y_3 + \frac{\alpha}{\omega} (l_{\gamma_0} - r_{\gamma_0}) \right) e^{-m(t-t_0)} - \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) \left( l_{\gamma_0} + r_{\gamma_0} \right) e^{m(t-t_0)} \right]
\]

is the \( \alpha \)-cut of the strong solution of the FODE (1).

**Case 3.1.2.2: when \( l_{\gamma_0} < r_{\gamma_0} \)**

Here \( \frac{\partial}{\partial t} [x_2(t, \alpha)] < 0 \).

In this case, the strong solution of FODE (1) will exist if \( \frac{\partial}{\partial t} [x_1(t, \alpha)] > 0 \)

i.e. \( t > t_0 + \frac{1}{2m} \log \left[ \frac{r_{\gamma_0} - l_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}} \right] \)

Hence

\[
\left[ \frac{1}{2} \left( y_1 + y_3 + \frac{\alpha}{\omega} (l_{\gamma_0} - r_{\gamma_0}) \right) e^{-m(t-t_0)} + \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) \left( l_{\gamma_0} + r_{\gamma_0} \right) e^{m(t-t_0)} \right]
\]

\[
\left[ \frac{1}{2} \left( y_1 + y_3 + \frac{\alpha}{\omega} (l_{\gamma_0} - r_{\gamma_0}) \right) e^{-m(t-t_0)} - \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) \left( l_{\gamma_0} + r_{\gamma_0} \right) e^{m(t-t_0)} \right]
\]

is the \( \alpha \)-cut of the strong solution of the FODE (1) if \( t > t_0 + \frac{1}{2m} \log \left[ \frac{r_{\gamma_0} - l_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}} \right] \).
Case 3.1.2.3: when \( l_{\gamma_0} > r_{\gamma_0} \)

Here \( \frac{\partial}{\partial \alpha} [x_1(t, \alpha)] > 0 \)

In this case the strong solution of the FODE (1) will exist if \( \frac{\partial}{\partial \alpha} [x_2(t, \alpha)] < 0 \)

i.e. if \( t > t_0 + \frac{1}{2m} \log \left[ \frac{r_{\gamma_0} - r_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}} \right] \).

Hence

\[
\left[ \frac{1}{2} \left( y_1 + \frac{\alpha}{\omega} (l_{\gamma_0} - r_{\gamma_0}) \right) e^{-m(t-t_0)} + \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) (l_{\gamma_0} + r_{\gamma_0}) e^{m(t-t_0)} \right]
\]

\[
\left[ \frac{1}{2} \left( y_1 + \frac{\alpha}{\omega} (l_{\gamma_0} - r_{\gamma_0}) \right) e^{-m(t-t_0)} - \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) (l_{\gamma_0} + r_{\gamma_0}) e^{m(t-t_0)} \right]
\]

is the \( \alpha \)-cut of the strong solution of the FODE (1) if \( t > t_0 + \frac{1}{2m} \log \left[ \frac{r_{\gamma_0} - r_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}} \right] \).

### 3.2 Solution Procedure of 1st Order Linear Homogeneous FODE of Type-II

Consider the initial value problem

\[
\frac{dx}{dt} = \dot{k}x
\]

with IC \( x(0) = \gamma \dot{k} = (\beta_1, \beta_2, \beta_3; \lambda) \)

Let \( \ddot{x}(t) \) be the solution of FODE (18)

Let \( x(t, \alpha) = [x_1(t, \alpha), x_2(t, \alpha)] \) be the \( \alpha \)-cut of the solution and the \( \alpha \)-cut of \( \dot{k} \) be

\( \dot{k} = [k_1(\alpha), k_2(\alpha)] = \left[ \beta_1 + \frac{\alpha \lambda k}{\lambda}, \beta_3 - \frac{\alpha \lambda k}{\lambda} \right] \forall \alpha \in [0, \lambda], 0 < \lambda \leq 1 \)

where \( l_k = \beta_2 - \beta_1 \) and \( r_k = \beta_3 - \beta_2 \)

Here we solve the given problem for \( \ddot{k} > 0 \) and \( \ddot{k} < 0 \) respectively

**Case 1: when \( \ddot{k} > 0 \)**

Let \( \ddot{k} = (\beta_1, \beta_2, \beta_3; \lambda) \)

Therefore, \( \dot{k} = [k_1(\alpha), k_2(\alpha)] = \left[ \beta_1 + \frac{\alpha \lambda k}{\lambda}, \beta_3 - \frac{\alpha \lambda k}{\lambda} \right] \forall \alpha \in [0, \lambda], 0 < \lambda \leq 1 \)

where \( l_k = \beta_2 - \beta_1 \) and \( r_k = \beta_3 - \beta_2 \)

