Interval King method to compute enclosure solutions of nonlinear equations

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Abstract
In this paper, a different variant of the classic King method for computing the enclosure solutions of a given nonlinear equation is introduced. Our proposed approach is based on interval analysis which was first invented by R. Moore. Also, error analysis and convergence will be discussed. Moreover, the proposed interval method will be compared with the interval Newton and Ostrowski methods. Some implemented examples with INTLAB are also included to illustrate the validity and applicability of the scheme.

Keywords: Interval analysis; King method; Enclosure algorithm; Error analysis.

1 Introduction
In practice, interval analysis provides rigorous enclosure of solutions to the given model equations. In fact, interval algorithms are designed to automatically provide rigorous bounds on accumulated rounding errors, approximation errors, and propagated uncertainties in initial data during the process of the computation [1, 3, 5, 9, 11]. Solving non-linear equations is one of the most important problems in numerical analysis. The classical Newton method is an important method with convergence of quadratic. To improve the local order of convergence and efficiency index, many methods have been proposed [4, 6, 12, 14]. One of these important and basic methods is the King method [6].

There are two ordinary families to find the roots of a given nonlinear equation \( f(x) = 0 \) generally. The first family is always convergent and low, e.g. bisection or bracket method, while the other family is not always convergent but fast, under the same conditions, iterative methods. The basic problem is, if we can modify the second family method so that it will have guaranteed convergency. Fortunately, some attempts to obtain guaranteed Newtons method have successfully been made by Moore [5] and Alefeld and Herzberger [1].

An interval Newton method has been developed for solving nonlinear equations. This verified approach enables us to compute interval enclosures for the exact values of the solution with sharp bounds [9]. Recently, classic two-point Ostrowski’s method developed to its interval method [7].

In the present article, a different variant interval method, based on the classic King method, for finding the enclosure roots of nonlinear equations is introduced. Convergence rate of the proposed method is also examined. Moreover, error bound and comparison are given. This approach is asymptotically convergent. Applicability and reliability of this algorithm will be investigated and justified through some examples implemented by using INTLAB, which is free

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to use [13].
This paper is organized as follows: In Section 2 some preliminaries are given. Construction and convergence analysis of the proposed method is presented in Section 3. Some numerical test problems as well as comparison with Interval Newton’s method are provided in Section 4. Section 5 deals with concluding remarks.

2 Preliminaries: notations and results

We first introduce some basic properties of interval arithmetic from [1, 8, 9]. An interval number is a closed set in $\mathbb{R}$ that includes the possible range of an unknown real number, where $\mathbb{R}$ denotes the set of real numbers. Therefore, a real interval is a set of the form $(x, \overline{x}]$, where $x$ and $\overline{x}$ are the lower and upper bounds (end-points) of the interval number $x$, respectively. The set of compact real intervals is denoted by $\mathbb{IR} = \{x = [x, \overline{x}] | x, \overline{x} \in \mathbb{R}, x \leq \overline{x}\}$. A real number $x$ is identified with a point interval $x = [x, x]$. The quality of interval analysis is measured by the width of the interval results, and a sharp enclosure for the exact solution is desirable. The midpoint and the width of an $x$ real interval is a set of the form $(x, \overline{x}]$, where $x$ and $\overline{x}$ are the lower and upper bounds (end-points) of the interval number $x$, respectively. The midpoint and the width of an interval are denoted by mid and wid, respectively. Considering $|x| = \max\{|x| | x \in x\}$, for any $x, y \in \mathbb{IR}$ and $a, b \in \mathbb{R}$ it can be concluded that [9]:

$$\text{wid}(ax + by) = |a| \text{wid}(x) + |b| \text{wid}(y),$$

$$\text{wid}(xy) \leq |x| \text{wid}(y) + |y| \text{wid}(x).$$

**Definition 2.1.** (See [9]) We say that $f$ is an interval extension of $f$, if for degenerate interval arguments, $f$ agrees with $f$, i.e. $f([x, x]) = f(x)$.

It should be noted that in general $f$ is not the set image of $f$. Generally, we expect, $f(x) \subseteq f(x)$. Besides, when $f$ is an inclusion function of $f$, then we can directly obtain lower and upper bounds of $f$ over any interval $x$ within the domain of $f$ just by taking $f_x(x)$ and $f_x(x)$, respectively.

**Definition 2.2.** (See [9]) An interval extension $f$ is said to be Lipschitz in $x^0$ if there is a constant $L$ such that $\text{wid}(f(x)) \leq L \text{wid}(x)$ for every $x \subseteq x^0$.

