A fuzzy dynamic DEA model

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Abstract
Dynamic DEA is a nonparametric technique for measuring the relative efficiencies of a set of peer decision making units (DMUs) which consume multiple inputs to produce multiple outputs over a certain time period. This paper provides a theoretical discussion about some fuzzy approaches for incorporating nondeterministic data in dynamic DEA framework.

Keywords: Dynamic DEA; Efficiency; Fuzzy Linear Programming.

1 Introduction

Data envelopment analysis (DEA) is a linear programming-based technique, first introduced by Charnes et al.[1], for measuring the relative efficiencies of a set of peer decision making units (DMUs) which consume multiple inputs to produce multiple outputs. Although basic DEA models often measure the efficiencies of units over a certain time-period, some scholars studied assessing the efficiency over time by dynamic DEA models, see, e.g.[2, 3, 4]. From applied point of view, DEA has allocated a wide variety of research to itself. There are many publications applying DEA as a tool for performance analysis in different systems. Although nowadays DEA has allocated to itself a wide variety of applied research, in the real world the data of the units under evaluation are often fuzzy rather than crisp. Some authors point out this matter, considering the natural uncertainty inherent to some production processes. One can find several fuzzy mathematical programming-based approaches to evaluate DMUs in the DEA literature, see e.g.[5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. This paper provides a theoretical discussion about some fuzzy dynamic DEA models, through establishing some theoretical results. Sections 2 and 3 of the paper provide some preliminaries and section 3 discusses about some fuzzy dynamic DEA models theoretically.

2 Dynamic DEA

Let us assume that we have $n$ decision making units $DMU_j$ $(j = 1, 2, \ldots, n)$ such that for $t = 1, 2, \ldots, T$, in the $t$th period $DMU_j$ uses two different groups of inputs $k_{t-1,j} \in \mathbb{R}_+^l$ (vector of quasi-fixed inputs) and $x_{t,j} \in \mathbb{R}_+^m$ (vector of variable inputs) to produce two different types of outputs $y_{t,j} \in \mathbb{R}_+^s$ (vector of goods) and $k_{t,j}$. Define $X_t = [x_{t,1}, \ldots, x_{t,n}]$, $K_t = [k_{t,1}, \ldots, k_{t,n}]$, and $Y_t = [y_{t,1}, \ldots, y_{t,n}]$ as $m \times n$, $l \times n$, and $s \times n$ matrices related to the inputs, quasi-fixed inputs, and outputs, respectively. Dynamic DEA technique incorporates variable inputs and quasi-fixed inputs into a framework of performance analysis. Considering $DMU_p$ as the DMU under assessment.*Corresponding author. Email address: kavoussoleimani@yahoo.com Tel: +989123205284.
and \(\bar{k}_0\) as an initial vector for \(k_0\). Soleimani-damaneh [3] provided the following model for evaluating DMUs under the variable RTS (VRS) assumption of production technology, in which \(w_{t,p} = (w_{1,t,p}, \ldots, w_{m,t,p}) \in \mathbb{R}_+^m\) and \(v_{t,p} = (v_{1,t,p}, \ldots, v_{l,t,p}) \in \mathbb{R}_+^l\) are the positive given weight vectors, corresponding to DMU \(_p\) in the \(r\)th period and \(e = (1, 1, \ldots, 1)\).

\[
\theta_p = \min \sum_{t=1}^T a_t \lambda_t + v_{1,p} \bar{k}_0
\]

s.t. \(K_0 \lambda_1 \leq \bar{k}_0\),

\[
Y_t \lambda_t \geq y_{t,p}, \quad t = 1, 2, \ldots, T \quad (1-a)
\]

\[
e \lambda_t = 1, \quad t = 1, 2, \ldots, T \quad (1-b)
\]

\[
K_t \lambda_t - K_{t+1} \lambda_{t+1} \geq 0, \quad t = 1, 2, \ldots, T - 1 \quad (1-c)
\]

\[
\lambda_t \geq 0, \quad t = 1, 2, \ldots, T \quad (1-d)
\]

where

\[
a_1 = w_{1,p} X_t, \quad (2.2)
\]

\[
a_t = w_{t,p} X_t + v_{t,p} K_{t-1}, \quad t = 2, 3, \ldots, T. \quad (2.3)
\]

