A note on the solution of fuzzy transportation problem using fuzzy linear system

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Abstract
In this paper, we discuss the solution of a fuzzy transportation problem, with fuzzy quantities. The problem is solved in two stages. In the first stage, the fuzzy transportation problem is reduced to crisp system by using the lower and upper bounds of fuzzy quantities. In the second stage, the crisp transportation problems are solved by usual simplex method. The procedure is illustrated with numerical examples

Keywords: Fuzzy number; Fuzzy linear system; Fuzzy linear programming; Fuzzy Transportation model.

1 Introduction
Transportation model deals with transportation of a product available at several origin (sources) to a number of different destinations (jobs). This model can be used for a wide variety of situations such as scheduling, production, investment, plant location, inventory control, employment scheduling and many others. The total movement from each origin and the total movement to each destination are given and it is desired to find how the associations can be made subject to the limitations on total. In such problems, sources can be divided among the jobs or jobs may be done with a combination of sources. The objective is to minimize the cost of transportation while meeting the requirement at the destination. The parameters unit cost of transportation, supply and demand is not always exactly known and stable. This paper deals with the case when the unit costs are known exactly, but the supply and demand values are only imprecise. We adopt to express the supply and demand is fuzzy number in parametric form. The first formulation of fuzzy linear programming is proposed by Zimmermann [13]. Thereafter, many authors considered various types of the fuzzy linear programming problems and proposed several approaches for solving these problems [4, 5, 9, 11, 12]. Usually, most of the methods are based on the concept of comparison of fuzzy numbers by use of ranking function [3, 8]. In this paper, we propose a new method and algorithm to solve fuzzy transportation problem. Initially, the fuzzy transportation problem is divided into four crisp transportation problems. The crisp transportation problems are solved by usual simplex method. The optimal solution of fuzzy transportation problem is obtained from the solutions of crisp transportation problem.

2 Preliminaries
We first review some known definitions which are relevant to this work.

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Definition 2.1. [7] A fuzzy number is a map \( u : \mathbb{R} \rightarrow I = [0, 1] \) which satisfies:

(i) \( u \) is upper semi-continuous.

(ii) \( u(x) = 0 \) outside some interval \([c, d] \subset \mathbb{R} \).

(iii) There exists real numbers \( a, b \) such that \( c \leq a \leq b \leq d \) where

1. \( u(x) \) is monotonic increasing on \([c, a] \).
2. \( u(x) \) is monotonic decreasing on \([b, d] \).
3. \( u(x) = 1 \), \( a \leq x \leq b \).

Also \( u \) is called symmetric fuzzy number if \( u(\alpha x + x) = u(\alpha x - x) \), \( \forall x \in \mathbb{R} \), where \( \alpha = \frac{a + b}{2} \).

Definition 2.2. [2] A fuzzy number \( \tilde{A} \) is called non-negative (positive), if its membership function \( \mu_{\tilde{A}}(x) \) satisfies \( \mu_{\tilde{A}}(x) = 0, \forall x \leq 0 \, (x < 0) \).

Definition 2.3. [1] An arbitrary fuzzy number in parametric form is represented by an ordered pair of functions \((\underline{u}(r), \overline{u}(r))\), \(0 \leq r \leq 1\), which satisfy the following requirements:

1. \( \underline{u}(r) \) is a bounded left-continuous non-decreasing function over \([0, 1]\).
2. \( \overline{u}(r) \) is a bounded left-continuous non-increasing function over \([0, 1]\).
3. \( \underline{u}(r) \leq \overline{u}(r), \quad 0 \leq r \leq 1 \).

Also \( u = (\underline{u}, \overline{u}) \) is called a symmetric fuzzy number in parametric form if \( \alpha = \frac{\overline{u} + \underline{u}}{2} \) is a real constant for all \( 0 \leq r \leq 1 \).

Definition 2.4. [1] A crisp number \( \alpha \) is simply represented by \( \underline{u}(r) = \overline{u}(r) = \alpha \), \( 0 \leq r \leq 1 \).

