Nth-order Fuzzy Differential Equations Under Generalized Differentiability

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Abstract
In this paper, the multiple solutions of Nth-order fuzzy differential equations by the equivalent integral forms are considered. Also, an Existence and uniqueness theorem of solution of Nth-order fuzzy differential equations is proved under Nth-order generalized differentiability in Banach space.

Keywords: Nth-order fuzzy differential equations; Nth-order generalized differentiability; Fuzzy-valued functions; Hukuhara difference.

1 Introduction
Fuzzy set theory is a powerful tool for modeling of uncertainty and for processing of vague or subjective information in mathematical models, whose main direction of development have been diversified and applied in many varied real problems. Therefore, for such mathematical modeling, using fuzzy differential equations are necessary. Important element of fuzzy differential equation is fuzzy derivative. The concept of the fuzzy derivative was first introduced by Chang and Zadeh [12]. It was followed up by many researchers [1, 16, 17, 18, 24]. Bede et. al in [8] proposed new concept of derivative based on the Hukuhara difference which so-called generalized Hukuhara differentiability.

Consequently, in [6], the solutions of second-order fuzzy initial value problems were investigated. In [10], nth-order fuzzy linear differential equation is discussed and in [15], Nth-order fuzzy differential equation is investigated generally.

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Naturally, investigation of existence and uniqueness of solution of fuzzy initial value problems is important. In [15], the existence and uniqueness of solution of fuzzy initial value problems in higher order by using classical form of Hukuhara differentiability has been discussed. Also, in [6], the existence and uniqueness of solution to second-order fuzzy differential equations was discussed by applying Nth-order generalized Hukuhara differentiability, which, we have been used such definition for our investigations. Also, some papers have been published about solving fuzzy differential equations in [2, 3, 4, 5].

Recently, Khastan et. al [19], investigated the multiple solution of Nth-order fuzzy differential equations under generalized differentiability. Also, Nieto et. al [23] obtained solution of first order fuzzy differential equations using generalized differentiability by interpreting the original fuzzy differential equations with two crisp ordinary differential equations. Also, an even higher order of generalization towards metric dynamical systems has been published [20, 21, 22].

In this paper, we are going to study solutions of Nth-order initial value problem

\[ x^{(n)}(t) = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)), \quad x(t_0) = k_1, \ldots, x^{(n-1)}(t_0) = k_n \]  

where the function \( f : [t_0, T] \times E \times \cdots \times E \rightarrow E \) is a fuzzy process with fuzzy initial values \( x^{(i-1)} = k_i, i = 1, \ldots, n \).

The structure of paper is as follows: In Section 2 the basic concepts is described. In Section 3, the uniqueness and existence of solution for Nth-order fuzzy differential equations under Nth-order strongly differentiability are studied. Finally, conclusion will be drawn in Section 4.

2 Preliminaries

A non-empty subset \( A \) of \( R \) is called convex if and only if \((1 - k)x + ky \in A\) for every \( x, y \in A \) and \( k \in [0, 1] \).

There are various definitions for the concept of fuzzy numbers ([13, 14])

**Definition 2.1.** A fuzzy number is a function such as \( u : R \rightarrow [0, 1] \) satisfying the following properties:

\( (i) \) \( u \) is normal, i.e. \( \exists x_0 \in R \) with \( u(x_0) = 1 \),

\( (ii) \) \( u \) is a convex fuzzy set i.e. \( u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\} \forall x, y \in R, \lambda \in [0, 1] \),

\( (iii) \) \( u \) is upper semi-continuous on \( R \),

\( (iv) \) \( \{x \in R : u(x) > 0\} \) is compact, where \( \overline{A} \) denotes the closure of \( A \).

The set of all fuzzy real numbers is denoted by \( E \). Obviously \( R \subseteq E \). Here \( R \subseteq E \) is understood as \( R = \{\chi_x : x \text{ is usual real number}\} \). For \( 0 < r \leq 1 \), it is denoted that
\([u]_r = \{x \in R; u(x) \geq r\}\) and \([u]_0 = \{x \in R; u(x) > 0\}\). Then it is well-known that for any \(r \in [0,1]\), \([u]_r\) is a bounded closed interval. For \(u, v \in E\), and \(\lambda \in R\), where sum \(u + v\) and the product \(\lambda u\) are defined by \([u + v]_r = [u]_r + [v]_r\), \([\lambda u]_r = \lambda [u]_r, \forall r \in [0,1]\), where \([u]_r + [v]_r = \{x + y : x \in [u]_r, y \in [v]_r\}\) means the conventional addition of two intervals (subsets) of \(R\) and \([\lambda u]_r = \{\lambda x : x \in [u]_r\}\) means the conventional product between a scalar and a subset of \(R\)(see e.g. [13, 25]).