As mentioned in [3], the above model is a reduced version of the model provided by Sueyoshi and Sekitani[4] and reduces the computational requirements for using the basic dynamic DEA models. In fact, model (2.1) has \((m + 2l + s + 1)T - l\) constraints, while that given in [4] has \((s + l + 1)T\) constraints. This shows that model (2.1) has \(mT + l(T - 1)\) constraints fewer than that given in [4], and hence utilizing model (2.1) is strongly economical from a computational point of view. Recall that since the memory size needed for keeping the basis (or its inverse) in the simplex algorithm is the square of the number of constraints, a reduction in the number of the constraints of LP models gives us a computational advantage.

3 Fuzzy numbers

Suppose that \(X\) is a crisp set and \(\tilde{A} : X \rightarrow [0, 1]\) is a fuzzy set on \(X\). \(\tilde{A}\) is called a fuzzy number if \(\tilde{A}\) is an upper-semi-continuous function on \(\mathbb{R}\), \(supp \tilde{A} = cl \{x \in X | \tilde{A}(x) > 0\}\) is a compact interval, say, \([a, b]\), and there exists \(c, d\) such that \(a < c < d < b\) and \(\tilde{A}\) is increasing on the interval \([a, c]\), equal to 1 on \([c, d]\), and decreasing on the interval \([d, b]\). For each \(\alpha \in (0, 1]\), the \(\alpha\)-cut set of \(\tilde{A}\), denoted by \([\tilde{A}]_\alpha\), is \([\tilde{A}]_\alpha = \{x \in \mathbb{R} | \tilde{A}(x) \geq \alpha\}\), while \([\tilde{A}]_0 = supp \tilde{A}\). The lower and upper endpoints of any \(\alpha\)-cut set, \([\tilde{A}]_\alpha\), are represented by \([\tilde{A}]_\alpha^L\) and \([\tilde{A}]_\alpha^U\), respectively. There are different methods for ranking the fuzzy numbers, see [13] among others.

Considering \(\tilde{A}, \tilde{B}\) as fuzzy numbers, along the lines of [13], we consider the following weighted signed distance:

\[
d(\tilde{A}, \tilde{B}) = \int_0^1 S(\alpha)(|\tilde{A}^L|_\alpha + |\tilde{A}^U|_\alpha - |\tilde{B}^L|_\alpha - |\tilde{B}^U|_\alpha) d\alpha,
\]

and along the lines of [13] we define a ranking system on the set of fuzzy number, \(F(\mathbb{R})\), as:

\[
\tilde{B} \prec \tilde{A} \text{ if } d(\tilde{A}, \tilde{B}) > 0, \tilde{B} \succ \tilde{A} \text{ if } d(\tilde{A}, \tilde{B}) < 0, \text{ and } \tilde{A} \approx \tilde{B} \text{ if } d(\tilde{A}, \tilde{B}) = 0.
\]

\(S(\alpha)\) is a reducing weight function, see [13, 14].

An LR-fuzzy number \(\tilde{a}\) can be described with the following membership function:

\[
\tilde{a}(x) = \begin{cases} 
L(\frac{\alpha - m}{p}) & m - \beta \leq x \leq m, \\
1 & m \leq x \leq m, \\
R(\frac{x - m}{\gamma}) & m \leq x \leq m + \gamma, \\
0 & \text{otherwise,}
\end{cases}
\]

where \(L,R:[0, 1] \rightarrow [0, 1]\), with \(L(0) = R(0) = 1\) and \(L(1) = R(1) = 0\), are non-increasing, continuous shape functions. The LR-fuzzy number is then denoted by \(\tilde{a} = (m, m, \beta, \gamma)_{LR}\), and \((m, m)\) is the peak of \(\tilde{a}\). Considering \(\tilde{a} = (m, m, \beta, \gamma)_{LR}\) and \(\tilde{b} = (m, m, \beta, \gamma)_{LR}\)
\((\bar{a}, \bar{b}, p, q)_{LR}\) as two LR-fuzzy numbers with the same shape functions, from [13, 14] it can be seen that

\[
d(\bar{a}, \bar{b}) = (m + \bar{m} - a - \bar{a}) \int_0^1 S(\alpha) d\alpha + (p - \bar{p}) \int_0^1 S(\alpha) L^{-1}(\alpha) d\alpha + (\gamma - q) \int_0^1 S(\alpha) R^{-1}(\alpha) d\alpha.
\] (3.6)

