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Research Article

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Returns To Scale In PL-DEA models

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Abstract

In this paper we consider piecewise linear data envelopment analysis (PL-DEA) model which incorporates piecewise linear functions of factors. Then, we expand the Piecewise linear (PL) DEA models and define the PL-BCC, PL-BCC-CCR, PL-CCR-BCC models and we discuss the RTS methods for the DMUs in the presently available types of PL-DEA models. Also, the mentioned methods are compared and numerical example of these methods is provided for illustration.

Keywords: Date envelopment analysis,Returns to scale,Marginal value, Piecewise linear function.

1 Introduction

Date envelopment analysis (DEA) is a non parametric technique for measuring and evaluating the relative efficiencies of decision making unit(DMU) with multiple inputs and multiple outputs. The economic concept of returns to scale (RTS) have been studied widely nowadays. Among diverse models of DEA, radial models are the most well known ones which include CCR and BCC each of which implies a radial model under constant and variable returns to scale. First contributions on characterizing the RTS were made by Fare and Grosskopf [1] and Fare et al.[2]. Under constant returns to scale technology different approaches for estimating type of the RTS are proposed note that they can determine the type of RTS but not its magnitude [3], [4]. Under variable returns to scale the RTS type can be determined by the sign of the u_o^*. Banker et al.[7] proposed method for estimating the non radial models, additive and multiplicative. Banker and Thrall [5] gave conditions for identifying the RTS for BCC model which is under variable returns to scale. Banker et al.[3] developed a number of modifications of the models by determining the maximum and minimum values of u_o^*, which enables their method to determine the RTS without surveying all the alternative multiple solutions. It was showed that possible alternative optimal solutions only affected the estimation of the RTS on DMUs which should be classified as constant returns to scale. Färe et al.[2] proposed a method for evaluating the RTS of a DMU for which three Lp problems must be solved. The significant advantage of their method as compared to other method is that there is no need for exploring alternative optimal solutions. Also in Kerstens et al.[10] method, three LP problems must be solved that seems to be onerous and in regard of computational point of view is problematic.
In many situations, differences in values may not be reflected adequately by linear pricing. We deal with these situations where for certain outputs in input-oriented model, either a nonincreasing or nondecreasing set of multipliers

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for larger magnitude of the factors describes the weight function. Cook and zhu examined such a situation[8], they verified that on the earlier work of Cook et al.[9] certain factors which were treated as behaving linearly, should be deemed as having a nonlinear impact on the efficiency. Lotfi et al. [6] proposed a modified PL-DEA model in order to determine the best possible target DMUs.
In this paper we propose methods for estimating the issue of RTS on DMUs which have decreasing set of multipliers. This paper unfolds as follows: in section two we give a review of piecewise linear DEA and in section three we introduce new PL-DEA models. In section four we investigate for the RTS of DMUs which contains DMV variables. The last two sections provides numerical example and conclusion.

2 piecewise linear DEA background

Let us assume that we have a set of DMUs consisting of $\text{DMU}_j,\ j=1,\dots,n$. Every DMU$j$ used m imputs to produce s outputs. All inputs and outputs are assumed to be nonnegative, but at least one input and one output are positive $x_j=(x_{1j},\dots,x_{mj})$, $x_j\neq 0$ and $y_j=(y_{1j},\dots,y_{sj})$, $y_j\neq 0$. In presence of those variables, which have nonlinear impact on efficiency, k segments are considered for the scale of the variable from the theory of piecewise linear programming. With this logic, the scale of variable should view as consisting of $k_r$ranges $[0,L_1]$, $[L_1,L_2]$, …, $[L_{k_r-1},L_{k_r}]$. let μ_{r_k} be the value which is given to the portion of y_{rj} that lies in the k^{th} range. If $y_{rj}\in[L_{k_j-1},L_{k_j}]$ then the parameters y^k_{rj} are defined as follows:

$$y^k_{rj}=\begin{cases}L_k, & \text{if} \quad k=1\\ L_k-L_{k-1}, & \text{if} \quad k=2,\dots,k_j-1\\ y_{rj}-L_{k_j-1}, & \text{if} \quad k=k_j\\ 0, & \text{if} \quad k>k_j\end{cases} \tag{2.1}$$