The FODE (18) becomes

\[
\frac{dx_1(t, \alpha)}{dt} = (\beta_1 + \frac{\alpha \lambda k}{\lambda}) x_1(t, \alpha)
\]

(19)

and

\[
\frac{dx_2(t, \alpha)}{dt} = (\beta_3 - \frac{\alpha \lambda k}{\lambda}) x_2(t, \alpha)
\]

(20)

Taking Laplace Transform both sides of (19) we get

\[
\mathcal{L} \left\{ \frac{dx_1(t, \alpha)}{dt} \right\} = \mathcal{L} \left\{ (\beta_1 + \frac{\alpha \lambda k}{\lambda}) x_1(t, \alpha) \right\}
\]

Or, \( s \{ x_1(t, \alpha) \} - x_1(0, \alpha) = (\beta_1 + \frac{\alpha \lambda k}{\lambda}) \{ x_1(t, \alpha) \} \)

Or, \( \{ x_1(t, \alpha) \} = \frac{\gamma}{s - (\beta_1 + \frac{\alpha \lambda k}{\lambda})} \)

(21)

Taking inverse Laplace Transform of (21) we get
\[ x_2(t, \alpha) = t^{-1} \left( \frac{\gamma}{s - (\beta_1 + \frac{\alpha t}{\lambda})} \right) = \gamma t^{-1} \left( \frac{1}{s - (\beta_1 + \frac{\alpha t}{\lambda})} \right) \]

Or, \( x_1(t, \alpha) = \gamma e^{(\beta_1 + \frac{\alpha t}{\lambda})t} \) \( \tag{22} \)

Taking Laplace Transform both sides of (20) we get
\[ l \left\{ \frac{dx_2(t, \alpha)}{dt} \right\} = l \{ (\beta_3 - \frac{\alpha r_k}{\lambda})x_2(t, \alpha) \} \]

Or, \( sl\{x_2(t, \alpha)\} - x_2(0, \alpha) = (\beta_3 - \frac{\alpha r_k}{\lambda})l\{x_2(t, \alpha)\} \]

Or, \( l\{x_2(t, \alpha)\} = \frac{\gamma}{s - (\beta_3 - \frac{\alpha r_k}{\lambda})} \) \( \tag{23} \)

Taking inverse Laplace Transform of (23) we get
\[ x_2(t, \alpha) = t^{-1} \left( \frac{\gamma}{s - (\beta_3 - \frac{\alpha r_k}{\lambda})} \right) = \gamma t^{-1} \left( \frac{1}{s - (\beta_3 - \frac{\alpha r_k}{\lambda})} \right) \]

Or, \( x_2(t, \alpha) = \gamma e^{(\beta_3 - \frac{\alpha r_k}{\lambda})t} \) \( \tag{24} \)

Here \( \frac{d}{d\alpha} [x_1(t, \alpha)] = \frac{\gamma t}{\lambda} (t - t_0) e^{(\beta_1 + \frac{\alpha t}{\lambda})(t-t_0)} > 0 \)

and \( \frac{d}{d\alpha} [x_2(t, \alpha)] = -\frac{\gamma t}{\lambda} (t - t_0) e^{(\beta_3 - \frac{\alpha r_k}{\lambda})(t-t_0)} < 0 \)

Hence the \( \alpha \)-cut of the strong solution of FODE (18) is
\[ x(t, \alpha) = \gamma \left[ e^{(\beta_1 + \frac{\alpha t}{\lambda})(t-t_0)}, e^{(\beta_3 - \frac{\alpha r_k}{\lambda})(t-t_0)} \right] \]

**Case 2: When \( \tilde{k} < 0 \)**

Let \( \tilde{k} = -\bar{m} \), where \( \bar{m} = (\beta_1, \beta_2, \beta_3; \lambda) \) is a positive GTFN.

So \( (\bar{m})_\alpha = [m_1(\alpha), m_2(\alpha)] = [\beta_1 + \frac{\alpha l_m}{\lambda}, \beta_3 - \frac{\alpha r_m}{\lambda}] \quad \forall \ \alpha \in [0, \lambda], \quad 0 < \lambda \leq 1 \)

where \( l_m = \beta_2 - \beta_1 \) and \( r_m = \beta_3 - \beta_2 \)

The FODE (1) becomes
\[ \frac{dx(t, \alpha)}{dt} = -(\beta_3 - \frac{\alpha r_m}{\lambda})x_2(t, \alpha) \] \( \tag{25} \)

and
\[ \frac{dx_2(t, \alpha)}{dt} = -(\beta_1 + \frac{\alpha l_m}{\lambda})x_1(t, \alpha) \] \( \tag{26} \)

Taking Laplace Transform both sides of (25) we get
\[ l \left\{ \frac{dx(t, \alpha)}{dt} \right\} = l \{ -(\beta_3 - \frac{\alpha r_m}{\lambda})x_2(t, \alpha) \} \]