Hence, the width of $f(x)$ approaches zero at least linearly with the width of $x$.

**Definition 2.3.** (See [9]) An interval sequence $\{x^{(k)}\}$ is nested if $x^{(k+1)} \subseteq x^{(k)}$ for all $k$.

**Lemma 2.1.** (See [9]) Suppose $\{x^{(k)}\}$ is such that there is a real number $x \in x^{(k)}$ for all $k$. Define $\{y^{(k)}\}$ by $y^{(1)} = x^{(1)}$ and $y^{(k+1)} = x^{(k+1)} \cap y^{(k)}$ for all $k = 1, 2, \ldots$. Then $y^{(k)}$ is nested with limit $y$, and

$$x \in y \subseteq y^{(k)}, \forall k.$$

**Lemma 2.2.** (See [9]) Every nested sequence $\{x^{(k)}\}$ converges and has the limit $\cap_{k=1}^{\infty} x^{(k)}$.

**Lemma 2.3.** (See [9]) If $f$ is a natural interval extension of a real rational function with $f(x)$ defined for $x \subseteq x^0$, where $x$ and $x^0$ are interval, then $f$ is Lipschitz in $x^0$; in other words:

$$\text{wid}(f(x)) \leq L \text{wid}(x).$$

(2.1)

2.1 Interval Newton method

Newton method is the well-known iterative method for finding a simple zero of function. Let $f$ be a real-valued function of a real variable $x$, and suppose that $f$ is continuously differentiable. Let $f'(x)$ be an inclusion monotonic interval extension of $f'(x)$ and consider the algorithm

$$x^{(k+1)} = x^{(k)} \cap N(x^{(k)}), \quad (k = 0, 1, 2, \ldots),$$

(2.2)

where

$$N(x) = \text{mid}(x) - \frac{f(\text{mid}(x))}{f'(x)}.$$  

(2.3)

This is well-known as interval Newton method [9].
Theorem 2.1 (See [9]). If an interval \( x^{(0)} \) contains a zero \( x^* \) of \( f(x) \), then so does \( x^{(k)} \) for all \( k = 0, 1, 2, \ldots \), defined by (2.2). Furthermore, the intervals \( x^{(k)} \) form a nested sequence converging to \( x^* \) if \( 0 \notin f'(x^{(0)}) \).

The interval Newton method (2.2) is asymptotically error squaring.

Theorem 2.2 (See [9]). Given a real rational function \( f \) of a single real variable \( x \) with rational extensions \( f, f' \), respectively, such that \( f \) has a simple zero \( x^* \) in an interval \( x^{(0)} \) for which \( f(x^{(0)}) \) is defined and \( f'(x^{(0)}) \) is defined and does not contain zero i.e. \( 0 \notin f'(x^{(0)}) \). Then there is a positive real number \( C \) such that

\[
\text{wid}(x^{(k+1)}) \leq C \left( \text{wid}(x^{(k)}) \right)^2.
\]

If \( 0 \notin f'(x^{(0)}) \), then \( 0 \notin f'(x^{(k)}) \) for all \( k \) and \( \text{mid}(x^{(k)}) \) is not contained in \( N(x^{(k)}) \), unless \( f(\text{mid}(x^{(k)})) = 0 \). So, convergence of the sequence follows [1, 2, 9, 10]. Some special cases of (2.2) have been discussed in [2] in more details.

2.2 Classic King method

The King method [6], for finding a simple root of a nonlinear equation, is written as

\[
x_{n+1} = K(x_n),
\]

where

\[
N(x) = \mid x - \text{mid} (x) \mid f'(x) / \left( f(\text{mid}(x)) + (\beta - 2)f(\text{mid}(y)) \right)
\]

\[
y_n = x_n - \left( f(x_n) / f'(x_n) \right),
\]

The order of convergence of this method is four.

3 Main Results

This section focuses on interval extension of the classic King method.

3.1 Interval King method

We can consider a natural interval extension of (2.4). Let \( x = [x, \bar{x}] \) be an interval in which we seek a solution of the equation

\[
f(x) = 0.
\]

Now natural interval extension of (2.4) can be considered as

\[
x^{(k+1)} = x^{(k)} \cap K(x^{(k)}), \quad k = 0, 1, 2, \ldots,
\]

where

\[
K(x) = \text{mid} (y) - \frac{f(\text{mid}(x)) + \beta f(\text{mid}(y))}{f(\text{mid}(x)) + (\beta - 2)f(\text{mid}(y))} \times \frac{f(\text{mid}(y))}{f'(x)},
\]

\[
y^{(k)} = x^{(k)} \cap N(x^{(k)}),
\]

\[
N(x) = \text{mid} (x) - \frac{f(\text{mid}(x))}{f'(x)}.
\]

Here we have considered the interval extension of \( f' \), i.e. \( f' \).

