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Research Article

The lower limit of interval efficiency in Data Envelopment Analysis

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Abstract
In data envelopment analysis technique, the relative efficiency of the homogenous decision making units is calculated. These calculations are done based on the classical model of linear programming such as CCR,BCC,…. Because of maximizing the weighted sum of outputs to that in inputs of one unit under certain conditions, the obtained efficiency in all of these models is the upper limit of exact relative efficiency. In other words, the efficiency is calculatedfrom the optimistic viewpoint. To caculated the lower limit of efficiency, i.e. the efficiency obtained from a pessimistic viewpoint for certain weights, the existing models cannot calculate the exact lower limit and in some cases, there exist some models that show an incorrect lower limit. Through the model introduced in the present study, we can calculate the exact lower limit of the interval efficiency. The designed model can be obtained by minimizing the ratio of weighted sum of outputs to that of inputs for every unit under certion conditions. The exact lower limit can be calculated in all states through our adopted model.

Keywords: Data envelopment analysis, Interval efficiency, Interval inefficiency, lower limit,upper limit.

1 Introduction

The data envelopment analysis (DEA) was introduced by Charnes et al.(1978) for the first time. It is an effective tool for decision making and management. DEA is a nonparametric technique for measuring and evaluating the relative efficiency of decision making units(DMU) with the common inputs and outputs. In DEA models, efficiency is the weighted sum of outputs to that of inputs. If the relative efficiency of the DMU under evaluation equal to one, it is said to be efficient DEA; otherwise, it is said to be inefficient DEA. Because every decision making unit can be evaluated from both optimistic and pessimistic viewpoints, it is better that the efficiencies be obtainedas intervals and in this way, the obtained efficiency would encompass the possible range of the efficiency values for each DMU. If the range of the interval efficiency is large, it means that the DMU can be good from the optimistic viewpoint and can be bad from the pessimistic one. The interval DEA studies the evaluation of the decision making units in more details. The conventional models DEA, CCR and BCC examine the model from optimistic viewpoint. Another model called as “Inverted DEA” (IDEA), examines each DMU from pessimistic viewpoint. The objective function of IDEA is the ratio of the weighted sum inputs to that of outputs. Doyle et al. (1995) studied the efficiency measurement both

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from optimistic and pessimistic viewpoints, and so Entani et al. (2002) to obtain the lower limit of interval efficiency, They introduced a model which was different from the IDEA model. Recently, Wang et al. (2007) have presented the bounded input oriented DEA models. In these models, the performance of the decision making units is evaluated in a range of the interval and the maximum input and output is used to obtain the best and worst relative efficiency. The present paper is to show the drawbacks of the Entani et al. model (2002) and to introduce a model which is free of these drawbacks. The adventage of the present authors' model over thire model is represented by an example. The rest of the paper covers 3 sections. In the section 2, the DEA models, In the section 3, the numerical examples will be represented,and the last section is the conclution part of the paper.

2 DEA models

In DEA, the maximum ratio of output to input is assumed as the efficiency which is calculated from the optimistic viewpoint for each DMU. Consider n decision making units (DMU_j : $j \in J=\{1,\dots,n\}$) each of which DMU_j is using inputs x_{ij} , i=1,…,m, to produce outputs y_{rj} , r=1,…,s. Let the input and output vectors for DMU_j be $X_j = (x_{1j},\dots,x_{mj})^t$ and $Y_j = (y_{1j},\dots,y_{sj})^t$, respectively. For DMU_j it has been assumed that $X_j \geq 0, X_j \neq 0$ and $Y_j \geq 0, Y_j \neq 0$. The efficiency of the DMUo,($o \in J$) relative to the others is obtained through the following fractional model:

$$
\begin{aligned}
& Max \quad \frac{U^t Y_o}{V^t X_o} \\
& S.t. \quad \frac{U^t Y_j}{V^t X_j} \leq 1, \quad j=1,\dots,n \qquad (2.1) \\
& U \geq 0, \quad V \geq 0.
\end{aligned}
$$

