Fuzzy integration using homotopy perturbation method

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Abstract
Complicated fuzzy integrals are difficult to solve, and cannot be expressed in terms of elementary functions or analytical formulae. In this paper, we calculate the fuzzy integrals by using homotopy perturbation method. Some examples are given, revealing its effectiveness and convenience.

Keywords: Fuzzy integral; homotopy perturbation method.

1 Introduction

Considering the approximation theory, the integration problem plays a major role in various areas such as mathematics, physics, statistics, engineering and social sciences. Since in many applications at least some of the system’s parameters and measurements are represented by fuzzy rather than crisp numbers, it is important to develop the integration of fuzzy function or fuzzy data and to solve them. The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Zadeh \cite{1} and others. The topic of fuzzy integration is discussed by Zimmerman \cite{2} and Allahviranloo\cite{3, 4}. In recent years, the homotopy perturbation method (HPM), first proposed by he \cite{5}, has successfully been applied to solve many types of linear and non-linear functional equations. This method which is a combination of homotopy in topology, and classic perturbation techniques, provides a convenient way to obtain analytic or approximate. Solutions to a wide variety of problems arising in different fields, see \cite{6, 10} and the references there in. In this work, we intend to use HPM for computing the first-order differential equations
\[
\frac{du}{dx} + p(x)u = Q(x); \quad u(0) = 0.
\]
The solution of equation can be expressed in the following integration \cite{9}:
\[
u(x)f(x) = \int Q(x)f(x)dx,
\]
where
\[
f(x) = e^{\int p(x)dx},
\]
this paper shows that some difficult fuzzy integrals can be easily calculated by the homotopy perturbation method.

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2 Preliminaries

We represent an arbitrary fuzzy number by an ordered pair functions \((u(r), \pi(r))\), which satisfy the following requirements:

a: \(u(r)\) is abounded monotonic increasing left continuous function,

b: \(\pi(r)\) is abounded monotonic decreasing left continuous function,

c: \(u(r) \leq \pi(r), 0 \leq r \leq 1\).

A crisp number \(\alpha\) is simply represented by \(u(r) = \alpha, 0 \leq r \leq 1\), we recall that for \(a < b < c\) with \(a, b, c \in R\), the triangular fuzzy number \(u = (a, b, c)\) determined by \(a, b, c\) is given such that \(u(r) = a + (b - c)r\) and \(\pi(r) = c - (c - b)r\) are the end points of the \(r\)-level sets. For all \(r \in [0, 1]\) for arbitrary \(u = (u(r), \pi(r))\), \(v = (v(r), \pi(r))\) and \(k > 0\) we define addition \(u \oplus v\), subtraction \(u \ominus v\) and scalar multiplication by kaleva [11].

a) Addition:
\[ u \oplus v = (u(r) + v(r), \pi(r) + \pi(r)), \]

b) subtraction:
\[ u \ominus v = (u(r) - \pi(r), \pi(r) - v(r)), \]

c) scalar multiplication:
\[ k \odot u = \begin{cases} (ku, k\pi) & k \geq 0 \\ (k\pi, ku) & k < 0 \end{cases}, \]

if \(k = -1\) then \(k \odot u = -u\).

Theorem 2.1. [12] Let \(f(x)\) be a fuzzy value function on \([a, b]\) and it is represented by \((\mathcal{f}(x, r), \mathcal{r}(x, r))\) for \(r \in [0, 1]\), assume \(\mathcal{r}(x, r)\) are Riemann-integrable on \([a, b]\) for every \(b \geq a\) and assume. There are two positive \(M(r)\) and \(\overline{M}(r)\) such that
\[ \int_a^b |\mathcal{f}(x, r)| dx \leq M(r), \]

and
\[ \int_a^b |\mathcal{r}(x, r)| dx \leq \overline{M}(r), \]

for every \(b \geq a\).

Then \(f(x)\) is improper fuzzy Riemann-integrable on \([a, \infty)\) and the Improper fuzzy Riemann-integrable is a fuzzy number: further more, we have:
\[ \int_a^\infty f(x) dx = \left( \int_a^\infty \mathcal{f}(x) dx, \int_a^\infty \mathcal{r}(x) dx \right). \]

Proposition 2.1. [12] If \(f(x)\) and \(g(x)\) are fuzzy value function and fuzzy Riemann-integrable on \([a, \infty)\) then \(f(x) + g(x)\) is fuzzy Riemann-integrable on \([a, \infty)\).