Definition 2.5. [6] The \( n \times n \) linear system

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= y_n
\end{align*}
\]

(2.1)

where the coefficients matrix \( A = (a_{ij}) \), \(1 \leq i, j \leq n\) is a crisp \( n \times n \) matrix and each \( y_i \), \(1 \leq i \leq n\), is fuzzy number in parametric form called a fuzzy linear system in parametric form.

Remark 2.1. [1] For any arbitrary fuzzy numbers \( x = (\underline{x}(r), \overline{x}(r)), y = (\underline{y}(r), \overline{y}(r)) \) in parametric form and scalar \( k \),

1. \( x = y \) iff \( \underline{x}(r) = \underline{y}(r) \) and \( \overline{x}(r) = \overline{y}(r) \).
2. \( x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r)) \).
3. \( kx = (k\underline{x}(r), k\overline{x}(r)) \) if \( k \) is non-negative and \( kx = (k\overline{x}(r), k\underline{x}(r)) \) if \( k \) is negative.

Definition 2.6. [10] Consider the following linear programming problem

\[
\text{Max} / \text{Min} \quad z = cx
\]

subject to

\[
A\bar{x} \leq \begin{pmatrix} \alpha & \geq \end{pmatrix} \bar{b} \quad (\text{or} \geq) \\
\bar{x} \geq 0
\]

(2.2)

where the coefficient matrix \( A = [a_{ij}]_{m \times n} \) and the vector \( c = (c_1, c_2, \ldots, c_n) \) are non-negative crisp matrix and vector respectively and \( \bar{x} = (\bar{x}_j) \), \( \bar{b} = (\bar{b}_j) \) are non-negative fuzzy vectors such that \( \bar{x}_j, \bar{b}_j \), \(1 \leq j \leq n\), \(1 \leq i \leq m\) are fuzzy number in parametric form is called a fuzzy linear programming problem in parametric form.
Definition 2.7. [10] A fuzzy vector $\bar{x}$ is a fuzzy feasible solution of $A\bar{x} = \bar{b}$, $\bar{x} \geq 0$ where $A$ and $\bar{b}$ are defined in (2.2), if $\bar{x}$ satisfies in system.

Definition 2.8. [10] A feasible solution fuzzy vector $\bar{x}$ optimizes the objective function $\bar{z} = \bar{c}\bar{x}$ then it’s said to be optimal solution to fuzzy linear programming problem.

Definition 2.9. Transportation Problem in Fuzzy Environment: In many practical situations, the constraints like supply and demand in transportation models are uncertain; they cannot be treated as crisp terms. In such situation, it is desirable to use some type of fuzzy transportation model.

One of the mathematical models of transportation problem to minimize the total transportation cost from $m$ sources of supply to $n$ destinations is stated as follows:

$$\text{Min } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij}X_{ij}$$

subject to

$$\sum_{j=1}^{n} X_{ij} = (b_j, \bar{b}_j), \quad j = 1, 2, \ldots, n.$$  
$$\sum_{i=1}^{m} X_{ij} = (a_i, \bar{a}_i), \quad i = 1, 2, \ldots, m.$$  

and $X_{ij} \geq 0$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

3 Solution of Fuzzy Transportation Model

The supply and demand are triangular fuzzy number in parametric form. Fuzzy transportation model is transformed into equivalent two different transportation model by replacing $r$ as 0 and 1 since $0 \leq r \leq 1$.

Proposition 3.1. When $r = 0$, the fuzzy transportation model is transformed into transportation model with crisp values in supply and demand. The crisp transportation model yields the solution as $X_{ij}^0$ and $X_{ij}^0$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. 

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Proposition 3.2. When \( r = 1 \), the fuzzy transportation model is transformed into transportation model with crisp values in supply and demand. The crisp transportation model yields the solution as \( X^1_{ij} \) and \( X^0_{ij} \) for all \( i = 1,2,\ldots,m \) and \( j = 1,2,\ldots,n \).

From the above results, we obtain the optimal solution for fuzzy transportation model is

\[
X_{ij} = \left( X^1_{ij} - X^0_{ij} \right) r + X^0_{ij}
\]

\[
X^*_{ij} = \left( X^T_{ij} - X^0_{ij} \right) r + X^0_{ij}
\]

The method to solve fuzzy transportation problem consists of the following steps:

Algorithm 3.1.

