Kernel iterative method for solving two-dimensional fuzzy Fredholm integral equations of the second kind

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Abstract
Using parametric form of fuzzy functions, we convert a linear two-dimensional fuzzy Fredholm integral equation of the second kind (2D-FFIE-2) to a linear system of Fredholm integral equations of the second kind with three variables in crisp case. In this paper an iterative method is presented to find the approximate solution of 2D-FFIE-2. Also a proof of convergence of this method is discussed in detail. Finally, the proposed method is illustrated by two numerical example.

Keywords: Two-dimensional fuzzy Fredholm integral equation; Parametric form of fuzzy integral equation; Kernel iterative method.

1 Introduction
The concept of integration of fuzzy functions was first introduced by Dubois and Prade. Alternative approaches were later suggested by Goetschel and Voxman [4], Kaleva [6], Matloka [7], Nanda [8] and others. One of the first applications of fuzzy integration was given by Wu and Ma (Wu et al. 1990), who investigated the fuzzy Fredholm integral equation of the second kind (FFIE-2). In recent years, some numerical methods have been introduced to solve (FFIE-2). These methods can be found in [5, 9, 13]. Also, some numerical methods are presented to solve two-dimensional fuzzy Fredholm integral equations of the second kind; for example, see in [10, 11]. In this work, we apply kernel iterative method for solving 2D-FFIE-2.

This paper is organized as follows: in section 2, we briefly mention the basic notations of fuzzy number, fuzzy function and fuzzy integral. 2D-FFIE-2 and it’s parametric form are discussed in section 3. In section 4, the proposed algorithm is illustrated, also a proof of convergence of the method is discuss in this section. We apply the method for two examples in section 5.

2 preliminaries

Definition 2.1. [4] The parametric form of a fuzzy number u is a pair of functions \((\mu(r), \pi(r))\), \(0 \leq r \leq 1\), which satisfies in the following requirements:

1. \(\mu(r)\) is a bounded, continuous, monotonic increasing function over \([0, 1]\).
2. \(\pi(r)\) is a bounded, continuous, monotonic decreasing function over \([0, 1]\).
3. \(\mu(r) \leq \pi(r), 0 \leq r \leq 1\).

\((\mu(r), \pi(r)), 0 \leq r \leq 1\), are called the r-cut sets of u.

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The set of all fuzzy numbers is denoted by $E^1$.

**Definition 2.2.** [12] For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and real number $k$, we have

\[
(u + v)(r) = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)),
\]

\[
(u - v)(r) = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)),
\]

\[
(ku)(r) = \begin{cases} (ku\underline{u}(r), ku\overline{u}(r)) & k \geq 0, \\ (ku\overline{u}(r), ku\underline{u}(r)) & k < 0. \end{cases}
\]

**Lemma 2.1.** [1] Let $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, be a given family of non-empty intervals. If

1) $(\underline{u}(r_1), \overline{u}(r_1)) \supseteq (\underline{u}(r_2), \overline{u}(r_2))$ for $0 \leq r_1 \leq r_2 \leq 1$,

2) $(\lim_{k \to +\infty} \underline{u}(r_k), \lim_{k \to +\infty} \overline{u}(r_k)) = (\underline{u}(r), \overline{u}(r))$, whenever $(r_k)$ is a non-decreasing converging sequence converges to $r \in [0, 1]$.

then the family $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, represent the $r$-cut sets of a fuzzy number $u \in E^1$.

Conversely, if $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, are the $r$-cut sets of a fuzzy number $u \in E^1$, then the conditions (1) and (2) hold.

**Definition 2.3.** [4] For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ the quantity

\[
D(u, v) = \max \left\{ \sup_{0 \leq r \leq 1} |\underline{u}(r) - \underline{v}(r)|, \sup_{0 \leq r \leq 1} |\overline{u}(r) - \overline{v}(r)| \right\},
\]

is called the distance between $u$ and $v$.

It is shown that $(E^1, D)$ is a complete metric space [7].