Then in stead of a single expression $\mu_r y_{rj}$, $\sum_{k=1}^{K_r}\mu_{r_k}y^k_{rj}$ is used. It is necessary that $\{\mu_{r_k}\}_{k=1}^{K}$ should form a decreasing sequence so the constraint (2a) is imposed.
It is noteworthy that a_{r_k} and b_{r_k} would take on values$>1(<1)$ for those variables have increasing and decreasing set of multipliers. The multiplier form of CCR model which contains variables with decreasing set of multipliers is as follows:

$$\begin{aligned}
Z_1=\max \quad & \sum_{r\in R_1}u_r y_{ro}+\sum_{r\in R_2}\sum_{k=1}^{K_r}u_{r_k}y^k_{rj}-\sum_{r\in R_2}\sum_{k=1}^{K_r}\mu_{r_k}\bar{y}^k_{rj}\\
s.t. \quad & \sum_{i\in I}v_i x_{io}=1\\
& \sum_{r\in R_1}u_r y_{rj}+\sum_{r\in R_2}\sum_{k=1}^{K_r}u_{r_k}y^k_{rj}-\sum_{i\in I}v_i x_{ij\leq 0} \qquad j\in J\\
& u_{r_{k+1}}a_{r_k}\leq u_{r_k}\leq u_{r_{k+1}}b_{r_k}, \qquad k=1,...,K_r, r\in R_2\\
& u_r\geq 0,\quad \mu_{r_k}\geq 0,\quad v\geq 0,\\
& u_{r_k}\geq 0,\quad J\in\{1,...,n\}
\end{aligned} \tag{2.2}$$

where:

$$\bar{y}^k_{rj}=\begin{cases}0, & \text{if} \quad k=1\\ 0, & \text{if} \quad k=2,\dots,k_j-1\\ L_k-y_{rj}, & \text{if} \quad k=k_j\\ L_k-L_{k_j-1}, & \text{if} \quad k>k_j\end{cases} \tag{2.3}$$

R_1 and R_2, respectively, are used to denote the sets of regular and DMV/IMV outputs. k_r represents number of intervals $[L_{k_j-1},L_{k_j}]$.

3 New PL-DEA models

Let us assume that we have a set of n DMUs consisting of $\mathrm{DMU}_j, j = 1,\ldots,n$ with input_output vector (x_j, y_j) in which $x_j = (x_{1j},\ldots,x_{mj})$ and $y_j = (y_{1j},\ldots,y_{sj})$.
With the presence of the variable with decreasing set of multipliers we introduce another PL-DEA model by adding the unrestricted u_o variable to the PL-CCR model, which we shall designate this new model as a multiplier form of PL-BCC model in the input orientation:

$$\begin{aligned} Z_2 = \max \quad & \sum_{r\in R_1} u_r y_{ro} + \sum_{r\in R_2}\sum_{k=1}^{K_r} u_{r_k} y_{rj}^k - \sum_{r\in R_2}\sum_{k=1}^{K_r} \mu_{r_k} \bar{y}_{rj}^k + u_0 \\ s.t. \quad & \sum_{i\in I} v_i x_{io} = 1 \\ & \sum_{r\in R_1} u_r y_{rj} + \sum_{r\in R_2}\sum_{k=1}^{K_r} u_{r_k} y_{rj}^k - \sum_{i\in I} v_i x_{ij+u_0\leq 0} \qquad \forall j\in J \\ & u_{r_{k+1}} a_{r_k} \leq u_{r_k} \leq u_{r_{k+1}} b_{r_k} \quad k=1,\ldots,K_r, r\in R_2 \\ & u_r \geq 0, \quad u_{r_k}\geq 0, \quad v\geq 0,, \quad j=\{1,..,n\} \\ & u_o \text{ unrestricted}, \quad \mu_{r_k}\geq 0. \end{aligned} \tag{3.4}$$

Theorem 3.1. *In optimal solution of model (3.4)* $\mu_{r_k} = 0, k = 1,\ldots,k_r$.