Also the presentation of fuzzy number in parametric form, is as follows:

**Definition 2.2.** [16]. An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions \((u(r), \overline{u}(r))\), \(0 \leq r \leq 1\), which satisfy the following requirements:

1. \(u(r)\) is a bounded left-continuous non-decreasing function over \([0,1]\).
2. \(\overline{u}(r)\) is a bounded left-continuous non-increasing function over \([0,1]\).
3. \(u(r) \leq \overline{u}(r), 0 \leq r \leq 1\).

A crisp number \(a\) is simply represented by \(u(r) = \overline{u}(r) = a, 0 \leq r \leq 1\). We recall that for \(a < b < c, a, b, c \in R\), the triangular fuzzy number \(u = (a, b, c)\) determined by \(a, b, c\) is given such that \(u(r) = a + (b - a)r\) and \(\overline{u}(r) = c - (c - b)r\) are the endpoints of the \(r\)-level sets, for all \(r \in [0,1]\). Here \(u(1) = \overline{u}(1) = b\) is denoted by \([u]_1\). For arbitrary \(u = (u(r), \overline{u}(r)), v = (\overline{u}(r), \overline{v}(r))\) we define addition and multiplication by \(k\) as

1. \((u + v)(r) = (u(r) + v(r)),\)
2. \((u + v)(r) = (\overline{u}(r) + \overline{v}(r)),\)
3. \((ku)(r) = ku(r), (\overline{u})(r) = k\overline{u}(r), k \geq 0,\)
4. \((ku)(r) = k\overline{u}(r), (ku)(r) = ku(r), k < 0.\)

In this paper, following [6], an arbitrary fuzzy number with compact support is represented by a pair of functions \((u(r), \overline{u}(r)), 0 \leq r \leq 1\). Also, we use the Hausdorff distance between fuzzy numbers. This fuzzy number space, as shown in [8], can be embedded into Banach space \(B = \overline{r}[0,1] \times \overline{r}[0,1]\) where the metric is usually defined as follows: Let \(E\) be the set of all upper semicontinuous normal convex fuzzy numbers with bounded \(r\)-level sets. Since the \(r\)-levels of fuzzy numbers are always closed and bounded, the intervals are written as \(u[r] = [u(r), \overline{u}(r)]\), for all \(r\). We denote by \(\omega\) the set of all non-empty compact subsets of \(R\) and by \(\omega\), the subsets of \(\omega\) consisting of non-empty convex compact sets. Recall that

\[\rho(x, A) = \min_{a \in A} \|x - a\|\]

is the distance of a point \(x \in R\) from \(A \in \omega\) and the Hausdorff separation \(\rho(A, B)\) of \(A, B \in \omega\) is defined as

\[\rho(A, B) = \max_{a \in A} \rho(a, B).\]
Note that the notation is consistent, since \( \rho(a, B) = \rho(\{a\}, B) \). Now, \( \rho \) is not a metric. In fact, \( \rho(A, B) = 0 \) if and only if \( A \subseteq B \). The Hausdorff metric \( d_H \) on \( \omega \) is defined by

\[
d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.
\]

The metric \( d_\infty \) is defined on \( E \) as

\[
d_\infty(u, v) = \sup\{d_H(u[r], v[r]) : 0 \leq r \leq 1\}, \quad u, v \in E.
\]

for arbitrary \( (u, v) \in E \times E \). The following properties are well-known. (see e.g. \[14, 25\])

(i) \( d_\infty(u + w, v + w) = d_\infty(u, v) \), \( \forall u, v, w \in E \),

(ii) \( d_\infty(ku, kv) = |k|d_\infty(u, v) \), \( \forall k \in R, u, v \in E \),

(iii) \( d_\infty(u + v, w + e) \leq d_\infty(u, w) + d_\infty(v, e) \), \( \forall u, v, w, e \in E \).

Theorem 2.1. \[7\].