**Definition 2.4.** [2] A function $f : R^2 \to E^1$ is called a fuzzy function in two-dimensional space. $f$ is said to be continuous, if for arbitrary fixed $t_0 \in R^2$ and $\varepsilon > 0$ a $\delta > 0$ exists such that

\[
||t - t_0|| < \delta \Rightarrow D(f(t), f(t_0)) < \varepsilon, \quad t = (x, y), \ t_0 = (x_0, y_0).
\]

**Definition 2.5.** [2] Let $f : [a, b] \times [c, d] \to E^1$.

For each partition $p = \{x_1, x_2, \ldots, x_m\}$ of $[a, b]$ and $q = \{y_1, y_2, \ldots, y_n\}$ of $[c, d]$ and for arbitrary $\xi_i : x_{i-1} \leq \xi_i \leq x_i$, $2 \leq i \leq m$, and for arbitrary $\eta_j : y_{j-1} \leq \eta_j \leq y_j$, $2 \leq j \leq n$, let

\[
R_p = \sum_{i=2}^{m} \sum_{j=2}^{n} f(\xi_i, \eta_j)(x_i - x_{i-1})(y_j - y_{j-1}),
\]

the definite integral of $f(x, y)$ over $[a, b] \times [c, d]$ is

\[
\int_{a}^{b} \int_{c}^{d} f(x, y) \, dx \, dy = \lim_{R \to 0} R_p,
\]

\[
(\max_{2 \leq i \leq m} |x_i - x_{i-1}|, \max_{2 \leq j \leq n} |y_j - y_{j-1}|) \to (0, 0),
\]

provided that this limit exists in metric $D$.

If the function $f(x, y)$ is continuous in the metric $D$, its definite integral exists. Furthermore,

\[
\int_{a}^{b} \int_{c}^{d} f(x, y, r) \, dx \, dy = \int_{c}^{d} \int_{a}^{b} f(x, y, r) \, dx \, dy,
\]

\[
\int_{c}^{d} \int_{a}^{b} f(x, y, r) \, dx \, dy = \int_{c}^{d} \int_{a}^{b} f(x, y, r) \, dx \, dy.
\]
Definition 2.6. The linear fuzzy Fredholm integral equation of the second kind (FFIE-2) is

\[ u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \quad x \in D, \]  
(2.1)

where \( u(x) \) and \( f(x) \) are fuzzy functions on \( D = [a,b] \) and \( k(x,t) \) is an arbitrary kernel function over \( T = [a,b] \times [a,b] \), and \( u \) is unknown on \( D \).

Theorem 2.1. [3] Let \( k(x,t) \) be a continuous function over \( T \) and \( f(x) \) be a fuzzy continuous function on \( D \). If

\[ |\lambda| < \frac{1}{M(b-a)}, \]  
(2.2)

where \( M = \max_{(x,t) \in T} |k(x,t)| \), then equation (2.1) has a fuzzy unique solution.

3 Two-dimensional fuzzy integral equation

The linear two-dimensional fuzzy Fredholm integral equation of the second kind (2D-FFIE-2) is

\[ u(x,y) = f(x,y) + \lambda \int_a^d \int_a^b k(x,y,s,t)u(s,t)dsdt, \quad (x,y) \in V, \]  
(3.3)

where \( u(x,y) \) and \( f(x,y) \) are fuzzy functions on \( V = [a,b] \times [c,d] \) and \( k(x,y,s,t) \) is an arbitrary kernel function over \( S = [a,b] \times [c,d] \times [a,b] \times [c,d] \), \( \lambda \) is an arbitrary real number and \( u \) is unknown on \( V \).

Now, suppose that \( (f(x,y),\overline{f}(x,y)) \) and \( (u(x,y),\overline{u}(x,y)) \), \( 0 \leq r \leq 1 \), \( (x,y) \in V \), are parametric form of fuzzy functions \( f(x,y) \) and \( u(x,y) \), respectively. By substituting these forms into (3.3), we have

\[ (u(x,y),\overline{u}(x,y)) = (f(x,y),\overline{f}(x,y)) + \lambda \int_a^d \int_a^b k(x,y,s,t)(u(s,t,r),\overline{u}(s,t,r))dsdt, \]

using the definition of fuzzy integral and fuzzy arithmetic for the above equation, we have the following cases for the parametric form of 2D-FFIE-2:

Case 1: If \( \lambda k(x,y,s,t) \geq 0 \), the parametric form of (3.3) is:

\[ u(x,y,r) = f(x,y,r) + \lambda \int_a^d \int_a^b k(x,y,s,t)u(s,t,r)dsdt, \]  
(3.4)

\[ \overline{u}(x,y,r) = \overline{f}(x,y,r) + \lambda \int_a^d \int_a^b k(x,y,s,t)\overline{u}(s,t,r)dsdt, \]  
(3.5)

where (3.4) and (3.5) are integral equations in three-dimensional space in crisp case.