Step 1. In the parametric form of fuzzy transportation model, replace \( r \) as zero in each supply \( (a_i(r), a_i(\overline{r})) \) and demand \( (b_j(r), b_j(\overline{r})) \), for \( i = 1,2,\ldots,m \) and \( j = 1,2,\ldots,n \).

Step 2. Divide the fuzzy transportation model into two different problems such that the supply and demand for the first problem are \( a_i(0) \) and \( b_j(0) \). For the second problem, the supply and demand are \( a_i(0) \) and \( b_j(0) \), for \( i = 1,2,\ldots,m \) and \( j = 1,2,\ldots,n \).

Step 3. Solving the above two different fuzzy transportation problems, we obtain \( X^0_{ij} \) and \( X^0_{ij} \), \( i = 1,2,\ldots,m \) and \( j = 1,2,\ldots,n \).

Step 4. Replace \( r \) by 1 in fuzzy transportation model and repeating the steps 2 and 3, we get \( X^1_{ij} \) and \( X^1_{ij} \), \( i = 1,2,\ldots,m \) and \( j = 1,2,\ldots,n \).

Step 5. From 4th and 5th step, we have

\[
X_{ij} = \left( X^1_{ij} - X^0_{ij} \right) r + X^0_{ij}
\]

\[
X^*_{ij} = \left( X^T_{ij} - X^0_{ij} \right) r + X^0_{ij}
\]

for \( i = 1,2,\ldots,m \) and \( j = 1,2,\ldots,n \).

Step 6. The optimal solution of the fuzzy transportation model is \( X_{ij} = \left( X_{ij}, X^*_{ij} \right) \) for \( i = 1,2,\ldots,m \) and \( j = 1,2,\ldots,n \).

4 Examples

Example 4.1.

Consider the following fuzzy transportation problem in parametric form

\[
\begin{array}{c|cc|c|cc}
\text{Warehouse} & W_1 & W_2 & \text{Supply} \\
\hline
\text{Factory} & F_1 & F_2 & \text{Supply} \\
\hline
\text{Factory} & W_1 & W_2 & \text{Supply} \\
W_1 & 5 & 4 & (3 + r, 5 - r) \\
W_2 & 2 & 6 & (8 + r, 10 - r) \\
\hline
\text{Demand} & (4 + r, 6 - r) & (7 + r, 9 - r) \\
\end{array}
\]
The equivalent Linear programming problem for the above fuzzy transportation model is

$$\text{Min } Z = 5X_{11} + 4X_{12} + 2X_{21} + 6X_{22}$$

subject to

$$X_{11} + X_{12} = (3 + r, 5 - r)$$
$$X_{21} + X_{22} = (8 + r, 10 - r)$$
$$X_{11} + X_{21} = (4 + r, 6 - r)$$
$$X_{12} + X_{22} = (7 + r, 9 - r)$$

and

$$X_{11}, X_{12}, X_{21}, X_{22} \geq 0$$

The above fuzzy transportation model is first divided into two models as,

**Model 1 (r=0)**

<table>
<thead>
<tr>
<th>Warehouse</th>
<th>Factory</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$F_1$</td>
<td>$(3, 5)$</td>
</tr>
<tr>
<td></td>
<td>$F_2$</td>
<td>$(8, 10)$</td>
</tr>
<tr>
<td>Demand</td>
<td></td>
<td>$(4, 6)$ (7, 9)</td>
</tr>
</tbody>
</table>

**Model 2 (r=1)**

<table>
<thead>
<tr>
<th>Warehouse</th>
<th>Factory</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$F_1$</td>
<td>$(4, 4)$ (9, 9)</td>
</tr>
<tr>
<td></td>
<td>$F_2$</td>
<td></td>
</tr>
<tr>
<td>Demand</td>
<td></td>
<td>(5, 5) (8, 8)</td>
</tr>
</tbody>
</table>