(i) If we define \( \tilde{0} = \chi_0 \), then \( \tilde{0} \in E \) is a neutral element with respect to addition, i.e. \( u + \tilde{0} = 0 + u = u \), for all \( u \in E \).

(ii) With respect to \( \tilde{0} \), none of \( u \in E \setminus R \), has opposite in \( E \).

(iii) For any \( a, b \in R \) with \( a, b \geq 0 \) or \( a, b \leq 0 \) and any \( u \in E \), we have \( (a+b)u = a.u + b.u \); however, this relation does not necessarily hold for any \( a, b \in R \), in general.

(iv) For any \( \lambda \in R \) and any \( u, v \in E \), we have \( \lambda(u + v) = \lambda.u + \lambda.v \);

(v) For any \( \lambda, \mu \in R \) and any \( u \in E \), we have \( \lambda.(\mu.u) = (\lambda.\mu).u \). (see \[25\])

Remark 2.1. \( d_\infty(u, \tilde{0}) = d_\infty(\tilde{0}, u) = \| u \| \).

Definition 2.3. \[25\]. Let \( \tilde{f}(x) \) be a fuzzy valued function on \( [a, b] \). Suppose that \( f(x, r) \) and \( \tilde{f}(x, r) \) are improper Riemann-integrable on \( [a, b] \) then we say that \( f(x) \) is improper on \( [a, b] \), furthermore,

\[
\left( \int_a^b f(x, r) \, dx \right) = \int_a^b f(x, r) \, dx,
\]

\[
\int_a^b \tilde{f}(x, r) \, dx = \int_a^b \tilde{f}(x, r) \, dx.
\]

Note that the continuity of function \( f : [t_0, T] \times E \rightarrow E \) provide the integrability of \( f \), and for \( f, g \) integrable function, \( d(f, g) \) is integrable and \( d_\infty(\int f, \int g) \leq \int d_\infty(f, g) \).

Definition 2.4. Consider \( x, y \in E \). If there exists \( z \in E \) such that \( x = y + z \), then \( z \) is called the \( H \)-difference of \( x \) and \( y \) and it is denoted by \( x \ominus y \).

In this paper, the sign \( \ominus \) always stands for \( H \)-difference and note that \( x \ominus y \neq x + (-y) \). Also throughout the paper is assumed that Hukuhara difference and strongly Hukuhara differentiability are exist. Let us recall the definition of strongly generalized differentiability introduced in \[8\].
**Definition 2.5.** [9]. Let \( f : (a, b) \to E \) and \( x_0 \in (a, b) \). We say that \( f \) is strongly generalized differentiable at \( x_0 \), if there exists an element \( f'(x_0) \in E \), such that

(i) for all \( h > 0 \) sufficiently small, \( \exists f(x_0 + h) \ominus f(x_0), \exists f(x_0) \ominus f(x_0 - h) \) and the limits (in the metric \( d_{\infty} \)):

\[
\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)
\]

or

(ii) for all \( h > 0 \) sufficiently small, \( \exists f(x_0) \ominus f(x_0 + h), \exists f(x_0 - h) \ominus f(x_0) \) and the limits (in the metric \( d_{\infty} \)):

\[
\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)
\]

or

(iii) for all \( h > 0 \) sufficiently small, \( \exists f(x_0 + h) \ominus f(x_0), \exists f(x_0 - h) \ominus f(x_0) \) and the limits (in the metric \( d_{\infty} \)):

\[
\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)
\]

or

(iv) for all \( h > 0 \) sufficiently small, \( \exists f(x_0) \ominus f(x_0 + h), \exists f(x_0) \ominus f(x_0 - h) \) and the limits (in the metric \( d_{\infty} \)):

\[
\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)
\]

(\( h \) and \( -h \) at denominators mean \( \frac{1}{h} \) and \( \frac{1}{-h} \), respectively)

**Proposition 2.1.** [13]. If \( f : (a, b) \to E \) is a continuous fuzzy valued function then \( g(x) = \int_a^x f(t)dt \) is differentiable with derivative \( g'(x) = f(x) \).