The convergent conditions of the generated interval sequence by (3.5) are discussed later. Let us introduce a computational algorithm for finding enclosure roots of a given nonlinear equation based on our proposed method by using relation (3.5).
AlGORITHM

To summarize the previous development, the following computational algorithm is produced:

**INPUT**
- given initial interval $x^{(0)}$ including one root;
- tolerance TOL;
- maximum number of iteration N;
- functions $f$, $f'$, $f''$.

for k=0: N-1

Compute $N(x^{(k)})$ from (3.8).

$y^{(k)} := N(x^{(k)}) \cap x^{(k)}$.

Compute $K(x^{(k)})$ from (3.6).

$x^{(k+1)} := K(x^{(k)}) \cap x^{(k)}$.

If $\text{wid}(x^{(k+1)}) \leq \text{TOL}$ or $\text{wid}(x^{(k+1)}) = \text{wid}(x^{(k)})$, then go to OUTPUT STEP

end

**OUTPUT** $(x^{(k+1)})$: (The procedure was successful.)

This confirms the Moore's algorithm which is given in [9].

### 3.2 Convergence analysis and error bounds

In this section, the convergence and error bound for interval method (3.5) are discussed. Unlike the classic King method, the interval version always displays a very regular behavior. To begin with, we will assume that $f : x \rightarrow R$ is a continuously differentiable function, and $x^* \in x$ is a root of $f$. We also assume that an interval extension of $f'$ exists and satisfies $0 \not\in f'(x)$. In particular, this implies that $f'(x) \neq 0$ for all $x \in x$. The sequence of interval King method has some nice and subtle properties.

**Theorem 3.1** (Interval King method). Assume $f \in C(x^{(0)})$ and $0 \not\in f'(x^{(k)})$ for $k = 0, 1, 2, \ldots$. If $x^{(0)}$ contains a root $x^*$ of $f$, then so do all intervals $x^{(k)}$, $k = 1, 2, \ldots$. Besides, the intervals $x^{(k)}$ form a nested sequence converging to $x^*$.

**Proof.** By induction, since $0 \not\in f'(x^{(k)})$, if $x^* \in x^{(0)}$ then $x^* \in x^{(k)}$ for $k = 1, 2, \ldots$. Also, Lemma (2.1) leads that the intervals $x^{(k)}$, generated by (3.5), form a nested sequence. Therefore, since for $k = 0, 1, 2, \ldots$ we have $x^* \in x^{(k)}$ then $x^* \in \cap_{k=0}^{\infty} x^{(k)}$ or $\lim_{n \rightarrow \infty} \cap_{k=0}^{n} x^{(k)} = x^*$ and the proof is completed.

One of the most useful properties of the interval King operator $K$ is that we are provided with means of detecting when a region does not contain a root of $f$. As this is a common situation, it is important that we can quickly discard a set on the grounds of it containing no roots. Another important contribution from the properties of $K$ is a simple verifiable condition that guarantees the existence of a unique root within an interval.

**Theorem 3.2.** Suppose $f \in C(x^{(0)})$ and $0 \not\in f'(x^{(k)})$ for $k = 0, 1, 2, \ldots$.

1. If $x^* \in x^{(0)}$ and $K(x^{(k)}) \subseteq x^{(k)}$, then $x^{(k)}$ contains exactly one zero of $f$.

2. If $x^{(k)} \cap K(x^{(k)}) = \emptyset$, then $x^{(k)}$ does not contain any zero of $f$.

**Proof.** First, part one is proved. Since $0 \not\in f'(x^{(k)})$, then $f'(x) \neq 0$ for all $x \in x^{(k)}$ and therefore $f$ is monotonic on $x^{(k)}$. In other words, it has at most one zero in $x$. Hence, it is sufficient to find a zero $x^* \in x^{(k)}$. Since $K(x^{(k)}) \subseteq x^{(k)}$, using the Theorem (3.1) it is obvious that $f$ has exactly one root in $x^{(k)}$.

To establish part (2), suppose $x^*$ is a zero of $f$ and $x^* \in x^{(0)}$, then Theorem (3.1) results $x^* \in K(x^{(k)})$. Consequently $x^* \in x^{(k)} \cap K(x^{(k)})$ which is contradiction. So the proof is completed. 