Case 2: if \( \lambda k(x,y,s,t) < 0 \), we have

\[ u(x,y,r) = f(x,y,r) + \lambda \int_a^d \int_a^b k(x,y,s,t)\overline{u}(s,t,r)dsdt, \]  
(3.6)

\[ \overline{u}(x,y,r) = \overline{f}(x,y,r) + \lambda \int_a^d \int_a^b k(x,y,s,t)u(s,t,r)dsdt, \]

where (3.6) is the parametric form of (3.3). It is clear that (3.6) is a system of three-dimensional integral equations in crisp case.
4 Kernel iterative method

In this section we illustrate kernel iterative method for solving (3.3). First, we consider case 1 and apply the method for equations (3.4) and (3.5). For equation (3.4) we define the sequence \( \{u_n(x,y,r)\}_{n=0}^{\infty} \) as follows:

\[
\begin{align*}
u_0(x,y,r) &= f(x,y,r), \\
u_n(x,y,r) &= \int f(x,y,r) + \lambda \int_{d}^{b} k(x,y,s,t)u_{n-1}(s,t,r)dsdt, \quad n \geq 1,
\end{align*}
\]

and we define the solution of (3.4) as:

\[
u(x,y,r) = \lim_{n \to \infty} u_n(x,y,r),
\]

Also, by this method the solution of (3.5) is

\[
\begin{align*}
u_{\tau}(x,y,r) &= \lim_{n \to \infty} \nu_n(x,y,r), \\
u(x,y,r) &= \lim_{n \to \infty} \nu_n(x,y,r),
\end{align*}
\]

where

\[
\begin{align*}
u_{\tau}(x,y,r) &= \int f(x,y,r), \\
u(x,y,r) &= \int f(x,y,r) + \lambda \int_{d}^{b} k(x,y,s,t)\nu_{n-1}(s,t,r)dsdt, \quad n \geq 1,
\end{align*}
\]

Now we consider case 2 and apply the method for the system of equations (3.6). We define sequences \( \{u_n(x,y,r)\}_{n=0}^{\infty} \), \( \{\nu(x,y,r)\}_{n=0}^{\infty} \) as follows:

\[
\begin{align*}
u_0(x,y,r) &= f(x,y,r), \\
u_0(x,y,r) &= \int f(x,y,r), \\
u_n(x,y,r) &= \int f(x,y,r) + \lambda \int_{d}^{b} k(x,y,s,t)\nu_{n-1}(s,t,r)dsdt, \\
u_n(x,y,r) &= \int f(x,y,r) + \lambda \int_{d}^{b} k(x,y,s,t)\nu_{n-1}(s,t,r)dsdt, \quad n \geq 1,
\end{align*}
\]

then the solution of (3.6) is

\[
u(x,y,r) = \lim_{n \to \infty} u_n(x,y,r), \\
u(x,y,r) = \lim_{n \to \infty} \nu_n(x,y,r),
\]

Now, we show \( u = (\nu(x,y,r), \nu(x,y,r)) \) that is obtained in case 2, is the parametric form of fuzzy valued function. So, in the following proposition we show that \( u \) satisfies in lemma 2.1.

**Proposition 4.1.** The \( u = (\nu(x,y,r), \nu(x,y,r)) \), \( (x,y) \in V, r \in [0,1] \), given by (4.12), are the \( r \)-cut sets of fuzzy valued function.

**Proof.** Let \( (x,y) \in V, 0 \leq r_1 \leq r_2 \leq 1 \), since \( f \) is a fuzzy function, we have

\[
f(x,y,r_1) \leq f(x,y,r_2) \leq \overline{f}(x,y,r_2),
\]

because \( \lambda k \) is a negative function:

\[
\begin{align*}
&f(x,y,r_1) + \lambda \int_{d}^{b} k(x,y,s,t)\overline{f}(s,t,r_1)dsdt \leq \overline{f}(x,y,r_1) + \lambda \int_{d}^{b} k(x,y,s,t)\overline{f}(s,t,r_1)dsdt \\
&\overline{f}(x,y,r_2) + \lambda \int_{d}^{b} k(x,y,s,t)f(s,t,r_2)dsdt \leq \overline{f}(x,y,r_2) + \lambda \int_{d}^{b} k(x,y,s,t)f(s,t,r_1)dsdt \\
&\overline{f}(x,y,r_1) + \lambda \int_{d}^{b} k(x,y,s,t)f(s,t,r_1)dsdt,
\end{align*}
\]