Or, \( sl\{x_1(t, \alpha)\} - x_1(0, \alpha) = -(\beta_3 - \frac{\alpha r_m}{\lambda})l\{x_2(t, \alpha)\} \]

Or, \( sl\{x_1(t, \alpha)\} + (\beta_3 - \frac{\alpha r_m}{\lambda})l\{x_2(t, \alpha)\} = \gamma \) \( \tag{27} \)

Taking Laplace Transform both sides of (26) we get
\[
\begin{align*}
\{\frac{dx(t,\alpha)}{dt}\} &= l\{-(\beta_1 + \frac{\alpha l_m}{\lambda})x_1(t, \alpha)\} \\
\text{Or, } s\{x_2(t, \alpha)\} - x_2(0, \alpha) &= -(\beta_1 + \frac{\alpha l_m}{\lambda})l\{x_1(t, \alpha)\} \\
\text{Or, } \left(\beta_1 + \frac{\alpha r_m}{\lambda}\right)l\{x_1(t, \alpha)\} + s\{x_2(t, \alpha)\} &= \gamma
\end{align*}
\]

Solving (27) and (28) we get
\[
\begin{align*}
l\{x_2(t, \alpha)\} &= \frac{\gamma s - (\beta_1 + \frac{\alpha l_m}{\lambda})}{s^2 - \left(\beta_1 + \frac{\alpha l_m}{\lambda}\right)\left(\beta_3 - \frac{\alpha r_m}{\lambda}\right)} \\
\text{and} \\
l\{x_1(t, \alpha)\} &= \frac{\gamma s - (\beta_3 - \frac{\alpha r_m}{\lambda})}{s^2 - \left(\beta_1 + \frac{\alpha l_m}{\lambda}\right)\left(\beta_3 - \frac{\alpha r_m}{\lambda}\right)}
\end{align*}
\]

Taking inverse Laplace Transform of (30) we get
\[
\begin{align*}
x_1(t, \alpha) &= \gamma^{-1}\left\{\frac{s}{s^2 - \left(\beta_1 + \frac{\alpha l_m}{\lambda}\right)\left(\beta_3 - \frac{\alpha r_m}{\lambda}\right)}\right\} - \gamma\left\{\frac{\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}}{\left(\beta_3 - \frac{\alpha r_m}{\lambda}\right)}\frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}}\right\}^{-1} - \gamma\left\{\frac{\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}}{\left(\beta_3 - \frac{\alpha r_m}{\lambda}\right)}\frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}}\right\}^{-1} \\
&= \gamma \cosh\left(\beta_1 + \frac{\alpha l_m}{\lambda}\right)\beta_3 - \frac{\alpha r_m}{\lambda} t - \gamma \frac{\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}}{\sqrt{\beta_3 - \frac{\alpha r_m}{\lambda}}} \sinh\left(\beta_1 + \frac{\alpha l_m}{\lambda}\right)\beta_3 - \frac{\alpha r_m}{\lambda} t \\
&= \frac{\gamma}{2}\left(1 - \frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}}\right)e^{\left(\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}\beta_3 - \frac{\alpha r_m}{\lambda} t\right)} + \left(1 + \frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}}\right)e^{-\left(\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}\beta_3 - \frac{\alpha r_m}{\lambda} t\right)}
\end{align*}
\]

Taking inverse Laplace Transform of (29) we get
\[
\begin{align*}
x_2(t, \alpha) &= \gamma^{-1}\left\{\frac{s}{s^2 - \left(\beta_1 + \frac{\alpha l_m}{\lambda}\right)\left(\beta_3 - \frac{\alpha r_m}{\lambda}\right)}\right\} - \gamma\left\{\frac{\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}}{\left(\beta_3 - \frac{\alpha r_m}{\lambda}\right)}\frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}}\right\}^{-1} - \gamma\left\{\frac{\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}}{\left(\beta_3 - \frac{\alpha r_m}{\lambda}\right)}\frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}}\right\}^{-1} \\
&= \gamma \cosh\left(\beta_1 + \frac{\alpha l_m}{\lambda}\right)\beta_3 - \frac{\alpha r_m}{\lambda} t - \gamma \frac{\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}}{\sqrt{\beta_3 - \frac{\alpha r_m}{\lambda}}} \sinh\left(\beta_1 + \frac{\alpha l_m}{\lambda}\right)\beta_3 - \frac{\alpha r_m}{\lambda} t \\
&= \frac{\gamma}{2}\left(\frac{\beta_1 + \frac{\alpha l_m}{\lambda}}{\beta_3 - \frac{\alpha r_m}{\lambda}}\right) - \frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}}\left(1 - \frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}}\right)e^{\left(\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}\beta_3 - \frac{\alpha r_m}{\lambda} t\right)} + \left(1 + \frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}}\right)e^{-\left(\sqrt{\beta_1 + \frac{\alpha l_m}{\lambda}}\beta_3 - \frac{\alpha r_m}{\lambda} t\right)}
\end{align*}
\]

3.3. Solution Procedure of 1st Order Linear Homogeneous FODE of Type-III

Consider the initial value problem \(\frac{dx}{dt} = \bar{k}x\) (33)

With fuzzy IC \(\bar{x}(0) = \bar{y}_0 = (y_1, y_2, y_3; \omega)\), where \(\bar{k} = (\beta_1, \beta_2, \beta_3; \lambda)\)

Let \(\bar{x}(t)\) be the solution of FODE (33).