Proof. Let $(u_r^*, v_i^*, \mu_{r_k}^*, u_o)$ be a optimal solution for model (3.4) Where $\bar{y}_{rj}^k \geq 0$. By contradiction suppose that $\mu_{r_k}^* > 0$ thus we can have a feasible solution such as:

$$(u_r = u_r^*, u_{r_k} = u_{r_k}^*, \mu_{r_k} = 0, u_o).$$

The objective function of this feasible solution is greater than that of optimal one. Therefore in optimal solution $\mu_{r_k}^* = 0$. □

The following model is a multiplier form of PL-CCR-BCC model in the input-orientation:

$$\begin{aligned} Z_3 = \max \quad & \sum_{r\in R_1} u_r y_{ro} + \sum_{r\in R_2}\sum_{k=1}^{K_r} u_{r_k} y_{rj}^k - \sum_{r\in R_2}\sum_{k=1}^{K_r} \mu_{r_k} \bar{y}_{rj}^k - u_0 \\ s.t. \quad & \sum_{i\in I} v_i x_{io} = 1 \\ & \sum_{r\in R_1} \mu_r y_{rj} + \sum_{r\in R_2}\sum_{k=1}^{K_r} \mu_{r_k} y_{rj}^k - \sum_{i\in I} v_i x_{ij-u_0\leq 0} \qquad \forall j\in J \\ & u_{r_{k+1}} a_{r_k} \leq \mu_{r_k} \leq \mu_{r_{k+1}} b_{r_k} \qquad K=1,\ldots,K_r, r\in R_2 \\ & u_r \geq 0, \quad u_{r_k}\geq 0, \quad v\geq 0, \quad j=\{1,..,n\} \\ & u_o \geq 0, \quad \mu_{r_k}\geq 0. \end{aligned} \tag{3.5}$$

It is noteworthy that the above theorem holds true here, as well. By our previous explanations we introduce the multiplier form of PL-BCC-CCR model as follows:

$$
\begin{aligned}
Z_4 = \max \quad & \sum_{r\in R_1} u_r y_{ro} + \sum_{r\in R_2}\sum_{k=1}^{K_r} u_{r_k} y_{rj}^k - \sum_{r\in R_2}\sum_{k=1}^{K_r} \mu_{r_k} \bar{y}_{rj}^k + u_0 \\
s.t. \quad & \sum_{i\in I} v_i x_{io} = 1 \\
& \sum_{r\in R_1} u_r y_{rj} + \sum_{r\in R_2}\sum_{k=1}^{K_r} u_{r_k} y_{rj}^k - \sum_{i\in I} v_i x_{ij} + u_0 \leq 0, \qquad j \in J \\
& u_{r_{k+1}} a_{r_k} \leq \mu_{r_k} \leq \mu_{r_{k+1}} b_{r_k} \quad k = 1,...,K_r, r \in R_2 \\
& u_{r_1} a_{r_1 r_2} y_{r_2 j} \leq \sum_{k=1}^{K_{r_2}} \mu_{r_{2k}} y_{r_2 j}^K \leq \mu_{r_1} b_{r_1 r_2} y_{r_2 j}, \quad j \in J, r_1 \in R_1, r_2 \in R_2 \\
& u_r \geq 0, \quad u_{r_k} \geq 0, \quad v \geq 0, J = \{1,..,n\} \\
& u_o \geq 0, \quad \mu_{r_k} \geq 0.
\end{aligned}
\tag{3.6}
$$

the mentioned details and theorem hold true in this model as well. Here particular weight restrictions are imposed and they do not cause incorrect assessment of the relative efficiency since the only weight restrictions are imposed on output weights, and by imposing such homogeneous weight restrictions the following models do not suffer from being infeasible [11], [12]. And, Since the weight restrictions are imposed on output multipliers the efficiency score will be greater than zero.

4 PL-Returns to scale

Let us assume that we have a set of DMUs consisting of $\text{DMU}_j,\ j = 1,\ldots,n$. Every DMU$j$ produces s outputs y_{rj} and uses m inputs x_{ij}. All inputs and outputs are assumed to be nonnegative, but at least one input and one output are positive $x_j = (x_{1j},\ldots,x_{mj})$, $x_j \neq 0$ and $y_j = (y_{1j},\ldots,y_{sj})$, $y_j \neq 0$.

We use the objective function of the PL-CCR, PL-BCC, PL-BCC-CCR, PL-CCR-BCC for recognizing the RTS status for a DMU under assessment.