using (4.11) and from (4.13), (4.14), we have

\[
\begin{align*}
u_0(x,y,r_1) \leq \nu_0(x,y,r_2) \leq \nu_0(x,y,r_2) \leq \nu_0(x,y,r_1), \\
u_1(x,y,r_1) \leq \nu_1(x,y,r_2) \leq \nu_1(x,y,r_2) \leq \nu_1(x,y,r_1),
\end{align*}
\]

by doing above procedure, we obtain

\[
u_n(x,y,r_1) \leq \nu_n(x,y,r_2) \leq \nu_n(x,y,r_2) \leq \nu_n(x,y,r_1), \quad n \geq 2,
\]
so \[
\lim_{n \to \infty} u_n(x, y, r_1) \leq \lim_{n \to \infty} u_n(x, y, r_2) \leq \lim_{n \to \infty} \pi_n(x, y, r_2) \leq \lim_{n \to \infty} \pi_n(x, y, r_1),
\]
that means \[
h(x, y, r_1) \leq u(x, y, r_2) \leq \pi(x, y, r_2) \leq \pi(x, y, r_1),
\]
therefore, condition 1 from lemma 2.1 holds. Now, suppose \( \{r_k\} \) is a non-decreasing converging sequence converges to \( r \in [0, 1] \). As \( f \) is a fuzzy function and integral is continuous, for every \((u_i(x, y, r), \pi(x, y, r)), i = 0, 1, \ldots \) condition 2 holds, and
\[
\text{conditions: (3.6), and prove the theorem. First, assume that } u
\]
and
\[
\text{without less of generality, we assume that } M
\]
Proof. In the following theorem we bring the sufficient condition for converging the mentioned method to the exact solution.

**Theorem 4.1.** Let \( k(x, y, s, t) \) be a continuous function over \( S \) and \( f(x, y) \) be a fuzzy continuous function on \( V \). If
\[
|\lambda| < \frac{1}{M(b - a)(d - c)},
\]
(4.15)
where \( M = \max_{(x, y, t) \in S} |k(x, y, s, t)| \), then kernel iterative method is convergence.

**Proof.** Without less of generality, we suppose that \( \lambda \) is a negative function on \( S \). So, we consider system of integral equations (3.6), and prove the theorem. First, assume that \( u(x, y, r) \) and \( \pi(x, y, r) \) is the exact solution of (3.6). We consider two sequences \( \{u_i(x, y, r)\}_{i=0}^{\infty}, \{\pi(x, y, r)\}_{i=0}^{\infty} \), that are defined in (4.11) and show that they converge to the solution of (3.6). Without less of generality, we assume that \( n \) is odd, so we have
\[
u_n(x, y, r) = f(x, y, r) + \ldots + \lambda^n \int_a^b \cdots \int_a^b k(x, y, s_1, t_1) \cdots k(s_{n-1}, t_{n-1}, s, t) f(s, t) ds_1 dt_1 \cdots ds_{n-1} dt_{n-1} ds dt,
\]
and
\[
u_{n-1}(x, y, r) = f(x, y, r) + \ldots + \lambda^{n-1} \int_a^b \cdots \int_a^b k(x, y, s_1, t_1) \cdots k(s_{n-2}, t_{n-2}, s, t) f(s, t) ds_1 dt_1 \cdots ds_{n-2} dt_{n-2} ds dt,
\]
therefore
\[
|u_n(x, y, r) - u_{n-1}(x, y, r)| = \left| \lambda^n \int_a^b \cdots \int_a^b k(x, y, s_1, t_1) \cdots k(s_{n-1}, t_{n-1}, s, t) f(s, t) ds_1 dt_1 \cdots ds_{n-1} dt_{n-1} ds dt \right| \leq N (|\lambda|M(b - a)(d - c))^n,
\]
where
\[
N = \sup_{V \times [0, 1]} |\pi(x, y, r)|,
\]
and since \( \sum_{n=1}^{\infty} (|\lambda|M(b - a)(d - c))^n \) is a convergent series, so \( u_n(x, y, r) \) is uniformly convergent. On the other hand
\[
|u(x, y, r) - u_n(x, y, r)| \leq |\lambda|M(b - a)(d - c) \sup_{V \times [0, 1]} |\pi(x, y, r) - u_{n-1}(x, y, r)| \leq \ldots \leq (|\lambda|M(b - a)(d - c))^n a_0,
\]
where
\[
\alpha_0 = \sup |\overline{u}(x,y,r) - \underline{u}(x,y,r)| = \sup \left| \lambda \int_c^d \int_a^b k(x,y,s,t)u(s,t,r) ds dt \right|
\]
so, we conclude \( \{u_i(x,y,r)\}^{\infty}_{i=0} \) converges uniformly to \( u(x,y,r) \), ie
\[
\lim_{n \to \infty} u_n(x,y,r) = u(x,y,r).
\]
Similarly, we can show \( \{\overline{u}_i(x,y,r)\}^{\infty}_{i=0} \) converges uniformly to \( \overline{u}(x,y,r) \), so
\[
\lim_{n \to \infty} \overline{u}_n(x,y,r) = \overline{u}(x,y,r).
\]