Moreover, we have:
\[ \int_a^x f(x) + g(x) dx = \int_a^x f(x) dx + \int_a^x g(x) dx. \]

Definition 2.1. Consider the fuzzy first-order differential equation \(\frac{du}{dx} + p(x)u = Q(x)\), \(u(0) = 0\). The solution of equation can be expressed in the following integration:
\[ u(x) f(x) = \int Q(x) f(x) dx, \quad (2.1) \]

where
\[ f(x) = e^{\int p(x) dx}, \]

If the crisp function \(f(x)\) is continuous in the metric \(D\), its definite integral exists. Furthermore
\[ (u(x, r) f(x)) = \left( \int Q(x, r) f(x) dx \right), \quad (2.2) \]
and
\[(\pi(x,r)f(x)) = (\int \overline{Q}(x,r)f(x)dx).\] (2.3)

It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach. More details about properties of the fuzzy integral are given in [2].

3 Homotopy perturbation method

To illustrate the homotopy perturbation method (HPM) for solving non-linear differential equations, He [5] considered the following non-linear differential equation:
\[A(u) = F(r) \quad r \in \Omega,\] (3.4)
subject to the boundary condition
\[B(u, \frac{\partial u}{\partial n}) = 0 \quad r \in \Gamma,\] (3.5)

Where A is a general differential operator, B is a boundary operator, f(r) is a known analytic function, \(\Gamma\) is the boundary of the domain \(\Omega\) and \(\frac{\partial}{\partial n}\) denotes differentiation along the normal vector drawn outwards from \(\Omega\). The operator A can generally be divided into two parts M and N therefore, (3.4) can be rewritten as follows:
\[M(u) + N(u) = F(r) \quad r \in \Omega.\] (3.6)

He [5, 8] constructed a homotopy \(v(r, p) : \Omega \times [0, 1] \rightarrow R\) which satisfies
\[H(v, p) = (1 - p)[M(v) - M(u_0)] + p[A(v) - f(r)] = 0 .\] (3.7)

Which is equivalent to
\[H(v, p) = M(v) - M(u_0) + pM(u_0) + p[A(v) - f(r)] = 0,\] (3.8)

where \(p \in [0, 1]\) is an embedding parameter, and \(u_0\) is an initial approximation of (3.4). Obviously, we have
\[H(v, 0) = M(v) - M(u_0) = 0 ; \quad H(v, 1) = A(v) - F(r) = 0 .\] (3.9)

The changing process of \(p\) from zero to unity is just that of \(H(v, p)\) from \(M(v) - M(u_0)\) to \(A(v) - F(r)\). In topology, this is called deformation and \(M(v) - M(u_0)\) and \(A(v) - F(r)\) are called homotopic. According to the homotopy Perturbation method, the parameter \(p\) is used as a small parameter, and the solution of Eq. (3.7) or Eq. (3.8) can be expressed as a series in \(p\) in the form
\[V = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots .\] (3.10)

When \(p \rightarrow 1\), Eq. (3.7) or Eq. (3.8) correspond to the original one. Eq. (3.6) and Eq. (3.10) becomes the approximate solution of Eq. (3.6), ie.,
\[U = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + v_3 + \cdots ,\] (3.11)

the convergence of the series in Eq. (3.11) is discussed by He in [5].

Analysis: Generally the integral of Eq. (2.2) and Eq. (2.3) are complicated and can not be expressed in term of elementary functions nor conveniently tabulated in open literature. However, this method is a powerful tool to calculate such difficult fuzzy integral. We construct the following homotopy with
\[M(u, r) = p(x)u(x, r) ,\]
\[N(u, r) = p(x)u(x, r) ,\]
and
\[M(u, r) = u(x, r) ,\]

\[N(u, r) = u(x, r) ,\]

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\( \mathcal{N}(u, r) = u^r(x, r), \)

\[
\begin{align*}
\begin{cases}
M(u, r) - M(u_0, r) + pM(u_0, r) + p[N(v_0, r) - \mathcal{Q}(x, r)] = 0, \\
\overline{M}(u, r) - \overline{M}(u_0, r) + p\overline{M}(u_0, r) + p[\mathcal{N}(v_0, r) - \mathcal{Q}(x, r)] = 0.
\end{cases}
\end{align*}
\]  

(3.12)