Definition 4.1. *Let us assume that the DMUO under assessment is PL-BCC efficient, first solve the PL-CCR model, if DMUo is PL-CCR efficient then CRS will prevail on DMUo else i.e.* $\theta^*_{PL-CCR} \neq 1$*, solve the PL-BCC-CCR model to see whether DMUo is PL-BCC-CCR efficient or not, if* $\theta^*_{PL-BCC-CCR} = 1$ *then IRS will prevail on DMUo, else if DMUo is PL-CCR-BCC efficient then DTRS will prevail on DMUo.*

Consider the multiplier form of PL-BCC model in input orientation, from the sign of u_o^* we characterize the RTS status as follow:

Theorem 4.1. *a) If* $u_0^* = 0$ *in any of the alternative optimal solutions then CRS prevail on DMUo.*

b) If $u_0^* > 0$ *for all alternative optimal solutions then IRS prevail on DMUo.*

c)If $u_0^* < 0$ *for all alternative optimal solution then DRS prevail on DMUo.*

Proof. If $u_0^* = 0$ since we have $\theta^*_{PL-BCC} = 1$ hence $\theta^*_{PL-BCC} = \theta^*_{PL-CCR}$ so CRS prevail on DMUo. If DMUo exhibits CRS so we have $\theta^*_{PL-BCC} = \theta^*_{PL-CCR} = 1$ and it implies that $u_0^* \neq 0$ can not hold true for any optimal solutions in PL-BCC model thus we get $u_0^* = 0$ and the proof (a) is completed. If $u_0^* > 0$ since we have $\theta^*_{PL-BCC} = 1$ therefore $\theta^*_{PL-BCC} \neq \theta^*_{PL-CCR}$ and $\theta^*_{PL-BCC} = \theta^*_{PL-BCC-CCR} = 1$ thus IRS prevail on DMUo. If DMUo exhibits IRS since we $\theta^*_{PL-BCC-CCR} = 1$ and also we have $\theta^*_{PL-BCC} = 1$ and $\theta^*_{PL-CCR} \neq 1$ and these two imply that $u_0^* = 0$ can not hold true for all optimal solutions of PL-BCC model so by comparing the objective functions of PL-BCC and PL-BCC-CCR, $u_0^* > 0$ is satisfied, and (b) is completed. If $u_0^* < 0$ since we have $\theta^*_{PL-BCC} = 1$ therefore $\theta^*_{PL-BCC} \neq \theta^*_{PL-CCR}$ and $\theta^*_{PL-BCC} = \theta^*_{PL-CCR-BCC} = 1$ thus DRS prevail on DMUo. If DMUo exhibits DRS since we $\theta^*_{PL-CCR-BCC} = 1$ and also we have $\theta^*_{PL-BCC} = 1$ and $\theta^*_{PL-CCR} \neq 1$ and these two imply that $u_0^* = 0$ can not hold true for all optimal solutions of PL-BCC model so, $u_0^* < 0$ is satisfied, and (c) is completed. □

The following corollary shows that for estimating the RTS there is no need to solve the above four models.

Theorem 4.2. *If DMUo is PL-BCC efficient:*
i) IRS prevail on DMUo if and only if

$$\theta^*_{PL-BCC-CCR} \geq \theta^*_{PL-CCR-BCC}.$$

ii) DRS prevail on DMUo if and only if

$$\theta^*_{PL-BCC-CCR} \leq \theta^*_{PL-CCR-BCC}.$$

iii) CRS prevail on DMUo if and only if

$$\theta^*_{PL-BCC-CCR} = \theta^*_{PL-CCR-BCC}.$$

So, from theorem 4.1 with help of the sign of u^*_o the RTS can be characterized. Estimating the RTS with this method involves generating all optimal solutions and it is computationally problematic. Another method for estimating the RTS in a easier way is to solve model(3.4)to see weather there exists alternative optimal solutions or not. In the case of having alternatives, instead of solving models(3.5)and(3.6) the following model can be solved, while we are given the existence of optimal solution $u^*_o > 0$.

$$\begin{array}{ll}
\min & \hat{u}_o \\
s.t. & \sum_{i\in I} v_i x_{io} = 1 \\
& \sum_{r\in R_1} \mu_r y_{rj} + \sum_{r\in R_2}\sum_{k=1}^{K_r} \mu_{r_k} y^k_{rj} - \sum_{i\in I} v_i x_{ij} + \hat{u}_o \leq 0 \quad j\in J \\
& \mu_{r_{k+1}} a_{r_k} \leq \mu_{r_k} \leq \mu_{r_{k+1}} b_{r_k}, \quad k=1,\dots,K_r, r\in R_2 \\
& \sum_{r\in R_1} \mu_r y_{ro} + \sum_{r\in R_2}\sum_{k=1}^{K_r} \mu_{r_k} y^k_{ro} + \hat{u}_o = Z_2, \\
& \mu_{r_1} a_{r_1 r_2} y_{r_2 j} \leq \sum_{k=1}^{K_{r_2}} \mu_{r_{2k}} y^K_{r_2 j} \leq \mu_{r_1} b_{r_1 r_2} y_{r_2 j} \quad j\in J, r_1\in R_1, r_2\in R_2 \\
& \mu_r \geq 0, \quad \mu_{r_k} \geq 0, \quad v \geq 0, \quad \hat{u}_o \geq 0, \quad j\in\{1,..,n\}
\end{array} \tag{4.7}$$

The third equation helps us to make sure that we are confined to the efficiency frontier.