5 Example

Example 5.1. Consider the following two-dimensional fuzzy Fredholm integral equation:

\[
u(x,y) = f(x,y) + \frac{1}{2\pi} \int_1^2 \int_1^2 xy \cos \left( \frac{\pi s}{2} \right) u(s,t) ds dt, \quad 1 \leq x, y \leq 2,
\]

where
\[
f(x,y)(r) = (\sin(x/2)(r^2 + r), x \sin(y/2)(4 - r^3 - r)), \quad 0 \leq r \leq 1,
\]

In this example \( |k(x,y,s,t)| \leq 4, \lambda = 1/\pi \) and \( (b-a)(d-c) = 1 \). So we have \( |\lambda| M(b-a)(d-c) < 1 \).

Since \( \lambda k \) is a negative function, the sequences (4.11) converge to the solution of the above integral equation. Some first terms of (4.11) are:

\[
u_0(x,y) = x \sin(y/2)(r^2 + r),
\]
\[
u_0(x,y) = x \sin(y/2)(4 - r^3 - r),
\]
\[
u_1(x,y) = x \sin(y/2)(r^2 + r) - 0.111858492681684xy(4 - r^3 - r),
\]
\[
u_1(x,y) = x \sin(y/2)(4 - r^3 - r) - 0.111858492681684xy(r^2 + r),
\]

we can approximate the solution by kernel iterative method as

\[
u(x,y) \approx \nu_2(x,y,r) = x \sin(y/2)(r^2 + r) + xy(-0.475116614798505 + 0.148323850263418 + 0.029544696563791r^2 + 0.118779153699626r^3),
\]
\[
u(x,y) \approx \nu_2(x,y,r) = x \sin(y/2)(4 - r^3 - r) + xy(0.118178786255165 - 0.118323850263418 - 0.029544696563791r^2 - 0.148323850263418).\]

And by direct method the exact solution is

\[
u(x,y) = x \sin(y/2)(r^2 + r) + xy(-0.476942288082044 + 0.148893796424640r + 0.029658224404129r^2 + 0.119235572020511r^3),
\]
\[
u(x,y) = x \sin(y/2)(4 - r^3 - r) + xy(0.118632897616517 - 0.148893796424640r - 0.119235572020511r^2 - 0.029658224404129r^3).\]

The exact and obtained solutions of 2D-FFIE-2 at \( (x,y) = (0.3, 0.6) \) for some variant values of \( r \) are compared in Table 1 and illustrated in Figure 1.
Consider the following two-dimensional fuzzy Fredholm integral equation:

\[
    u(x, y) = f(x, y) + \int_0^1 \int_0^1 \frac{\pi}{2} \sin(2\pi s) u(s, t) ds dt, \quad 0 \leq x, y \leq 1,
\]

where

\[
    f(x, y) = \pi y(x^2 + r) + \frac{2}{15} (4 - r^2 - r),
\]

\[
    f(x, y) = \pi y(2r^2 + r) + \frac{13}{15} (4 - r^2 - r), \quad 0 \leq r \leq 1.
\]