Let \(x(t, \alpha) = [x_1(t, \alpha), x_2(t, \alpha)]\) be the \(\alpha\)-cut of the solution.

Here \((\bar{y}_0)_\alpha = [x(0, \alpha), x(0, \alpha)] = [y_1 + \frac{\alpha l_0}{\omega}, y_3 - \frac{\alpha r_0}{\omega}] \quad \forall \alpha \in [0, \omega], \quad 0 < \omega \leq 1\)

where \(l_0 = y_2 - y_1\) and \(r_0 = y_3 - y_2\)

Let \(\eta = \min(\lambda, \omega)\)

Here we solve the given problem for \(\bar{k} > 0\) and \(\bar{k} < 0\) respectively.
Case 1: when $\tilde{k} > 0$

where $\tilde{k} = (\beta_1, \beta_2, \beta_3; \lambda)$

Here $\left(\tilde{k}\right)_\alpha = [k_1(\alpha), k_2(\alpha)] = \left[\beta_1 + \frac{\alpha l_k}{\lambda}, \beta_3 - \frac{\alpha r_k}{\lambda}\right] \quad \forall \alpha \in [0, \lambda]$, \quad 0 < \lambda \leq 1

where $l_k = \beta_2 - \beta_3$ and $r_k = \beta_3 - \beta_2$

The FODE (33) becomes

$$\frac{dx_1(t,\alpha)}{dt} = k_1(\alpha)x_1(t,\alpha) \quad (34)$$

and

$$\frac{dx_2(t,\alpha)}{dt} = k_2(\alpha)x_2(t,\alpha) \quad (35)$$

Taking Laplace Transform both sides of (34) we get

$$l\left(\frac{dx_1(t,\alpha)}{dt}\right) = l\{k_1(\alpha)x_1(t,\alpha)\}$$

Or, $s\{x_1(t,\alpha)\} - x_1(0,\alpha) = k_1(\alpha)l\{x_1(t,\alpha)\}$

Or, $l\{x_1(t,\alpha)\} = \frac{(y_1 + \frac{\alpha l_y}{\eta})}{s - k_1(\alpha)} \quad (36)$

Taking inverse Laplace Transform of (36) we get

$$x_1(t, \alpha) = l^{-1}\left(\frac{y_1 + \frac{\alpha l_y}{\eta}}{s - k_1(\alpha)}\right) = \left(y_1 + \frac{\alpha l_y}{\eta}\right)l^{-1}\left(\frac{1}{s - k_1(\alpha)}\right)$$

Or, $x_1(t, \alpha) = \left(y_1 + \frac{\alpha l_y}{\eta}\right)e^{k_1(\alpha)t} = \left(y_1 + \frac{\alpha l_y}{\eta}\right)e^{(\beta_1 + \frac{\alpha l_k}{\eta})t} \quad (37)$

Taking Laplace Transform both sides of (35) we get

$$l\left(\frac{dx_2(t,\alpha)}{dt}\right) = l\{k_2(\alpha)x_2(t,\alpha)\}$$

Or, $s\{x_2(t,\alpha)\} - x_2(0,\alpha) = k_2(\alpha)l\{x_2(t,\alpha)\}$

Or, $l\{x_2(t,\alpha)\} = \frac{(y_3 - \frac{\alpha r_y}{\eta})}{s - k_2(\alpha)} \quad (38)$

Taking inverse Laplace Transform of (38) we get

$$x_2(t, \alpha) = l^{-1}\left(\frac{y_3 - \frac{\alpha r_y}{\eta}}{s - k_2(\alpha)}\right) = \left(y_3 - \frac{\alpha r_y}{\eta}\right)l^{-1}\left(\frac{1}{s - k_2(\alpha)}\right)$$

Or, $x_2(t, \alpha) = \left(y_3 - \frac{\alpha r_y}{\eta}\right)e^{k_2(\alpha)t} = \left(y_3 - \frac{\alpha r_y}{\eta}\right)e^{(\beta_3 - \frac{\alpha r_k}{\eta})t} \quad (39)$

Case 2: when $\tilde{k} < 0$

Let $\tilde{k} = -\tilde{m}$ where $\tilde{m} = (\beta_1, \beta_2, \beta_3; \lambda)$ is a positive GTFN.

