A note on “ranking efficient DMUs based on a single virtual inefficient DMU in DEA”

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Abstract
In this paper we give a comment on paper by Shetty and Pakkala [U. Shetty, T.P.M. Pakkala, (2010), Ranking efficient DMUs based on single virtual inefficient DMU in DEA. OPSEARCH, 47 (1):50-72]. They proposed an approach to rank the efficient decision making units based on a virtual DMU in the constant return to scale DEA model. The input and output levels of this virtual DMU are the average of input and outputs of all DMUs. We obtain another method for select virtual DMU as negative ideal DMU. Our approach doesn’t need to existence at least an inefficient DMU in the set DMUs. In addition, it can be used in constant and variable return to scale DEA models. This brief comment provides an alternative approach for their work.

Keywords: Data envelopment analysis; Decision making units; Negative Ideal DMU.

1 Introduction

DEA is commonly used to measure the relative efficiency of comparable decision making units (DMUs) that employ common inputs to produce a common set of outputs. Data envelopment analysis (DEA) was originally developed to measure the relative efficiency of peer decision making units (DMUs) in multiple input-multiple output settings (Charnes et al. [1], 1978).
DEA identifies an efficient frontier where all DMUs have a unity score. Ranking DEA efficient units has become the interests of many DEA researchers and a variety of models (called super-efficiency models) were developed to rank DEA efficient units. In 1985, Charnes et al. [2], counted the number of times that an efficient DMU play the role of benchmark unit for others, and used this norm to rank these units. In 1986, Sexton et al. [3] developed the cross-evaluation Matrix and this approach was extended by Doyle and Green [4], both of which are referred to as the cross efficiency ranking methods. Andersen and Petersen [5] modify the CCR model to allow for a ranking of the efficient units. A procedure using DEA to rank DMUs of interests is proposed by combining multivariate statistic approaches such as discriminant analysis of ratios [6] and canonical correlation analysis [7]. Wang and Luo [8] proposed a ranking approach by incorporating DEA into the technique for order preference by similarity to ideal solution (TOPSIS, 1981). The idea of the ideal point is also adopted to develop the ‘‘worst practice DEA’’ [9], where the evaluation of each DMU relative to the NIP treats inputs as outputs and outputs as inputs. Before and after 2006, also many methods were proposed for ranking. In 2010, Du et al. [10] proposed a new DEA-based method for fully ranking all decision-making units. In the same year, Shetty and Pakalla [11] proposed a method for ranking efficient DMUs based on a single virtual inefficient DMU in DEA. In 2012, Xu and Dan [12] introduced two alternative efficiency measures by using efficient and anti-efficient frontiers in DEA and proposed a new ranking system for all DMUs. In the same year, Rezai Balf et al. [13] proposed a method for ranking extreme efficient decision making units. Their method used \( L_{\infty} \) (or Tchebycheff) Norm. In 2013, Chen et al. [14] proposed a super-efficiency based on a modified directional distance function. In this journal, Shetty and Pakkala [11] proposed an approach to rank the efficient decision making units based on a virtual DMU. They used of a virtual DMU for ranking Efficient DMUs. The input and output levels of virtual DMU are the average of inputs and outputs of all DMUs. We obtain another method for select virtual DMU as negative ideal DMU. Our approach doesn’t need to existence at least an inefficient DMU in the set DMUs. In addition, it can be used in constant and variable return to scale DEA models.

2 Shetty and Pakkala method

The initial problem is usually expressed as: \( n \) DMUs to be assessed with \( m \) inputs and \( s \) outputs indices. For each DMU, say \( DMU_j, \quad j = 1,\ldots,n \), the given values of indices are denoted as \((x_{ij}, x_{2j},\ldots,x_{mj})\) and \((y_{1j}, y_{2j},\ldots,y_{sj})\), respectively. Shetty and Pakkala [11] defined a virtual DMU and calculated the efficiency of the virtual DMU by deleting inefficient DMUs one by one. The virtual DMU and Their proposed model given as follows, respectively

\[
(x_{Av}, y_{Av}) = \left( \frac{1}{n} \sum_{j=1}^{n} x_{ij}, \frac{1}{n} \sum_{j=1}^{n} y_{ij} \right), \quad i = 1,\ldots,m; \quad r = 1,\ldots,s.
\]

and
\[ \delta_{Av,b} = \max \sum_{r=1}^{s} u_r y_{Av} \]

s.t.
\[ \sum_{i=1}^{m} v_i x_{Av} = 1, \]
\[ \sum_{r=1}^{s} u_r y_{Av} - \sum_{i=1}^{m} v_i x_{Av} \leq 0, \]
\[ \sum_{r=1}^{s} u_r y_{ij} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j \in J - b \]
\[ u_i \geq \varepsilon, v_i \geq \varepsilon, \quad \forall i, r. \]

Where \( J = \{1, ..., n\} \) is the set of DMUs, \( b \in E \) where \( E \) is set of CRS efficient DMUs.