**Theorem 2.2.** [11]. Let \( f : R \to E \) be a function and denote \( f(t) = (\underline{f}(t, r), \overline{f}(t, r)) \), for each \( r \in [0, 1] \). Then

(1) If \( f \) is differentiable in the first form (i), then \( \underline{f}(t, r) \) and \( \overline{f}(t, r) \) are differentiable functions and

\[ f'(t) = (\underline{f}'(t, r), \overline{f}'(t, r)). \]

(2) If \( f \) is differentiable in the second form (ii), then \( \overline{f}(t, r) \) and \( \underline{f}(t, r) \) are differentiable functions and

\[ f'(t) = (\overline{f}'(t, r), \underline{f}'(t, r)). \]
3 Nth-order fuzzy differential equation

We define a Nth-order fuzzy differentiable equation by

\[ x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t)) \]

where \( x(t) = ((\bar{x}(t), \tilde{x}(t), r(t), \ell(t))) \) is a fuzzy function of \( t \). \( f(t, x(t), \ldots, x^{(n-1)}(t)) \) is a fuzzy \(-\)-valued function. If initial values \( x(t_0) = k_1, x'(t_0) = k_2, \ldots, x^{(n-1)}(t_0) = k_n \) are given, we obtain a fuzzy Cauchy problem of the Nth-order

\[ x^{(n)}(t) = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)), \quad x(t_0) = k_1, \ldots, x^{(n-1)}(t_0) = k_n. \tag{3.2} \]

In this section, the uniqueness and existence of solutions of Nth-order fuzzy differential equation are studied. To this end, Nth-order strongly generalized differentiability is proposed as follow:

**Definition 3.1.** [6]. Let \( f : (t_0, T) \times E \times \ldots \times E \rightarrow E \) and \( x_0 \in (t_0, T) \). We define the nth-order differential of \( f \) as follow: Let \( f : (t_0, T) \rightarrow E \) and \( x_0 \in (t_0, T) \). We say that \( f \) is strongly generalized differentiable of the nth-order at \( x_0 \). If there exists an element \( f^{(s)}(x_0) \in E, \forall s = 1, \ldots, n \), such that

(i) for all \( h > 0 \) sufficiently small, \( \exists f^{(s-1)}(x_0+h) \odot f^{(s-1)}(x_0), \exists f^{(s-1)}(x_0) \odot f^{(s-1)}(x_0-h) \) and the limits (in the metric \( d_\infty \))

\[ \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0+h) \odot f^{(s-1)}(x_0)}{h} = \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0) \odot f^{(s-1)}(x_0-h)}{h} = f^{(s)}(x_0) \]

or

(ii) for all \( h > 0 \) sufficiently small, \( \exists f^{(s-1)}(x_0) \odot f^{(s-1)}(x_0+h), \exists f^{(s-1)}(x_0-h) \odot f^{(s-1)}(x_0) \) and the limits (in the metric \( d_\infty \))

\[ \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0) \odot f^{(s-1)}(x_0+h)}{-h} = \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0-h) \odot f(x_0)}{-h} = f^{(s)}(x_0) \]

or

(iii) for all \( h > 0 \) sufficiently small, \( \exists f^{(s-1)}(x_0+h) \odot f^{(s-1)}(x_0), \exists f^{(s-1)}(x_0-h) \odot f^{(s-1)}(x_0) \) and the limits (in the metric \( d_\infty \))

\[ \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0+h) \odot f^{(s-1)}(x_0)}{h} = \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0-h) \odot f^{(s-1)}(x_0)}{-h} = f^{(s)}(x_0) \]

or

(iv) for all \( h > 0 \) sufficiently small, \( \exists f^{(s-1)}(x_0) \odot f^{(s-1)}(x_0+h), \exists f^{(s-1)}(x_0) \odot f^{(s-1)}(x_0-h) \) and the limits (in the metric \( d_\infty \))

\[ \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0) \odot f^{(s-1)}(x_0+h)}{-h} = \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0) \odot f^{(s-1)}(x_0-h)}{h} = f^{(s)}(x_0), \]

(\( h \) and \(-h\) at denominators mean \( \frac{1}{h} \) and \(-\frac{1}{h}\), respectively \( \forall s = 1 \ldots n \))
We denote the space of continuous functions \( x : I = [t_0, T] \to E \) by \( C(I, E) \). \( C(I, E) \) is a complete metric space with the distance

\[
H(x, y) = \sup_{t \in I} \{d_{\infty}(x(t), y(t)) e^{-\rho t}\},
\]

where \( \rho \in R \) is fixed \cite{15}. Also, by \( C^n(I, E) \), we denote the set of continuous functions \( x : I \to E \) whose \( x^{(i)} : I \to E, \ i = 1, \ldots, n \) exist as continuous functions. For \( x, y \in C^n(I, E) \), consider the following distance:

\[
H_n(x, y) = H(x, y) + H(x', y') + \ldots + H(x^{(n)}, y^{(n)}).
\]

**Lemma 3.1.** \cite{15}. \( (C^n(I, E), H_n) \) is a complete metric space.

**Remark 3.1.** Please notice that, one can easily prove the complete metric space of \( H_n \) under generalized differentiability.