Model 1 further classified into two linear programming problems such as,

**Problem 1**

$$\text{Min } Z = 5X_{11}^0 + 4X_{12}^0 + 2X_{21}^0 + 6X_{22}^0$$

subject to

$$X_{11}^0 + X_{12}^0 = 3$$
$$X_{21}^0 + X_{22}^0 = 8$$
$$X_{11}^0 + X_{21}^0 = 4$$
$$X_{12}^0 + X_{22}^0 = 7$$

and

$$X_{11}^0, X_{12}^0, X_{21}^0, X_{22}^0 \geq 0$$

**Problem 2**

$$\text{Min } Z = 5X_{11}^0 + 4X_{12}^0 + 2X_{21}^0 + 6X_{22}^0$$
subject to
\[
\begin{align*}
X_{01}^1 + X_{12}^1 &= 5 \\
X_{21}^0 + X_{22}^0 &= 10 \\
X_{11}^0 + X_{21}^0 &= 6 \\
X_{12}^0 + X_{22}^0 &= 9
\end{align*}
\]
and
\[
X_{11}^0, X_{12}^0, X_{21}^0, X_{22}^0 \geq 0
\]
Solving the above two problems, we have
\[
\begin{align*}
X_{11}^0 &= 0, \\
X_{12}^0 &= 3, \\
X_{21}^0 &= 4, \\
X_{22}^0 &= 4
\end{align*}
\]
Similarly the Model 2 also further divided into two models and then solving that model, we have
\[
\begin{align*}
X_{11}^1 &= 0, \\
X_{12}^1 &= 4, \\
X_{21}^1 &= 5, \\
X_{22}^1 &= 4
\end{align*}
\]
From the above solutions, we have the following optimal solution
\[
\begin{align*}
X_{11} &= 0, X_{12} = 3 + r, X_{21} = 4 + r, X_{22} = 4. \\
X_{11} &= 0, X_{12} = 5 - r, X_{21} = 6 - r, X_{22} = 4.
\end{align*}
\]
That is
\[
X_{11} = (0, 0), X_{12} = (3 + r, 5 - r), X_{21} = (4 + r, 6 - r) \text{ and } X_{22} = (4, 4).
\]
The above solution can be represented graphically as follows:

Figure 1: The minimum transportation cost for the given problem is
\[
\begin{align*}
\text{Min } Z &= 5(0, 0) + 4(3 + r, 5 - r) + 2(4 + r, 6 - r) + 6(4, 4) = (44 + 6r, 56 - 6r).
\end{align*}
\]
Example 4.2.
Consider the following fuzzy transportation problem in parametric form

<table>
<thead>
<tr>
<th>Warehouse</th>
<th>F1</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>W1</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>W2</td>
<td>6</td>
<td>(13 + 2r, 17 - 2r)</td>
</tr>
<tr>
<td>W3</td>
<td>5</td>
<td>(8 + 3r, 11)</td>
</tr>
</tbody>
</table>

| Demand     | (3 + 2r, 7 - 2r) | (7 + 2r, 9) | (11 + r, 12) |

The equivalent Linear programming problem for the above fuzzy transportation model is

\[
\text{Min } Z = 9X_{11} + 6X_{12} + 12X_{13} + 5X_{21} + 7X_{22} + 10X_{23}
\]

subject to

\[
\begin{align*}
X_{11} + X_{12} + X_{13} & = (13 + 2r, 17 - 2r) \\
X_{21} + X_{22} + X_{23} & = (8 + 3r, 11) \\
X_{11} + X_{21} & = (3 + 2r, 7 - 2r) \\
X_{12} + X_{22} & = (7 + 2r, 9) \\
X_{13} + X_{23} & = (11 + r, 12)
\end{align*}
\]

and

\[X_{11}, X_{12}, X_{13}, X_{21}, X_{22}, X_{23} \geq 0.\]

The above fuzzy transportation model is first divided into two models as,

Model 1 \((r=0)\)

<table>
<thead>
<tr>
<th>Factory</th>
<th>F1</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>W1</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>W2</td>
<td>6</td>
<td>(13, 17)</td>
</tr>
<tr>
<td>W3</td>
<td>5</td>
<td>(8, 11)</td>
</tr>
</tbody>
</table>

| Demand     | (3, 7) | (7, 9) | (11, 12) |

Model 2 \((r=1)\)