An LR fuzzy number is named a triangular fuzzy number if

\[
\text{l}_1 \leq a \leq \text{r}_1,
\]

\[
\text{b}_1 \leq a \leq \text{d}_1.
\]

An LR fuzzy number is named a symmetric triangular fuzzy number if

\[
\text{l}_1 \leq a \leq \text{r}_1,
\]

\[
\text{d}_1 = \text{r}_1 - \text{l}_1.
\]

As mentioned in [10], in the context of fuzzy linear programming, the min T-norm is the most applied tool to evaluate a linear combination of fuzzy quantities \(\lambda_1 \bar{a}_1 \oplus \lambda_2 \bar{a}_2 \oplus \ldots \oplus \lambda_n \bar{a}_n\), when the fuzzy numbers are noninteractive. In particular, for a given set of fuzzy numbers \(\bar{a}_j = (m_j, \alpha_j, \beta_j)_{LR}; j = 1, \ldots, n\) with common shape functions \((L \text{ and } R)\) and for some positive scalars \(\lambda_j; j = 1, \ldots, n\) we have

\[
\lambda_1 \bar{a}_1 \oplus \lambda_2 \bar{a}_2 \oplus \ldots \oplus \lambda_n \bar{a}_n = \sum_{j=1}^{n} \lambda_j \bar{a}_j = (\sum_{j=1}^{n} \lambda_j m_j, \sum_{j=1}^{n} \lambda_j \alpha_j, \sum_{j=1}^{n} \lambda_j \beta_j)_{LR}.
\] (4.7)

and hence

\[
\left[\sum_{j=1}^{n} \lambda_j \bar{a}_j\right]_\alpha = \left[\sum_{j=1}^{n} \lambda_j (m_j - L^{-1}(\alpha)) \alpha_j, \sum_{j=1}^{n} \lambda_j (m_j + R^{-1}(\alpha)) \beta_j\right]_\alpha.
\] (4.8)

Let us assume that we have \(n\) decision making units \(DMU_j (j = 1, \ldots, n)\) such that for \(t = 1, 2, \ldots, T\), in the \(t\)th period \(DMU_j\) uses two different groups of inputs \(\bar{k}_{t-1,j}\) (\(l\)-vector of positive quasi-fixed fuzzy inputs) and \(\bar{x}_{t,j}\) (\(m\)-vector of positive variable fuzzy inputs) to produce two different types of outputs \(\bar{y}_{t,j}\) (\(s\)-vector of positive fuzzy goods) and \(\bar{k}_{t,j}\). Define \(\bar{X}_t = [x_{t,1}, \ldots, x_{t,n}], \bar{K}_t = [k_{t,1}, \ldots, k_{t,n}],\) and \(\bar{K}_t = [y_{t,1}, \ldots, y_{t,n}]\) as \(m \times n, l \times n\), and \(s \times n\) matrices related to the inputs, quasi-fixed inputs, and outputs, respectively. By these considerations, Model (2.1) to evaluate \(DMU_o\) can be extended to be the following fuzzy dynamic DEA model.

\[
\theta_{\mathbb{P}} = \min_{\lambda_1, \ldots, \lambda_n} \sum_{t=1}^{T} \dot{\bar{a}}_t \lambda_t + v_{1,p} \bar{K}_0
\]

s.t. \(\bar{K}_0 \lambda_1 \leq \bar{K}_0\),

\[
\dot{\bar{Y}}_t \lambda_t \geq \bar{Y}_{t-1,p}, \quad t = 1, 2, \ldots, T
\]

\[
e \lambda_t = 1, \quad t = 1, 2, \ldots, T
\]

\[
\bar{K}_t \lambda_t - \bar{K}_t \lambda_{t-1} \geq 0, \quad t = 1, 2, \ldots, T - 1
\]