Theorem 4.3. *Given the existence of the optimal solution u^*_o from PL-BCC model:*
i)$\hat{u}^*_o = 0$ if and only if CRS prevail on DMUo.
ii)$\hat{u}^*_o > 0$ if and only if IRS prevail on DMUo.

Proof. Having the optimal solution of model(4.7)we have

$$\sum_{r\in R_1} \mu^*_r y_{ro} + \sum_{r\in R_2}\sum_{k=1}^{K_r} \mu^*_{r_k} y^k_{ro} + \hat{u}^*_o = 1$$

$$\sum_{i\in I} v^*_i x_{io} = 1$$

so we have

$$\sum_{r\in R_1} \mu^*_r y_{ro} + \sum_{r\in R_2}\sum_{k=1}^{K_r} \mu^*_{r_k} y^k_{ro} + \hat{u}^*_o = \sum_{i\in I} v^*_i x_{io}$$

$$\sum_{r\in R_1} \mu^*_r y_{ro} + \sum_{r\in R_2}\sum_{k=1}^{K_r} \mu^*_{r_k} y^k_{ro} = \sum_{i\in I} v^*_i x_{io} - \hat{u}^*_o.$$

$$\sum_{r\in R_1}\mu_r^* y_{ro}+\sum_{r\in R_2}\sum_{k=1}^{K_r}\mu_{r_k}^* y_{ro}^k-\sum_{i\in I}\nu_i^* x_{io}+\hat{u}_o^*=0$$

If $\hat{u}_o^*=0$ then the first condition of theorem 4.1 is satisfied. If $\hat{u}_o^*>0$ then the RTS at DMUo is increasing as it is set forth in the second condition of theorem 4.1. □

As you have seen with this method examining all alternative optimal solutions has been avoided. Note that when we are given $u_o^*\leq 0$ as an optimal solution of the PL−BCC model, we alter the objective of model(4.7) to maximization and also $\hat{u}_o\leq 0$ is replaced with $\hat{u}_o\geq 0$ thus we get the following result;

a) The CTS of DMUo are CRS if and only if $\hat{u}_o^*=0$.
b) The DTS of DMUo are DRS if and only if $\hat{u}_o^*<0$.

It is noteworthy to say that as you have seen it is possible to estimate the RTS on each DMU without having to utilize the projection while given the existence of optimal solution.
Although with the above method exploring all optimal solutions has been avoided, this method involves solving two models.
We will show that for estimating the RTS there is no need to solve these two models.
By solving the following model

$$\begin{aligned}
u_o^*=\min\quad & u_o\\
s.t.\quad & \sum_{i\in I}\nu_i x_{io}=1\\
& \sum_{r\in R_1}\mu_r y_{rj}+\sum_{r\in R_2}\sum_{k=1}^{K_r}\mu_{r_k} y_{rj}^k-\sum_{i\in I}\nu_i x_{ij}+u_0\leq 0\quad j\in J\\
& \sum_{r\in R_1}\mu_r y_{ro}+\sum_{r\in R_2}\sum_{k=1}^{K_r}\mu_{r_k} y_{ro}^k+u_0=1\\
& \mu_{r_{k+1}}a_{r_k}\leq\mu_{r_k}\leq\mu_{r_{k+1}}b_{r_k}\quad k=1,\dots,K_r,r\in R_2\\
& \mu_{r_1}a_{r_1r_2}y_{r_2j}\leq\sum_{k=1}^{k_{r_2}}\mu_{r2_k}y_{r_2j}^k\leq\mu_{r_1}b_{r_1r_2}y_{r_2j}\quad j\in J,r_1\in R_1,r_2\in R_2\\
& \mu_r\geq 0,\quad \mu_{r_k}\geq 0,\quad \nu\geq 0,\quad u_o\geq 0,\quad j\in\{1,..,n\}
\end{aligned}\tag{4.8}$$

we can say:

Theorem 4.4. *a) If $u_0^*>0$ then IRS prevail on DMUo.*
b) If $u_0^*=0$ then CRS prevail on DMUo.
c) If model is infeasible then $u_0^*<0$ and DRS prevail DMUo.