**Theorem 3.1.** Let \( f : [t_0, T] \times E \times E \times \ldots \times E \to E \) be continuous. Consider the initial value problem (3.2). A mapping \( x : [t_0, T] \to E \) is a solution of (3.2) if and only if the function \( x \in C^n(I, E) \) satisfies the following integral equation for all \( z_1 \in [t_0, T] : \)

\[
x(z_1) = k_1 + sign(x)(k_2(z_1 - t_0)) + sign(x')(k_4 \int_{t_0}^{z_1} (z_2 - t_0) dz_2
+ sign(x'')(k_4 \int_{t_0}^{z_1} (z_3 - t_0) dz_3 dz_2 + sign(x''')k_5 \int_{t_0}^{z_1} \int_{t_0}^{z_2} (z_4 - t_0) dz_4 dz_3 dz_2
+ \ldots + sign(x^{(n-2)})(k_n \int_{t_0}^{z_1} \ldots \int_{t_0}^{z_{n-2}} (z_{n-1} - t_0) dz_{n-1} \ldots dz_2
+ sign(x^{(n-1)}) \int_{t_0}^{z_1} \ldots \int_{t_0}^{z_n} f(s, x(s), \ldots, x^{(n-1)}(s)) ds dz_n \ldots dz_2 \ldots
\]

where for \( j = 0, \ldots, n-1 \) is defined:

\[
\min(x^{(j)}) = \begin{cases} +1, & \text{if } x^{(j)} \text{ is } i - \text{differentiable} \\ \ominus(-1), & \text{if } x^{(j)} \text{ is } ii - \text{differentiable} \end{cases} \quad (3.3)
\]

**Proof.** For \( z_n \in [t_0, T] \), we have

\[
x^{(n-1)}(z_n) = k_n + sign(x^{(n-1)}) \int_{t_0}^{z_n} f(s, x(s), \ldots, x^{(n-1)}(s)) ds
\]

and for \( z_{n-1} \in [t_0, T] \),

\[
x^{(n-2)}(z_{n-1}) = k_{n-1} + sign(x^{(n-2)})(k_n(z_{n-1} - t_0) + sign(x^{(n-1)}) \int_{t_0}^{z_{n-1}} \int_{t_0}^{z_n} f(s, x(s), \ldots, x^{(n-1)}(s)) ds dz_n).
\]

By recurrence, for \( z_2 \in [t_0, T] \), we get

\[
x'(z_2) = k_2 + sign(x)(k_3(z_2 - t_0)) + sign(x')(k_4 \int_{t_0}^{z_2} (z_3 - t_0) dz_3 + \ldots
+ \ldots sign(x^{(n-2)})(k_n \int_{t_0}^{z_2} \ldots \int_{t_0}^{z_{n-2}} (z_{n-1} - t_0) dz_{n-1} \ldots dz_3
+ sign(x^{(n-1)}) \int_{t_0}^{z_2} \ldots \int_{t_0}^{z_n} f(s, x(s), \ldots, x^{(n-1)}(s)) ds dz_n \ldots dz_2 \ldots).
\]
The expression is obtained integrating last equality from $t_0$ to $z_1 \in [t_0, T]$ with respect to $z_2$.

Consequently, if all of derivatives $x, x', \ldots, x^{(n-1)}$ are (i)-differentiable or (ii)-differentiable then the solution $x(z_1)$ is converted to

$$x(z_1) = k_1 + k_2(z_1 - t_0) + k_3 \int_{t_0}^{z_1} (z_2 - t_0)dz_2 + k_4 \int_{t_0}^{z_1} \int_{t_0}^{z_2} (z_3 - t_0)dz_3dz_2$$

$$+ k_5 \int_{t_0}^{z_1} \int_{t_0}^{z_2} \int_{t_0}^{z_3} (z_4 - t_0)dz_4dz_3dz_2 + \ldots + k_n \int_{t_0}^{z_1} \ldots \int_{t_0}^{z_{n-2}} (z_{n-1} - t_0)dz_{n-1} \ldots dz_2$$