Then $\left(\tilde{m}\right)_\alpha = \left[\beta_1 + \frac{\alpha l_m}{\lambda}, \beta_3 - \frac{\alpha r_m}{\lambda}\right] \forall \alpha \in [0, \lambda], \quad 0 < \lambda \leq 1$

where $l_m = \beta_2 - \beta_1$ and $r_m = \beta_3 - \beta_2$

Let $\eta = \min(\lambda, \omega)$

The FODE (33) becomes

$$\frac{dx_1(t,\alpha)}{dt} = -\left(\beta_3 - \frac{\alpha r_m}{\eta}\right)x_2(t,\alpha) \quad (40)$$
and 
\[ \frac{dx_2(t, \alpha)}{dt} = -\left( \beta_1 + \frac{\alpha t_m}{\eta} \right) x_1(t, \alpha) \]  
(41)

Taking Laplace Transform both sides of (40) we get 
\[ l\left\{ \frac{dx_1(t, \alpha)}{dt} \right\} = l\left\{ -\left( \beta_3 - \frac{\alpha r_m}{\eta} \right) x_2(t, \alpha) \right\} \]
Or, 
\[ sl\{x_1(t, \alpha)\} - y(0, \alpha) = -\left( \beta_3 - \frac{\alpha r_m}{\eta} \right) l\{x_2(t, \alpha)\} \]
Or, 
\[ sl\{x_1(t, \alpha)\} + \left( \beta_3 - \frac{\alpha r_m}{\eta} \right) l\{x_2(t, \alpha)\} = \left( \gamma_1 + \frac{\alpha \gamma_0}{\eta} \right) \]
(42)

Taking Laplace Transform both sides of (41) we get 
\[ l\left\{ \frac{dx_2(t, \alpha)}{dt} \right\} = l\left\{ -\left( \beta_1 + \frac{\alpha t_m}{\eta} \right) x_1(t, \alpha) \right\} \]
Or, 
\[ sl\{x_2(t, \alpha)\} - \gamma(0, \alpha) = -\left( \beta_1 + \frac{\alpha t_m}{\eta} \right) l\{x_1(t, \alpha)\} \]
Or, 
\[ \left( \beta_1 + \frac{\alpha t_m}{\eta} \right) l\{x_1(t, \alpha)\} + sl\{x_2(t, \alpha)\} = \left( \gamma_3 - \frac{\alpha \gamma_0}{\eta} \right) \]
(43)

Solving (42) and (43) we get 
\[ l\{x_2(t, \alpha)\} = \frac{s\left( \gamma_3 - \frac{\alpha \gamma_0}{\eta} \right) - \left( \beta_1 + \frac{\alpha t_m}{\eta} \right) \left( \gamma_1 + \frac{\alpha \gamma_0}{\eta} \right)}{s^2 - \left( \beta_1 + \frac{\alpha t_m}{\eta} \right) \left( \beta_3 - \frac{\alpha r_m}{\eta} \right)} \]  
(44)

and 
\[ l\{x_1(t, \alpha)\} = \frac{s\left( \gamma_1 + \frac{\alpha \gamma_0}{\eta} \right) - \left( \beta_3 - \frac{\alpha r_m}{\eta} \right) \left( \gamma_3 - \frac{\alpha \gamma_0}{\eta} \right)}{s^2 - \left( \beta_1 + \frac{\alpha t_m}{\eta} \right) \left( \beta_3 - \frac{\alpha r_m}{\eta} \right)} \]  
(45)