5 An application of PL-DEA models for estimating the RTS

In this section we consider twenty DMUs with four outputs and three inputs. Among these outputs, two outputs behave in nonlinear manner.In standard model and PL-DEA models we impose on the multipliers μ_1, μ_2, $\mu_3 and \mu_4$, the AR constraints

$$3\mu_2\leq\mu_1\quad and\quad 3\mu_3\leq\mu_2$$

and

$$5\mu_1\leq\mu_4\leq\mu_1.$$

Also in piecewise models to invoke the requirement that each of the third and the fourth outputs should form a nonincreasing set of multipliers, we impose on the multipliers μ_{41}, μ_{42}, μ_{43} and μ_{31}, μ_{32}, μ_{33}, respectively the constraints

$$4\mu_{32}\leq\mu_{31}\leq 8\mu_{32}\quad and\quad 4\mu_{33}\leq\mu_{32}\leq 8\mu_{33}$$
$$4\mu_{42}\leq\mu_{41}\leq 8\mu_{42}\quad and\quad 4\mu_{43}\leq\mu_{42}\leq 8\mu_{43}.$$

Here we take $k_r = 3$, and we use the ranges $[0,200)$, $[200,300)$, $[300,500]$ for the third output and $[0,800)$, $[800,900)$, $[900,1000]$ for the fourth output. By using pl-CCR-BCC and pl-BCC-CCR models the RTS of the following units can be carachterized. The PL-BCC, PL-BCC-CCR and PL-CCR-BCC scores are displayed respectively in columns of Table 2. And the estimated RTS is displayed in the last column of Table 2. In the following the data are displayed in Table 1.

Table 1: Data (Inputs and Outputs)

DMU number	o1	o2	o3	o4	I1	I2	I3
1	85	450	210	850	685	284	715
2	72	400	295	821	510	245	525
3	69	425	385	800	785	425	680
4	41	417	370	750	545	380	660
5	73	490	261	690	688	325	665
6	102	520	351	782	796	322	604
7	120	385	422	823	436	388	712
8	98	376	299	924	167	413	668
9	95	296	256	826	556	325	678
10	90	455	250	427	335	312	677
11	27	421	270	1000	799	248	715
12	65	419	286	950	212	275	525
13	92	410	292	975	565	425	690
14	81	491	321	890	825	390	670
15	125	521	480	765	208	305	665
16	110	372	190	496	666	342	604
17	107	350	195	852	446	378	722
18	99	292	180	793	527	433	678
19	77	500	215	842	336	365	688
20	86	320	340	953	745	322	678

Table 2: Results

DMUnumber	PL-BCC	PL-BCC-CCR	PL-CCR-BCC	RTS
1	0.86	0.86	0.7	-
2	1	1	1	CRS
3	0.98	0.79	0.98	-
4	0.98	0.81	0.98	-
5	0.78	0.78	0.77	-
6	0.94	0.87	0.94	-
7	1	0.76	1	DRS
8	1	1	1	CRS
9	0.77	0.77	0.76	-
10	0.85	0.85	0.84	-
11	0.98	0.98	0.97	-
12	1	1	1	CRS
13	0.76	0.76	0.76	-
14	0.88	0.79	0.88	-
15	0.98	0.98	0.97	-
16	0.86	0.86	0.64	-
17	0.72	0.72	0.55	-
18	0.77	0.77	0.54	-
19	0.76	0.76	0.64	-
20	0.98	0.78	0.98	-

As you see four units are PL-BCC efficient, and by comparing PL-BCC-CCR and PL-CCR-BCC scored the RTS of units 2, 7, 8 and 12 respectively CRS, DRS, CRS and CRS. The following table contains the RTS of the mentioned units, which are estimated with model (4.8).

Table 3: RTS status

DMUnumber	2	7	8	12
RTS	CRS	DRS	CRS	CRS

6 Conclusion

The current study gives a method for characterizing the RTS in presence of variables which behave in nonlinear manner. To capture this idea special weight restrictions are imposed to the model. By adding such weight restrictions the status of returns to scale may change. Considering this discussion we introduced methods for estimating the RTS in presence of those variables for which a set of decreasing multipliers define the weight function .

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