$$+ \int_{t_0}^{z_1} \ldots \int_{t_0}^{z_n} f(s, x(s), \ldots, x^{(n-1)}(s))dsdz_n \ldots dz_2,$$

or

$$x(z_1) = k_1 \ominus (-1)(k_2(z_1 - t_0) \ominus (-1)(k_3 \int_{t_0}^{z_1} (z_2 - t_0)dz_2$$

$$\ominus (-1)(k_4 \int_{t_0}^{z_1} \int_{t_0}^{z_2} (z_3 - t_0)dz_3dz_2 \ominus (-1)(k_5 \int_{t_0}^{z_1} \int_{t_0}^{z_2} \int_{t_0}^{z_3} (z_4 - t_0)dz_4dz_3dz_2$$

$$\ominus (-1) \ldots \ominus (-1)(k_n \int_{t_0}^{z_1} \ldots \int_{t_0}^{z_{n-2}} (z_{n-1} - t_0)dz_{n-1} \ldots dz_2$$

$$\ominus (-1) \int_{t_0}^{z_1} \ldots \int_{t_0}^{z_n} f(s, x(s), \ldots, x^{(n-1)}(s))dsdz_n \ldots dz_2)\ldots),$$

respectively.

Before stating the main result, the following lemma which is needed in the proof of theorem is brought. The proof is completely similar to the case $n = 2$ proposed in [6].

**Lemma 3.2.** Let $(u_1, \ldots, u_{n-1}, u_n), (u_1, \ldots, u_{n-1}, v_n) \in E^n$ then,

$$d_\infty(u_1 \ominus (u_2 \ominus \ldots \ominus (u_{n-1} \ominus u_n) \ldots), (u_1 \ominus (u_2 \ominus \ldots \ominus (u_{n-1} \ominus v_n) \ldots) = d_\infty(u_n, v_n),$$

for all $u_i, v_i \in E, \ i = 1, \ldots, n$.

We introduce the following functions defined on $[t_0, T]$, which is applying later as follows:

$$\Theta_{p,0}(z_n) = \int_{t_0}^{z_n} e^{\rho s} ds = \frac{e^{\rho z_n} - e^{\rho t_0}}{\rho},$$

$$\Theta_{p,1}(z_{n-1}) = \int_{t_0}^{z_{n-1}} \int_{t_0}^{z_n} e^{\rho s} dsdz_n = \frac{e^{\rho z_{n-1}} - e^{\rho t_0} - \rho(z_{n-1} - t_0)e^{\rho t_0}}{\rho}$$

and, for $n - 1 \geq j \geq 2$,

$$\Theta_{p,j}(z_{n-j}) = \int_{t_0}^{z_{n-j}} \int_{t_0}^{z_{n-j+1}} \ldots \int_{t_0}^{z_n} e^{\rho s} dsdz_n \ldots dz_{n-j+1}$$

Now, we prove main theorem of this paper under (i)-differentiability and (ii)-differentiability (two important cases for such differentiability).
Theorem 3.2. Let $f : [t_0, T] \times E \times E \times \ldots \times E \to E$ be continuous, and suppose that there exist $M_1, M_2, \ldots, M_n > 0$ such that

$$d_\infty(f(t, x_1, x_2, \ldots, x_n), f(t, y_1, y_2, \ldots, y_n)) \leq \sum_{i=1}^{n} M_i d_\infty(x_i, y_i)$$

(3.4)

for all $t \in [t_0, T]$, $x_i, y_i \in E$, $i = 1, \ldots, n$. Then the initial value problem (3.2) has a unique solution on $[t_0, T]$ in each sense of differentiability.