Taking inverse Laplace transform of (44) we get 
\[ x_1(t, \alpha) = \left( \gamma_1 + \frac{\alpha \gamma_0}{\eta} \right) \left( \frac{s}{s^2 - \left( \beta_1 + \frac{\alpha t_m}{\eta} \right) \left( \beta_3 - \frac{\alpha r_m}{\eta} \right)} \right)^{-1} \]
\[ \left( \beta_3 - \frac{\alpha r_m}{\eta} \right) \left( \beta_1 + \frac{\alpha t_m}{\eta} \right)^{-1} \left( \frac{s}{s^2 - \left( \beta_1 + \frac{\alpha t_m}{\eta} \right) \left( \beta_3 - \frac{\alpha r_m}{\eta} \right)} \right)^{-1} \]
\[ \left( \gamma_3 - \frac{\alpha \gamma_0}{\eta} \right) \left( \beta_1 + \frac{\alpha t_m}{\eta} \right)^{-1} \left( \frac{s}{s^2 - \left( \beta_1 + \frac{\alpha t_m}{\eta} \right) \left( \beta_3 - \frac{\alpha r_m}{\eta} \right)} \right)^{-1} \]
\[ = \left( \gamma_1 + \frac{\alpha \gamma_0}{\eta} \right) \cosh \left( \beta_3 - \frac{\alpha r_m}{\eta} \right) \left( \beta_1 + \frac{\alpha t_m}{\eta} \right) \right) t - \left( \gamma_3 - \frac{\alpha \gamma_0}{\eta} \right) \left( \beta_1 + \frac{\alpha t_m}{\eta} \right) \left( \beta_3 - \frac{\alpha r_m}{\eta} \right) \right) t \]
\[ = \left[ \gamma_1 + \frac{\alpha \gamma_0}{\eta} - \left( \frac{\beta_3 - \frac{\alpha r_m}{\eta}}{\left( \beta_1 + \frac{\alpha t_m}{\eta} \right)} \left( \gamma_3 - \frac{\alpha \gamma_0}{\eta} \right) \right) \right] \left( \beta_1 + \frac{\alpha t_m}{\eta} \right) \left( \beta_3 - \frac{\alpha r_m}{\eta} \right) \right) t \]
\[ = E \left[ \gamma_1 + \frac{\alpha \gamma_0}{\eta} + \frac{\beta_3 - \frac{\alpha r_m}{\eta}}{\left( \beta_1 + \frac{\alpha t_m}{\eta} \right)} \left( \gamma_3 - \frac{\alpha \gamma_0}{\eta} \right) \right] \left( \beta_1 + \frac{\alpha t_m}{\eta} \right) \left( \beta_3 - \frac{\alpha r_m}{\eta} \right) \right) t \]
\[ = E \left( \gamma_1 + \frac{\alpha \gamma_0}{\eta} \right) \left( \beta_1 + \frac{\alpha t_m}{\eta} \right) \left( \beta_3 - \frac{\alpha r_m}{\eta} \right) \right) t \]  
(46)

Similarly from (43) we get,
Application

4.1 Bank Account Problem:

The balance $B(t)$ of a bank account grows under continuous process given by $\frac{dB}{dt} = rB$, where $r$ the constant of proportionality is the annual interest rate. If there are initially $B_0$ balance, solve the above problem in fuzzy environment when

(i) $\tilde{B}_0 = (950, 1000, 1100; 0.8)$ and $r = 4\%$
(ii) $B_0 = 1000$ and $\tilde{r} = (3.5, 4, 4.7; 0.7)%$
(iii) $\tilde{B}_0 = (850, 1000, 1100; 0.8)$ and $\tilde{r} = (3.7, 4, 4.5; 0.7)%$

Solution:

i. Here $\tilde{B}_0 = [950 + 62.5\alpha, 1100 - 125\alpha]$

Therefore the solution of the problem is $B_1(t, \alpha) = (950 + 62.5\alpha)e^{0.04t}$ and $B_2(t, \alpha) = (1100 - 125\alpha)e^{-0.04t}$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$B_1(t, \alpha)$</th>
<th>$B_2(t, \alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1071.1220</td>
<td>1240.2465</td>
</tr>
<tr>
<td>0.1</td>
<td>1078.1689</td>
<td>1226.1528</td>
</tr>
<tr>
<td>0.2</td>
<td>1085.2157</td>
<td>1212.0591</td>
</tr>
<tr>
<td>0.3</td>
<td>1092.2626</td>
<td>1197.9654</td>
</tr>
<tr>
<td>0.4</td>
<td>1099.3094</td>
<td>1183.8717</td>
</tr>
<tr>
<td>0.5</td>
<td>1106.3563</td>
<td>1169.7780</td>
</tr>
<tr>
<td>0.6</td>
<td>1113.4031</td>
<td>1155.6843</td>
</tr>
<tr>
<td>0.7</td>
<td>1120.4500</td>
<td>1141.5906</td>
</tr>
<tr>
<td>0.8</td>
<td>1127.4969</td>
<td>1127.4969</td>
</tr>
</tbody>
</table>

From the above table we see that for this particular value of $t$, $B_1(t, \alpha)$ is an increasing function, $B_2(t, \alpha)$ is a decreasing function and $B_1(t, 0.8) = B_2(t, 0.8)$. Hence we get that this is a strong solution.
ii. Here \((\hat{r})_\alpha = [3.5 + 0.71\alpha, 4.7 - \alpha]\)

Therefore the solution of the problem is
\[
B_1(t, \alpha) = 1000e^{(3.5+0.71\alpha)t} \quad \text{and} \quad B_2(t, \alpha) = 1000e^{(4.7-0.6\alpha)t}
\]