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If we start with an \(x^{(0)}\) such that \(K(x^{(0)}) \subseteq x^{(0)}\), then Theorems (3.1–3.2) guarantee a nested sequence of intervals \(\{x^{(k)}\}\) convergent to an interval \(x^*\) such that \(x^* \in x^{(0)}\) and \(x^* = K(x^*)\) and \(x^* \subseteq x^{(k)}\) for all \(k = 0, 1, 2, \ldots\). On a computer, the procedure can be stopped when \(x^{(k+1)} = x^{(k)}\) or \(\text{wid}(x^{(k)}) \leq \varepsilon\); using interval arithmetic (shorter 1A), with a specific number of digits, this yields the narrowest possible interval containing \(x^*\), inclusion solution.

The sequence (3.5) converges to \(x^*\) if the assumptions of the Theorems (3.1–3.2) are hold. Under conditions similar to those of Theorem (2.2), it is possible to show that the convergence rate is 4. A formal statement of this property is as follows.

**Theorem 3.3.** Suppose \(f \in C(x)\), \(f, f'\) are bounded on \(x\), \(0 \notin f'(x)\), and the iteration (3.5) converges. Then

\[
\text{wid}(x^{(k+1)}) = O\left((\text{wid}(x^{(k)}))^4\right).
\]

**Proof.** By Mean Value Theorem we have

\[
f(\text{mid}(x^{(k)})) = f'(\xi)[\text{mid}(x^{(k)}) - x^*],
\]

which \(\xi\) is between \(\text{mid}(x^{(k)})\) and \(x^*\). Since \(K(x^{(k)}, y^{(k)}) \subseteq x^{(k)}\), thus from (3.6), we can write

\[
x^{(k+1)} = \text{mid}(x^{(k)}) - \frac{[\text{mid}(x^{(k)}) - x^*]|f'(\xi_1) + \beta|\text{mid}(y^{(k)}) - x^*|f'(\xi_2)}{[f(\text{mid}(x^{(k)})) + (\beta - 2)f(\text{mid}(y^{(k)}))]} + (\beta - 2)f(\text{mid}(y^{(k)}))\]

\[
\times \frac{[\text{mid}(y^{(k)}) - x^*]|f'(\xi_2)}{f'(x^{(k)})},
\]

and

\[
\text{wid}(x^{(k+1)}) = \frac{|\text{mid}(x^{(k)}) - x^*||f'(\xi_1)| + \beta|\text{mid}(y^{(k)}) - x^*||f'(\xi_2)|}{[f(\text{mid}(x^{(k)})) + (\beta - 2)f(\text{mid}(y^{(k)}))]} \times \frac{|\text{mid}(y^{(k)}) - x^*||f'(\xi_2)|}{\text{wid}(f'(x^{(k)}))}.
\]

It is clear that

\[
|m(\text{mid}(x^{(k)}) - x^*)| \leq \text{wid}(x^{(k)}).
\]

Furthermore, since \(y^{(k)}\) is generated from (3.8), Theorem (2.2) leads

\[
|m(\text{mid}(y^{(k)}) - x^*)| \leq \text{wid}(y^{(k)}) \leq \left(\text{wid}(x^{(k)})\right)^2.
\]

Let \(|f'(\xi)| \leq M_i, i = 1, 2\) and \(|f(\text{mid}(x^{(k)})) + (\beta - 2)f(\text{mid}(y^{(k)}))| \leq M_3\), then by Theorem (2.3), since \(\text{wid}(1/f'(x^{(k)})) = O(\text{wid}(x^{(k)}))\), we have the following error bound

\[
\text{wid}(x^{(k+1)}) \leq M\left(\text{wid}(x^{(k)})\right)^4, \quad \text{M} = \frac{(M_1 + \beta M_2 \text{wid}(x^{(k)})) M_2}{M_3}.
\]

Therefore, the proof is completed.

\[\Box\]

**4 Numerical implementations**

In this section, the newly developed method is applied to solve some examples. Also the computed results are compared and the accuracy and applicability of the mentioned algorithm and theorem in the previous section are justified. In fact, the findings are illustrated by applying three methods on some examples. Numerical results are computed by using INTLAB toolbox created by Rump [13].
Remark 4.1. *In the last column of Table 2, the following abbreviation has been used*

\[4.306584728220697, 4.306584728220700]\]