Substituting (3.10) into (3.12). And equating coefficients of like power of \( p \), we obtain:

\[
\begin{align*}
p^0 : & \quad \begin{cases}
M(v_0, r) - M(u_0, r) = 0, \\
\overline{M}(v_0, r) - \overline{M}(u_0, r) = 0,
\end{cases} \\
p^1 : & \quad \begin{cases}
M(v_1, r) + M(u_0, r) + N(v_0, r) - f(x, r) = 0, \\
\overline{M}(v_1, r) + \overline{M}(u_0, r) + \overline{N}(v_0, r) - \overline{f}(x, r) = 0,
\end{cases} \\
p^2 : & \quad \begin{cases}
M(v_2, r) + N(v_1, r) = 0, \\
\overline{M}(v_2, r) + \overline{N}(v_1, r) = 0,
\end{cases} \\
& \vdots \\
p^{n+1} : & \quad \begin{cases}
M(v_{n+1}, r) + N(v_n, r) = 0, \\
\overline{M}(v_{n+1}, r) + \overline{N}(v_n, r) = 0.
\end{cases}
\end{align*}
\]

Therefore, we obtain:

\[
\int \mathcal{Q}(x, r)f(x)dx = \sum_{i=0}^{\infty} V_i(x, r)f(x),
\]

\[
\int \overline{\mathcal{Q}}(x, r)f(x)dx = \sum_{i=0}^{\infty} \overline{V}_i(x, r)f(x).
\]

4 Numerical examples

**Example 4.1.** Consider the following fuzzy integral:

\[
\int_0^\infty (r, 2 - r)e^{-rx}dx,
\]

the exact solution is \( (\frac{x}{2}, \frac{2 - x}{s}) \).

Suppose \( p(x) = -s \) and \( \mathcal{Q}(x) = (r, 2 - r) \) then \( f(x) = e^{-sx} \) by choosing \( u_0(x) = 0 \), we have:

\[
v_0(x) = 0,
\]

\[
v_1(x) = -\frac{\mathcal{Q}(x)}{s} = \frac{(r2 - r)}{s} = (\frac{x}{2}, \frac{2 - x}{s}),
\]

\[
v_2(x) = v_3(x) = v_4(x) = \cdots = 0,
\]

\[
\int_0^\infty (r, 2 - r)e^{-sx}dx = \sum_{i=0}^{\infty} V_i(x)e^{-sx}|_0^{\infty} = e^{-sx}(0 + \frac{x}{2} + 0 + \frac{2 - x}{s}|_0^{\infty}) = (\frac{x}{2}, \frac{2 - x}{s}).
\]

**Example 4.2.** Consider the following fuzzy integral:

\[
\int_0^1 (r^2 + r, 4 - r - r^3)x^2dx,
\]

the exact solution is \( \frac{1}{4}(r^2 + r, 4 - r - r^3) \).

Suppose \( p(x) = \frac{x}{2} \) and \( \mathcal{Q}(x) = (r^2 + r, 4 - r - r^3) \) then \( f(x) = x^2 \) by choosing \( u_0(x) = 0 \), we have:

\[
v_0(x) = 0,
\]
Example 4.4. Consider the following fuzzy integral:
\[
\int_0^1 (r, 2 - r)x^5e^xdx,
\]
the exact solution is \((3r - 1, 3 - r)(5e^4 - 1)\).
Suppose \(p(x) = 2x\) and \(Q(x) = (3r - 1, 3 - r)x^5\) then \(f(x) = e^x\) by choosing \(u_0(x) = 0\), we have:
\[
v_0(x) = 0,
\]
\[
v_1(x) = \frac{(3r - 1, 3 - r)x^5}{5} = \frac{1}{6}(3r - 1, 3 - r)x^4,
\]
\[
v_2(x) = -\frac{2(3r - 1, 3 - r)x^3}{5} = -(3r - 1, 3 - r)x^2,
\]
\[
v_3(x) = \frac{2(3r - 1, 3 - r)x}{5} = (3r - 1, 3 - r),
\]
\[
v_4(x) = v_5(x) = v_6(x) = \cdots = 0.
\]
\[
\int_0^1 (3r - 1, 3 - r)x^5e^xdx = \sum_{i=0}^3 V_i(x)e^x|_0^1 + \sum_{i=0}^2 V_i(x)e^x|_0^1
\]
\[
= [(3r - 1, 3 - r)x - (3r - 1, 3 - r)]e^x|_0^1 + [(3r - 1, 3 - r)x - (3r - 1, 3 - r)]e^x|_0^1 = (3r - 1, 3 - r)e + (2(3r - 1, 3 - r)e - 1 + (3r - 1, 3 - r)e - 1).
\]

5 Conclusion
In this paper, we apply homotopy perturbation method to calculate certain fuzzy integrals. It is a simple and very effective tool for calculating certain difficult fuzzy integrals or in deriving new fuzzy integration formulate.
contrast to the usual methods which need numerical fuzzy integration, homotopy perturbation method requires only simple differentiation to deduce the fuzzy integration formulae.

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References