### 3 Alternative approach for ranking efficient DMUs

We assume that the set of decision making units (DMUs) are given. For ranking efficient DMUs, the virtual DMU \((x_{Av}, y_{Av})\) is replaced by the negative ideal point (NIP), defined as

\[
\text{NIP} = (x_{NIP}, y_{NIP}) = \left( \max_{1 \leq j \leq n} (x_{j}), \min_{1 \leq j \leq n} (y_{j}) \right), \quad i = 1, ..., m; \quad r = 1, ..., s.
\]

This definition means that a NIP is also a virtual DMU consuming the most inputs but produces the least outputs. Therefore, Virtual DMU is denoted by \((x_{Av}, y_{Av}) = \text{NIP} = (x_{NIP}, y_{NIP})\).

**Definition 3.1.** Suppose that \(DMU_A(x_A, y_A)\) and \(DMU_B(x_B, y_B)\) be two units. Then we say \(DMU_A\) dominates \(DMU_B\) if and only if \(x_A \leq x_B\) and \(y_A \geq y_B\) and strictly inequality holds for at least one component.

A note according to Definition 3.1 is given as follows:

**Note.** If \(x_A \leq x_B\) and \(y_A \geq y_B\) and strictly inequality holds for all component. Then, \(DMU_B\) will be inefficient in CCR and BCC models. But if it hold for some component (not all), then \(DMU_B\) is also inefficient in CCR and weak efficient in BCC.

**Theorem 3.1.** A NIP is always inefficient in CCR model and inefficient or weak efficient in BCC model.

**Proof.** Suppose \(DMU_j(x_j, y_j), \quad j \in J = \{1, ..., n\}\) be \(n\) distinguish units. According to (3.3) we have

\[
x_{NIP} = \max_{1 \leq j \leq n} (x_{j}) \geq x_{j}, \quad i = 1, ..., m
\]
\[
y_{NIP} = \min_{1 \leq j \leq n} (y_{j}) \leq y_{j}, \quad r = 1, ..., s
\]

Then there exists at least a \(k \in J\) such that \(x_{NIP} \geq x_{ik}\), \(y_{NIP} \leq y_{ik}\) and strictly inequality hold for at least component. Otherwise all \(n\) units are same, that is it is a contradiction. Therefore \(DMU_A\) dominates NIP. Hence, NIP is inefficient in CCR model and inefficient or weak efficient in BCC model. Also it is interesting and trivial that we derive virtual DMU \((x_{Av}, y_{Av})\) in (2.1) dominates NIP in (3.3). Because, \(\forall i, \forall r\) we have
x_{NIP} = \max_{1 \leq j \leq n} (x_{ij}) \geq x_A \Rightarrow x_{NIP} = \sum_j x_{NIP} \geq x_A \Rightarrow x_{NIP} \geq \sum_j x_{NIP} \geq n x_{NIP} = \sum_j x_A \Rightarrow x_{NIP} \geq \frac{1}{n} \sum_j x_A

y_{NIP} = \min_{1 \leq j \leq n} (y_{ij}) \leq y_A \Rightarrow y_{NIP} = \sum_j y_{NIP} \leq y_A \Rightarrow y_{NIP} \leq \sum_j y_{NIP} \leq n y_{NIP} = \sum_j y_A \Rightarrow y_{NIP} \leq \frac{1}{n} \sum_j y_A

Thereby the virtual DMU \((x_{Av}, y_{Av})\) dominates NIP.

However, we can use idea and model (2.2) proposed by Shetty et al. [11] for ranking efficient DMUs to respect to NIP.

4 Conclusions

In this paper we focused on article Shetty and Pakkala. They proposed a method for ranking efficient DMUs based on single virtual inefficient DMU in DEA. Their approach for making virtual inefficient DMU is related to presence an inefficient DMU in production possibility set (PPS). Otherwise, it is possible that virtual inefficient DMU made by them, be efficient. However, we suggested another approach for assess virtual inefficient DMU as NIP. In this case, we made a NIP (virtual inefficient DMU) without presence any inefficient DMU.

References

https://doi.org/10.1016/0377-2217(78)90138-8

https://doi.org/10.1007/BF01874734

https://doi.org/10.1002/ev.1441

https://doi.org/10.1057/jors.1994.84

https://doi.org/10.1287/mnsc.39.10.1261

https://doi.org/10.1016/S0377-2217(97)00313-5
https://doi.org/10.1016/S0377-2217(97)84108-2

https://doi.org/10.1016/j.amc.2005.04.023

https://doi.org/10.1023/B:PROD.0000016870.47060.0b

https://doi.org/10.1111/j.1468-0394.2010.00553.x

https://doi.org/10.1007/s12597-010-0004-3


https://doi.org/10.1016/j.apm.2010.11.077

https://doi.org/10.1016/j.omega.2012.06.006