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(B_1(t, \alpha))</th>
<th>(B_2(t, \alpha))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1150.2738</td>
<td>1206.8335</td>
</tr>
<tr>
<td>0.1</td>
<td>1153.5452</td>
<td>1202.0158</td>
</tr>
<tr>
<td>0.2</td>
<td>1156.8259</td>
<td>1197.2174</td>
</tr>
<tr>
<td>0.3</td>
<td>1160.1160</td>
<td>1192.4381</td>
</tr>
<tr>
<td>0.4</td>
<td>1163.4154</td>
<td>1187.6778</td>
</tr>
<tr>
<td>0.5</td>
<td>1166.7242</td>
<td>1182.9366</td>
</tr>
<tr>
<td>0.6</td>
<td>1170.0424</td>
<td>1178.2143</td>
</tr>
<tr>
<td>0.7</td>
<td>1173.3701</td>
<td>1173.5109</td>
</tr>
</tbody>
</table>

Table 2: Value of \(x_1(t, \alpha)\) and \(x_2(t, \alpha)\) for different \(\alpha\) and \(t = 4\)

From the above table we see that for this particular value of \(t\), \(B_1(t, \alpha)\) is an increasing function, \(B_2(t, \alpha)\) is a decreasing function and \(B_1(t, 0.7) < B_2(t, 0.7)\). Hence we get that this is a strong solution.

iii. Here \((\overline{B}_0)_\alpha = [850 + 214.28\alpha, 1100 - 142.85\alpha]\) and \((\overline{r})_\alpha = [3.7 + 0.43\alpha, 4.5 - 0.71\alpha]\)

Therefore the solution of the problem is
\[
B_1(t, \alpha) = (850 + 187.5\alpha)e^{(3.7+0.43\alpha)t} \quad \text{and} \quad B_2(t, \alpha) = (1100 - 125\alpha)e^{(4.5-0.71\alpha)t}
\]

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(B_1(t, \alpha))</th>
<th>(B_2(t, \alpha))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1022.7357</td>
<td>1377.5550</td>
</tr>
<tr>
<td>0.1</td>
<td>1047.5458</td>
<td>1357.0748</td>
</tr>
<tr>
<td>0.2</td>
<td>1072.4580</td>
<td>1336.7224</td>
</tr>
<tr>
<td>0.3</td>
<td>1097.4726</td>
<td>1316.4973</td>
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<td>1122.5899</td>
<td>1296.3987</td>
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<tr>
<td>0.5</td>
<td>1147.8103</td>
<td>1276.4260</td>
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<tr>
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<td>1173.1340</td>
<td>1256.5786</td>
</tr>
<tr>
<td>0.7</td>
<td>1198.5614</td>
<td>1236.8558</td>
</tr>
</tbody>
</table>

Table 3: Value of \(B_1(t, \alpha)\) and \(B_2(t, \alpha)\) for different \(\alpha\) and \(t = 5\)

From the above table we see that for this particular value of \(t\), \(B_1(t, \alpha)\) is an increasing function, \(B_2(t, \alpha)\) is a decreasing function and \(B_1(t, 0.7) < B_2(t, 0.7)\). Hence we get that this is a strong solution.

4.2. Drug concentration in blood: The drug Theophylline is administered for asthma the concentration satisfied the differential equation \(\frac{dC}{dt} = -kC\), where \(k\) is the constant of proportionality and \(t\) is measured in hours. If the concentration initially \(C_0\) mg/liter. Determine the concentration after \(t\) hours in fuzzy environment when

(i) \(\overline{C}_0 = (12, 14, 17; 0.8)\)mg/liter and \(k = 0.17\).

(ii) \(C_0 = 14\) mg/liter and \(\overline{k} = (0.14, 0.17, 0.19; 0.7)\).

(iii) \(\overline{C}_0 = (12, 14, 15; 0.8)\)mg/liter and \(\overline{k} = (0.15, 0.17, 0.18; 0.7)\).
Solution:

i. Here \( (\tilde{C}_0)_\alpha = [12 + 2.5\alpha, 17 - 3.75\alpha] \)

Therefore the solution of the problem is

\[
C_1(t, \alpha) = (14.5 - 0.625\alpha)e^{-0.17t} + (3.12 - 2.5\alpha)e^{0.17t}\]

and

\[
C_2(t, \alpha) = (14.5 - 0.625\alpha)e^{-0.17t} - (3.12 - 2.5\alpha)e^{0.17t}\]

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( C_1(t, \alpha) )</th>
<th>( C_2(t, \alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>12.2806</td>
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<td>0.1</td>
<td>2.9954</td>
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<td>4.1638</td>
<td>10.3381</td>
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<td>4.7480</td>
<td>9.6906</td>
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<tr>
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<td>5.3322</td>
<td>9.0431</td>
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<tr>
<td>0.6</td>
<td>5.9164</td>
<td>8.3956</td>
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<td>6.5006</td>
<td>7.7480</td>
</tr>
<tr>
<td>0.8</td>
<td>7.0847</td>
<td>7.1005</td>
</tr>
</tbody>
</table>