Proof. Let $I = [t_0, T]$, consider the complete metric space $(C^n(I, E), H_n)$, and define the operator

$$G_{n-1} : C^{n-1}(I, E) \to C^{n-1}(I, E)$$

$$x \to G_{n-1}x$$

Consider operator $G_{n-1}$, given by the right-hand side in the integral expression obtained in Theorem (3.1) as follows:

$$(G_{n-1}x)(z_1) = k_1 + \text{sign}(x)(k_2(z_1 - t_0) + \text{sign}(x')(k_3 \int_{t_0}^{z_1} (z_2 - t_0) dz_2

+ \ldots + \text{sign}(x^{(n-2)})(k_n \int_{t_0}^{z_1} \ldots \int_{t_0}^{z_{n-1}} (z_{n-1} - t_0) dz_{n-1} \ldots dz_2

+ \text{sign}(x^{(n-1)}) \int_{t_0}^{z_1} \ldots \int_{t_0}^{z_n} f(s, x(s), \ldots, x^{(n-1)}(s)) ds dz_n \ldots dz_2 \ldots).$$

We prove that $G_{n-1}$ is a contraction mapping, for an appropriate $\rho > 0$. Indeed, choose $\rho > 0$ such that

$$\max\{M_1, \ldots, M_n\} \sum_{j=0}^{n-1} \frac{1}{j!}(T - t_0)^j \frac{1 - e^{-\rho(T-t_0)}}{\rho} < 1$$

(3.5)

which is possible since the terms in the finite sum have limited equal to zero as $\rho$ tends to infinite. Also, notice that $G_{n-1}x$ and $G_{n-1}y$ are operators which are applied on the some $x, y$ such that $\text{sign}(x^{(i-1)}) = \text{sign}(y^{(i-1)})$, $i = 1, \ldots, n$. Therefore, we have

$$H_{n-1}(G_{n-1}x, G_{n-1}y) = \sum_{i=0}^{n-1} H((G_{n-1}x)^{(i)}), (G_{n-1}y)^{(i)})$$

$$= \sup_{z_1 \in I} \{d_\infty(k_1 + \text{sign}(x)(k_2(z_1 - t_0) + \text{sign}(x')(k_3 \int_{t_0}^{z_1} (z_2 - t_0) dz_2

+ \text{sign}(x^{(n-2)})(k_n \int_{t_0}^{z_1} \ldots \int_{t_0}^{z_{n-1}} (z_{n-1} - t_0) dz_{n-1} \ldots dz_2

+ \text{sign}(x^{(n-1)}) \int_{t_0}^{z_1} \ldots \int_{t_0}^{z_n} f(s, x(s), \ldots, x^{(n-1)}(s)) ds dz_n \ldots dz_2 \ldots),

k_1 + \text{sign}(x)(k_2(z_1 - t_0) + \text{sign}(x')(k_3 \int_{t_0}^{z_1} (z_2 - t_0) dz_2

+ \text{sign}(x^{(n-2)})(k_n \int_{t_0}^{z_1} \ldots \int_{t_0}^{z_{n-1}} (z_{n-1} - t_0) dz_{n-1} \ldots dz_2

+ \text{sign}(x^{(n-1)}) \int_{t_0}^{z_1} \ldots \int_{t_0}^{z_n} f(s, x(s), \ldots, x^{(n-1)}(s)) ds dz_n \ldots dz_2 \ldots),$$
\[
\begin{align*}
+ \text{sign}(x'')(k_4 \int_0^{x_1} \int_0^{x_2} (z_3 - t_0)dz_3dz_2) \\
+ \text{sign}(x'')(k_5 \int_0^{x_1} \int_0^{x_2} \int_0^{x_3} (z_4 - t_0)dz_4dz_3dz_2) \\
+ \ldots + \text{sign}(x^{(n-2)})(k_6 \int_0^{x_1} \ldots \int_0^{x_{n-1}} (z_{n-1} - t_0)dz_{n-1} \ldots dz_2) \\
+ \text{sign}(x^{(n-1)}) \int_0^{x_1} \ldots \int_0^{x_n} f(s, y(s), \ldots, y^{(n-1)}(s))dsdz_n \ldots dz_2) e^{-\rho z_1} \\
= \sup_{z_2 \in I} \{ d_\infty (k_2 + \text{sign}(x'))(k_3(z_2 - t_0) + \text{sign}(x'')(k_4 \int_0^{z_3} (z_3 - t_0)dz_3) \\
+ \text{sign}(x'')(k_5 \int_0^{z_2} \int_0^{z_3} (z_4 - t_0)dz_4dz_3 + \ldots \\
+ \text{sign}(x^{(n-1)}(\int_0^{z_2} \ldots \int_0^{z_{n-1}} f(s, x(s), \ldots, x^{(n-1)}(s))dsdz_n \ldots dz_3) e^{-\rho z_2}) + \ldots \\
+ \sup_{z_n \in I} \{ d_\infty (\int_0^{z_n} f(s, x(s), \ldots, x^{(n-1)}(s))ds, \int_0^{z_n} f(s, y(s), \ldots, y^{(n-1)}(s))ds)e^{-\rho z_n} \\
= \ldots \}
\end{align*}
\]
Consider the following Nth-order fuzzy differential equation

Example 3.1. Also, for the case of H-differentiability, Theorem 3.2 coincide with the result in [15].