<table>
<thead>
<tr>
<th>Factory</th>
<th>F1</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>W1</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>W2</td>
<td>6</td>
<td>(15, 15)</td>
</tr>
<tr>
<td>W3</td>
<td>5</td>
<td>(11, 11)</td>
</tr>
</tbody>
</table>

| Demand     | (5, 5) | (9, 9) | (12, 12) |

Model 1 further classified into two Linear programming problems such as,

Problem 1

\[
\text{Min } Z = 9X_{11}^0 + 6X_{12}^0 + 12X_{13}^0 + 5X_{21}^0 + 7X_{22}^0 + 10X_{23}^0
\]

subject to

\[
\begin{align*}
X_{11}^0 + X_{12}^0 + X_{13}^0 & = 13 \\
X_{21}^0 + X_{22}^0 + X_{23}^0 & = 8 \\
X_{11}^0 + X_{21}^0 & = 3 \\
X_{12}^0 + X_{22}^0 & = 7 \\
X_{13}^0 + X_{23}^0 & = 11
\end{align*}
\]
and

\[ \mathbf{X}^0_{11}, \mathbf{X}^0_{12}, \mathbf{X}^0_{13}, \mathbf{X}^0_{21}, \mathbf{X}^0_{22}, \mathbf{X}^0_{23} \geq 0. \]

**Problem 2**

Minimize

\[ Z = 9X^0_{11} + 6X^0_{12} + 12X^0_{13} + 5X^0_{21} + 7X^0_{22} + 10X^0_{23} \]

subject to

\[
\begin{align*}
X^0_{11} + X^0_{12} + X^0_{13} &= 17 \\
X^0_{21} + X^0_{22} + X^0_{23} &= 11 \\
X^0_{11} + X^0_{21} &= 7 \\
X^0_{12} + X^0_{22} &= 9 \\
X^0_{13} + X^0_{23} &= 12
\end{align*}
\]

and

\[ \mathbf{X}^0_{11}, \mathbf{X}^0_{12}, \mathbf{X}^0_{13}, \mathbf{X}^0_{21}, \mathbf{X}^0_{22}, \mathbf{X}^0_{23} \geq 0. \]

Solving the above two problems, we have

\[
\begin{align*}
X^0_{11} &= 0, X^0_{12} = 7, X^0_{13} = 6, X^0_{21} = 3, X^0_{22} = 0, X^0_{23} = 5. \\
X^0_{11} &= 0, X^0_{12} = 9, X^0_{13} = 8, X^0_{21} = 7, X^0_{22} = 0, X^0_{23} = 4.
\end{align*}
\]

Similarly the Model 2 also further divided into two models and then solving that model, we have

\[
\begin{align*}
X^1_{11} &= 0, X^1_{12} = 9, X^1_{13} = 6, X^1_{21} = 5, X^1_{22} = 0, X^1_{23} = 6. \\
X^1_{11} &= 0, X^1_{12} = 9, X^1_{13} = 6, X^1_{21} = 5, X^1_{22} = 0, X^1_{23} = 6.
\end{align*}
\]

From the above solutions, we have the following optimal solution

\[
\begin{align*}
X_{11} &= 0, X_{12} = 7 + 2r, X_{13} = 6, X_{21} = 3 + 2r, X_{22} = 0, X_{23} = 5 + r. \\
X_{11} &= 0, X_{12} = 9, X_{13} = 8 - 2r, X_{21} = 7 - 2r, X_{22} = 0, X_{23} = 4 + 2r.
\end{align*}
\]

That is

\[
\begin{align*}
X_{11} = (0,0), \ X_{12} = (7+2r,9), \ X_{13} = (6,8-2r) \\
X_{21} = (3+2r,7-2r), \ X_{22} = (0,0), \ X_{23} = (5+r,4+2r).
\end{align*}
\]

The above solution can be represented graphically as follows:

![Graph](image-url)

Figure 2: The minimum transportation cost for the given problem is Min Z = \((179 + 32r, 225 - 14r)\)
5 Conclusions

We proposed a new method and algorithm to solve fuzzy transportation problem. Initially, the fuzzy transportation problem was divided into four crisp transportation problems. These crisp transportation problems are solved by existing simplex methods. The optimal solution of the fuzzy transportation problem was obtained from the solutions of crisp transportation problems.

References