Table 4: Value of \( C_1(t, \alpha) \) and \( C_2(t, \alpha) \) for different \( \alpha \) and \( t = 4 \)

From the above graph we see that for this particular value of \( t \), \( C_1(t, \alpha) \) is an increasing function, \( C_2(t, \alpha) \) is a decreasing function and \( C_1(t, 0.8) < C_2(t, 0.8) \) Hence this is strong solution.

ii. Here \( (\tilde{k})_\alpha = [0.14 + 0.42\alpha, 0.19 - 0.029\alpha] \)

Therefore the solution of the problem is

\[
C_1(t, \alpha) = \frac{14}{2} \left\{ \left( 1 - \sqrt{0.19 - 0.029\alpha} \right) e^{\sqrt{(0.14+0.042\alpha)(0.19-0.029\alpha)} t} \right\} + \\
\left\{ 1 + \sqrt{0.19 - 0.029\alpha} \right\} e^{-\sqrt{(0.14+0.042\alpha)(0.19-0.029\alpha)} t} \right\} \]

and

\[
C_2(t, \alpha) = \frac{14}{2} \sqrt{0.19 - 0.029\alpha} \left\{ \left( 1 - \sqrt{0.19 - 0.029\alpha} \right) e^{\sqrt{(0.14+0.042\alpha)(0.19-0.029\alpha)} t} \right\} + \\
\left\{ \frac{0.19 - 0.029\alpha}{0.14+0.042\alpha} \right\} e^{-\sqrt{(0.14+0.042\alpha)(0.19-0.029\alpha)} t} \right\} \]

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( C_1(t, \alpha) )</th>
<th>( C_2(t, \alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>11.7777</td>
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<td>5.8842</td>
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<td>6.2960</td>
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<td>7.1271</td>
</tr>
</tbody>
</table>

Table 5: Value of \( C_1(t, \alpha) \) and \( C_2(t, \alpha) \) for different \( \alpha \) and \( t = 4 \)

From the above graph we see that for this particular value of \( t \), \( C_1(t, \alpha) \) is an increasing function, \( C_2(t, \alpha) \) is a decreasing function and \( C_1(t, 0.7) < C_2(t, 0.7) \) Hence this is strong solution.
iii. Here \( (\bar{C}_1)_{\alpha} = [12 + 2.86\alpha, 15 - 1.43\alpha] \) and \( (\bar{k})_{\alpha} = [0.15 + 0.029\alpha, 0.18 - 0.014\alpha] \)

Therefore the solution of the problem is

\[
C_1(t, \alpha) = \frac{1}{2}[(12 + 2.86\alpha) - \sqrt{\frac{0.18 - 0.014\alpha}{0.15 + 0.029\alpha}} (15 - 1.43\alpha)]e^{\sqrt{\frac{0.18 - 0.014\alpha}{0.15 + 0.029\alpha}}t} + \frac{1}{2}[(12 + 2.86\alpha) + \sqrt{\frac{0.18 - 0.014\alpha}{0.15 + 0.029\alpha}} (16 - 3.75\alpha)]e^{-\sqrt{\frac{0.18 - 0.014\alpha}{0.15 + 0.029\alpha}}t}
\]

and

\[
C_2(t, \alpha) = \frac{1}{2}\sqrt{\frac{0.15 + 0.029\alpha}{0.18 - 0.014\alpha}}(12 + 2.86\alpha - \sqrt{\frac{0.18 - 0.014\alpha}{0.15 + 0.029\alpha}} (15 - 1.43\alpha)]e^{-\sqrt{\frac{0.18 - 0.014\alpha}{0.15 + 0.029\alpha}}t} - \frac{1}{2}\sqrt{\frac{0.15 + 0.029\alpha}{0.18 - 0.014\alpha}}(12 + 2.86\alpha + \sqrt{\frac{0.18 - 0.014\alpha}{0.15 + 0.029\alpha}} (16 - 3.75\alpha)]e^{\sqrt{\frac{0.18 - 0.014\alpha}{0.15 + 0.029\alpha}}t}
\]

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( C_1(t, \alpha) )</th>
<th>( C_2(t, \alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<tr>
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<tr>
<td>0.7</td>
<td>7.0918</td>
<td>7.0798</td>
</tr>
</tbody>
</table>

From the above graph we see that for this particular value of \( t \), \( C_1(t, \alpha) \) is an increasing function, \( C_2(t, \alpha) \) is a decreasing function and \( C_1(t, 0.7) > C_2(t, 0.7) \). Hence this is a weak solution.

8 Conclusion

In this paper, we have used Laplace transform to obtain the solution of first order linear homogeneous ordinary differential equation in fuzzy environment. Here all fuzzy numbers are taken as GTFNs. The method was discussed by different examples. Further research is in progress to apply and extend the Laplace transform to solve n-th order FDEs as well as a system of FDEs.

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