In this example the kernel function is positive on \([0, 1] \times [0, 1] \times [0, \frac{1}{2}] \times [0, 1]\) and it is negative on \([0, 1] \times [0, 1] \times (\frac{1}{2}, 1] \times [0, 1]\). The parametric form of this integral equation is

\[
    u(x, y, r) = f(x, y, r) + \int_0^1 \int_0^1 \frac{\pi}{2} \sin(2\pi s) u(s, t, r) ds dt + \int_0^1 \int_0^1 \frac{2}{15} \sin(2\pi s) \xi(s, t, r) ds dt,
\]

\[
    \xi(x, y, r) = \int_0^1 \int_0^1 \frac{1}{2} (2x + r) \sin(2\pi s) u(s, t, r) ds dt + \int_0^1 \int_0^1 \frac{1}{2} \sin(2\pi s) \xi(s, t, r) ds dt,
\]

by using this method we define \(u_i(x, y, r)\) \(i = 0, 1, \ldots\), \(\xi_i(x, y, r)\) \(i = 0, 1, \ldots\) as follows:

\[
    u_0(x, y, r) = f(x, y, r),
\]

\[
    \xi_0(x, y, r) = f(x, y, r),
\]

\[
    u_i(x, y, r) = f(x, y, r) + \int_0^1 \int_0^1 \frac{1}{2} k(x, y, s, t) u_{i-1}(s, t, r) ds dt + \int_0^1 \int_0^1 k(x, y, s, t) \xi_{i-1}(s, t, r) ds dt,
\]

\[
    \xi_i(x, y, r) = f(x, y, r) + \int_0^1 \int_0^1 \frac{1}{2} k(x, y, s, t) \xi_{i-1}(s, t, r) ds dt + \int_0^1 \int_0^1 k(x, y, s, t) u_{i-1}(s, t, r) ds dt,
\]

Table 1: Exact and approximate solution for \((x, y) = (0.3, 0.6)\)

<table>
<thead>
<tr>
<th>(r)</th>
<th>(u_0(x, y, r))</th>
<th>(u(x, y, r))</th>
<th>(u_2(x, y, r))</th>
<th>(\xi(x, y, r))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.085521</td>
<td>-0.085850</td>
<td>0.375986</td>
<td>0.375978</td>
</tr>
<tr>
<td>0.1</td>
<td>-0.073024</td>
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<td>0.364053</td>
<td>0.364124</td>
</tr>
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<td>0.2</td>
<td>-0.058520</td>
<td>-0.058827</td>
<td>0.351218</td>
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</tr>
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<td>0.3</td>
<td>-0.041880</td>
<td>-0.042173</td>
<td>0.336828</td>
<td>0.336872</td>
</tr>
<tr>
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Figure 1: Exact and approximate solution for \((x, y) = (0.3, 0.6)\)
so we can approximate solution as

$$\begin{align*}
u_7(x, y, r) &= \pi xy \left( \frac{133}{2993} r^3 + \frac{8315}{8909} r^2 + \frac{1451}{1484} r - \frac{532}{2993} \right), \\
\overline{u}_7(x, y, r) &= -\pi xy \left( \frac{8315}{8909} r^3 + \frac{133}{2993} r^2 + \frac{1451}{1484} r - \frac{8329}{2231} \right).
\end{align*}$$

The exact solution is

$$\begin{align*}
u(x, y, r) &= \pi xy \left( \frac{2}{45} r^3 + \frac{14}{45} r^2 + \frac{44}{45} r - \frac{8}{45} \right), \\
\overline{u}(x, y, r) &= -\pi xy \left( \frac{14}{15} r^3 + \frac{2}{45} r^2 + \frac{44}{45} r - \frac{56}{15} \right).
\end{align*}$$

Table 2 shows exact and approximate solution at $(x, y) = (0.3, 0.6)$ for some variant values of $r$.

Table 2: Exact and approximate solution for $(x, y) = (0.3, 0.6)$

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\nu_7(x, y, r)$</th>
<th>$\overline{u}_7(x, y, r)$</th>
<th>$\nu(x, y, r)$</th>
<th>$\overline{u}(x, y, r)$</th>
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</tr>
</tbody>
</table>

6 Conclusion

In this work, we present a numerical method for solving 2D-FFIE-2. We can see that, solving a linear two-dimensional fuzzy Fredholm integral equation of the second kind is converted to solving a system of linear Fredholm integral equations of the second kind with three variables. We show that solution that is obtained in this method is a fuzzy solution. Moreover, the convergence theorem has been presented for the mentioned method.

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