Then, for every continuous function $F$ on $[a, b]$,

$$
\int_a^x dx_n \int_a^x dx_{n-1} \cdots \int_a^x dx_1 F(x_1) dx_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} F(t) dt.
$$

Then, for the following expression involving $j + 1$ integrals, we get, for $z_{n-j} \in I$

$$
\Theta_{p,j}(z_{n-j}) = \int_{t_0}^{z_{n-j+1}} \int_{t_0}^{z_{n-j}} \cdots \int_{t_0}^{z_{n-j}} e^{p s} dsdz_n \cdots dz_{n-j+1}
= \frac{1}{j!} \int_{t_0}^{z_{n-j}} (z_{n-j} - s)^j e^{p s} ds, \quad j = 0, \ldots, n - 1.
$$

Then, for $j = 0, 1, \ldots, n - 1$, we have

$$
\sup_{z_{n-j} \in I} \{ \Theta_{p,j}(z_{n-j}) e^{-p z_{n-j}} \} = \frac{1}{j!} \sup_{z_{n-j} \in I} \{ \int_{t_0}^{z_{n-j}} (z_{n-j} - s)^j e^{p s} ds e^{-p z_{n-j}} \}
\leq \frac{1}{j!} \sup_{z_{n-j} \in I} \{ \int_{t_0}^{z_{n-j}} (T - t_0)^j e^{p s} ds e^{-p z_{n-j}} \}
= \frac{1}{j!} \sup_{z_{n-j} \in I} \{ (T - t_0)^j \frac{1 - e^{-p(T - t_0)}}{\rho} \} \rightarrow 0, \quad \text{if } \rho \rightarrow \infty
$$

and

$$
H_{n-1}(G_{n-1}x, G_{n-1}y) \leq \max \{ M_1, \ldots, M_n \} H_{n-1}(x, y) \sum_{j=0}^{n-1} (T - t_0)^j \frac{1 - e^{-p(T - t_0)}}{\rho^j}.
$$

The choice of $\rho$ provides that $G_{n-1}$ is a contraction, therefore, there exists a unique solution for problem (2).

Note that the unique fixed point of $G$ is in the space $C(I, E)$.

Also, for the case of H-differentiability, Theorem 3.2 coincide with the result in [15].

**Example 3.1.** Consider the following Nth-order fuzzy differential equation

$$
x^{(n)}(t) = f(t, x(t), x'(t), x''(t), \ldots, x^{(n-1)}(t))
$$
with initial conditions \(x(0) = k_1, x'(0) = k_2, x''(0) = k_3, \ldots, x^{(n-1)}(0) = k_n\) which are fuzzy numbers. Then, we assumed that the fuzzy-valued function \(f\) is a bounded fuzzy-valued function such that \(d_\infty(f, 0) \leq M_{n+1}\), where \(M_{n+1}\) is a positive real number, using Theorem (3.2), leads to obtain the bound of solution as following:

\[
d_\infty(x(z_1), \tilde{0}) \leq d_\infty(k_1, \tilde{0}) + d_\infty(k_2(z_1), \tilde{0}) + \ldots + d_\infty\left(\int_0^{z_1} \int_0^{z_2} \ldots \int_0^{z_n} f(s, x(s), \ldots, x^{(n-1)}(s)) \, ds \, dz_n \ldots \, dz_2, \tilde{0}\right)
\]

\[
\leq M_1 + M_2 + \ldots + M_n + \prod_{j=1}^{n} z_i M_{n+1},
\]

where \(M_i\) is the bound of \(i\)th term on the above right hand side inequality for all \(i = 1, 2, \ldots, n\) and \(z_1 \in [0, T]\).

### 4 Conclusion

In this paper, we considered the solutions of Nth-order fuzzy differential equations under generalized differentiability. However, we proposed the integral form of Nth-order fuzzy differential equations. This form of solution is useful to investigate the bound of solutions. Moreover, one can use it for obtaining the approximate solution and also constructing some operator methods to solve original problem [2].

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