The results of these examples show that the interval King method (IKM) for \( \beta = 0, 1 \) is faster than interval Newton (INM). It can be seen that if we chose initial interval containing a zero of a given nonlinear equation, then the speed of convergence is increased, too. However, we can chose initial guess suitably without worrying about its convergence problem. Furthermore, we varied the parameter \( \beta \) of King’s interval family and observed that it did not affect the convergence rate. So we reported only two cases i.e. \( \beta = 0, 1 \). Finally, we should say one important thing. Due to interval dependency, we can not compare interval methods with classic methods, numerically. It is always asked what is the differences between interval methods and classic methods? We try to answer this question briefly. Interval methods always converge provided that initial interval contains a zero of a given nonlinear equations while classic methods suffer to provide guaranteed convergence and rely on initial guess very much. Besides, when one tries to construct higher order of classic methods, for example three-point methods, the instability happens and so that it is needed to chose initial guess very close to a zero! Interval methods can provide a tiny desired interval which contains a zero.

<table>
<thead>
<tr>
<th>Test functions</th>
<th>Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1(x) = \exp(x) - 4x^2 )</td>
<td>4.3065847282206988</td>
</tr>
<tr>
<td>( f_2(x) = x^2 - \exp(x) - 3x + 2 )</td>
<td>0.2575302854398607</td>
</tr>
<tr>
<td>( f_3(x) = \exp(-x) + \cos(x) )</td>
<td>1.7461395304080122</td>
</tr>
<tr>
<td>( f_4(x) = x^2 - 3 )</td>
<td>1.7320508075688772</td>
</tr>
<tr>
<td>( f_5(x) = (x + 2)\exp(x) - 1 )</td>
<td>-0.442854401002388</td>
</tr>
<tr>
<td>( f_6(x) = x^3 + x^4 + 4x^2 - 15 )</td>
<td>1.3474280989683043</td>
</tr>
<tr>
<td>( f_7(x) = \cos(x) - x )</td>
<td>0.7390851332151606</td>
</tr>
<tr>
<td>( f_8(x) = x^3 - 2x - 5 )</td>
<td>2.0945514815423265</td>
</tr>
<tr>
<td>( f_9(x) = \sin(x) - x/3 )</td>
<td>2.2788626660758279</td>
</tr>
</tbody>
</table>
Table 2: Interval Newton and King families with $\beta = 0, 1$ solutions

<table>
<thead>
<tr>
<th>$f_i(x)$</th>
<th>$x^{(0)}$</th>
<th>Iterations</th>
<th>Enclosure</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x^{(0)}$</td>
<td>INM</td>
<td>IOM $\beta = 0$</td>
</tr>
<tr>
<td>$f_1$</td>
<td>$[4,5]$</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$[4,4,5]$</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$[4,3,4,4]$</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$f_2$</td>
<td>$[0,1]$</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$[0,5]$</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$[.24,.26]$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$f_3$</td>
<td>$[1,2]$</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$[1.5,2]$</td>
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<td>2</td>
</tr>
<tr>
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<td>$[1.6,1.8]$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
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<td>$[1,2]$</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$[1.5,2]$</td>
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<td></td>
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<td>$[.7, .8]$</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
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<td>$[2,3]$</td>
<td>6</td>
<td>3</td>
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<tr>
<td></td>
<td>$[2,2,5]$</td>
<td>6</td>
<td>3</td>
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<tr>
<td></td>
<td>$[2,2,1]$</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>$f_9$</td>
<td>$[2,3]$</td>
<td>6</td>
<td>3</td>
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<tr>
<td></td>
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<td>4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$[2,2,2,3]$</td>
<td>4</td>
<td>2</td>
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</tbody>
</table>

5 Conclusion

In this paper, a new enclosure method, interval King method, was developed to find the interval solution of a given nonlinear equation. A fundamental distinction between the interval King method and the ordinary King method is that the former uses computation with sets instead of computation with points. Again, this permits us to find all zeros of a function in a given staring interval. Whereas the ordinary King method is prone to erratic behavior, the interval version practically always converge. The difference in performance of the two methods can be dramatic. This method has the local order of convergence equal to 4 like classic King method. Moreover, necessary and sufficient conditions about the convergency were discussed in details. Also, error bound and convergence rate were studied. To verify the theory, this algorithm was then tested using some examples via INTLAB. Furthermore, the suggested method was compared with the interval Newton method. As expected, according to the discussed theory, this method was better than the interval Newton method.

It is worth mentioning that interval method do not support Kung-Traub conjecture due to interval dependency. This field has many basic problems which encourages interested researchers to develop it. Here we mention some basic problems which can be consider for the future works. There are many forth optimal methods of two-point methods which can be developed to their interval cases. Interval methods can be developed to find all zeros of a given nonlinear equations.
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