\[
\lambda_t \geq 0, \quad t = 1, 2, \ldots, T,
\] (4.9)

in which

\[
\dot{\bar{a}}_1 = w_{1,p} \bar{x}_1,
\] (4.10)

\[
\dot{\bar{a}}_t = w_{1,p} \bar{x}_t \oplus v_{1,p} \bar{K}_{t-1}, \quad t = 2, 3, \ldots, T,
\] (4.11)

are obtained using the above mentioned T-norm. Along the lines of [5,10,11], we consider the input-output levels as trapezoidal fuzzy numbers, as follows:

\[
\bar{k}_{t,j} = (k_{t,j,1}^1, k_{t,j,1}^2, k_{t,j,2}^1, k_{t,j,2}^2)
\]

\[
\bar{x}_{t,j} = (x_{t,j,1}^1, x_{t,j,1}^2, x_{t,j,2}^1, x_{t,j,2}^2)
\]

\[
\bar{y}_{t,j} = (y_{t,j,1}^1, y_{t,j,1}^2, y_{t,j,2}^1, y_{t,j,2}^2)
\]

\[
\bar{y}_{t,j} = (y_{t,j,1}^1, y_{t,j,1}^2, y_{t,j,2}^1, y_{t,j,2}^2).
\]
Hence $a_l$ and $\tilde{k}_0$ are trapezoidal fuzzy numbers too. Assuming that $a_l = (a_l^1, a_l^2, \delta_l^1, \delta_l^2)$, $\tilde{k}_0 = (\tilde{k}_0^1, \tilde{k}_0^2, \tilde{\beta}_0^1, \tilde{\beta}_0^2)$. Also note that $R(x) = L(x) = 1 - x$.

**Theorem 4.1.** $\tilde{0}$ is a distance based lower bound of the objective function of model (4.9) in the sense of [11].

**Proof.** Using (3.6)-(4.8), we have

\[
d(\sum_{t=1}^{T} \tilde{a}_t \lambda_t + v_1, p, \tilde{k}_0, 0) = (\sum_{t=1}^{T} (\tilde{a}_t^1 + \tilde{a}_t^2) \lambda_t + v_1, p, (\tilde{k}_0^1 + \tilde{k}_0^2)) \int_0^1 S(\alpha) d\alpha
\]

\[
+ (v_1, p, \tilde{\beta}_0^2 + \sum_{t=1}^{T} \delta_t^2 \lambda_t - v_1, p, \tilde{\beta}_0^1 - \sum_{t=1}^{T} \delta_t^1 \lambda_t) \int_0^1 S(\alpha)L^{-1}(\alpha) d\alpha.
\]

Since $\tilde{X}_t$ and $\tilde{K}_t$ are positive fuzzy vectors and $w_{t,p}$ and $v_{t,p}$ are positive crisp vectors, $\tilde{a}_t$ is a positive fuzzy vector. Also, $\lambda_t$ is a positive vector, hence $\sum_{t=1}^{T} \tilde{a}_t \lambda_t + v_1, p, \tilde{k}_0$ is a positive fuzzy number. Therefore

\[
|\sum_{t=1}^{T} \tilde{a}_t \lambda_t + v_1, p, \tilde{k}_0|_{L^1(\alpha)} > 0, \quad \forall \alpha \in [0, 1]
\]

and

\[
|\sum_{t=1}^{T} \tilde{a}_t \lambda_t + v_1, p, \tilde{k}_0|_{U^1(\alpha)} > 0, \quad \forall \alpha \in [0, 1].
\]

Also $S(\alpha)$ is a nonnegative function, hence

\[
d(\sum_{t=1}^{T} \tilde{a}_t \lambda_t + v_1, p, \tilde{k}_0, 0) = \int_0^1 S(\alpha)(|\sum_{t=1}^{T} \tilde{a}_t \lambda_t + v_1, p, \tilde{k}_0|_{L^1(\alpha)} + |\sum_{t=1}^{T} \tilde{a}_t \lambda_t + v_1, p, \tilde{k}_0|_{U^1(\alpha)}) d\alpha \geq 0.
\]

This implies that $\tilde{0}$ is a distance based lower bound of the objective function of model (4.9) in the sense of [11]. \qed

Regarding the above theorem we can replace the objective function of model (4.9) by

\[
\min(\sum_{t=1}^{T} (\tilde{a}_t^1 + \tilde{a}_t^2) \lambda_t + v_1, p, (\tilde{k}_0^1 + \tilde{k}_0^2)) \int_0^1 S(\alpha) d\alpha
\]

\[
-(v_1, p, \tilde{\beta}_0^2 + \sum_{t=1}^{T} \delta_t^2 \lambda_t - v_1, p, \tilde{\beta}_0^1 - \sum_{t=1}^{T} \delta_t^1 \lambda_t) \int_0^1 S(\alpha)L^{-1}(\alpha) d\alpha.
\]

See [10,11] for more details. Due to the above discussion and using (3.6)-(4.8), model (4.9) can be converted to the
Now defining $\pi^1 = \int_0^1 S(\alpha)d\alpha$, and $\pi^2 = \int_0^1 S(\alpha)L^{-1}(\alpha)d\alpha$, the above model is equivalent to the following LP:

\[
\begin{align*}
\text{min} & \quad \left(\sum_{t=1}^T (a_t^1 + a_t^2)\lambda_t + v_{1,p}(\tilde{k}_0^1 + \tilde{k}_0^2)\right)\pi^1 + (v_{1,p}\tilde{b}_0^2 + \sum_{t=1}^T \delta_t^2 \lambda_t - v_{1,p}\tilde{b}_0^1 - \sum_{t=1}^T \delta_t^1 \lambda_t)\pi^2 \\
\text{s.t.} & \quad (\tilde{k}_0^1 + \tilde{k}_0^2 - \sum_{j=1}^n \lambda_{t_j}K_{t_j}^1 - \sum_{j=1}^n \lambda_{t_j}K_{t_j}^2)\int_0^1 S(\alpha)d\alpha \\
& \quad + (\sum_{j=1}^n \lambda_{t_j}\beta_{j_0}^1 - \sum_{j=1}^n \lambda_{t_j}\beta_{j_0}^2 + \tilde{b}_0^2 - \tilde{b}_0^1)\int_0^1 S(\alpha)L^{-1}(\alpha)d\alpha \geq 0, \\
& \quad (\sum_{j=1}^n \lambda_{t_j}\gamma_{j_0}^1 + \sum_{j=1}^n \lambda_{t_j}\gamma_{j_0}^2 - \gamma_{t_0}^1 - \gamma_{t_0}^2)\int_0^1 S(\alpha)d\alpha \\
& \quad + (\sum_{j=1}^n (\lambda_{t_j} - \lambda_{t_0+1})(\beta_{j_0}^1 - \beta_{j_0}^2))\int_0^1 S(\alpha)L^{-1}(\alpha)d\alpha \geq 0, \quad t = 1, 2, \ldots, T, \\
& \quad \sum_{j=1}^n \lambda_{t_j} = 1, \quad t = 1, 2, \ldots, T, \\
& \quad \sum_{j=1}^n (\lambda_{t_j} - \lambda_{t_0+1})(K_{t_j}^1 + K_{t_j}^2)\int_0^1 S(\alpha)d\alpha \\
& \quad + (\sum_{j=1}^n (\lambda_{t_j} - \lambda_{t_0+1})(\beta_{j_0}^1 - \beta_{j_0}^2))\int_0^1 S(\alpha)L^{-1}(\alpha)d\alpha \geq 0, \quad t = 1, 2, \ldots, T - 1, \\
& \quad \lambda_t \geq 0, \quad t = 1, 2, \ldots, T.
\end{align*}
\]

Now defining

\[\pi^1 = \int_0^1 S(\alpha)d\alpha, \quad \text{and} \quad \pi^2 = \int_0^1 S(\alpha)L^{-1}(\alpha)d\alpha,\]

This model is an LP and can be solved by current codes for solving LPs, e.g. Simplex method and interior point approaches. It is evident that the provided approach can be used to convert all fuzzy DEA models (CCR, BCC, SBM, FDH, RAM, etc.) to crisp ones which have optimal values without any preconditions.
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