where U and V are the weight vectors for the input vectors X_j and output vectors Y_j , respectively. This fractional model will be changed into the following linear model call CCR by the Charnes-Cooper conversion:

$$
\begin{aligned}
& Max \quad U^t Y_o \\
& S.t. \quad V^t X_o = 1, \qquad (2.2) \\
& -V^t X + U^t Y \leq 0, \\
& U \geq 0, \quad V \geq 0.
\end{aligned}
$$

where $X \in R^{m\times n}$, $Y \in R^{s\times n}$ are the input and output matrix respectively. When the optimal value of the objective function equals to one, DMU_o is rated as efficient and otherwise is rated as inefficient.
Entani et al.(2002), the upper limit of interval efficiency is formulated as follows:

$$
\begin{aligned}
& \theta_o^{E^*} = \underset{u,v}{Max} \frac{\dfrac{U^t Y_o}{V^t X_o}}{\underset{j}{Max} \dfrac{U^t Y_j}{V^t X_j}} \qquad (2.3) \\
& S.t. \quad U \geq 0, \quad V \geq 0.
\end{aligned}
$$

with the change of variable $Max \frac{U^t Y_j}{V^t X_j} = \frac{1}{t}$, constraints $\frac{U^t Y_j}{V^t X_j} \leq 1$ are created.

The constrains $\frac{U^tY_j}{V^tX_j} \le 1$ or $U^tY_j - V^tX_j \le 0$ for any j, this guarantees that the relative efficiency is calculated. That is, the hyperplane we obtain be supported on production possibility set (PPS). Otherwise, the relation (2.3) will not be relative efficiency. Then, by the Charnes-Cooper conversion, we have:

$$\theta_o^{E^*} = \underset{u,v}{Max} \frac{U^tY_o}{V^tX_o}$$

$$S.t. \quad \underset{j}{Max} \frac{U^tY_j}{V^tX_j} = 1 \tag{2.4}$$

$$U \ge 0, V \ge 0.$$

They proved that the optimal value of (2.1) and (2.4) are equivalent. If the objective function of CCR model is minimization, to obtain the efficiency from the pessimistic viewpoint, it happens that U=0, $V \ne 0$ and the value of the objective function becames zero (0) for all DMU_s. This is why Entani et al.(2002) haved considered the minimization problem of (2.3) to obtain the lower limit of interval efficiency for DMU_o :

$$\theta_{o^*}^{E} = \underset{u,v}{Min} \frac{\frac{U^tY_o}{V^tX_o}}{\underset{j}{Max} \frac{U^tY_j}{V^tX_j}} \tag{2.5}$$

$$S.t. \quad U \ge 0, \quad V \ge 0.$$

In fact, by model (2.5) we obtain weights through which the relative efficiency is calculated and the worst state occurs to the unit under evaluation (DMU_o).

They obtained the following solutions problem by using the Charnes-Cooper conversion which can be represented as follows:

$$\theta_{o^*}^{E} = \underset{u,v}{Min} \frac{U^tY_o}{V^tX_o}$$

$$S.t. \quad \underset{j}{Max} \frac{U^tY_j}{V^tX_j} = 1, \tag{2.6}$$

$$U \ge 0, \quad V \ge 0.$$

The problem (2.6) can not be replaced with the equivalent linear programming problem. By assuming that $\frac{U^tY_j}{V^tX_j} = 1$ for each j, (2.6) can be divided into the following n problem :

$$\theta_{oj}^{E} = \underset{u,v}{Min} \frac{U^tY_o}{V^tX_o} \qquad (j = 1, \ldots, n)$$

$$S.t. \quad \frac{U^tY_j}{V^tX_j} = 1, \tag{2.7}$$

$$U \ge 0, \quad V \ge 0.$$

The condition $\frac{U^tY_j}{V^tX_j} = 1$ for a given j cannot be replaced with $\underset{j}{Max}\{\frac{U^tY_j}{V^tX_j}\}$

becames by considering other constraints, this can be a separable problem. To clarify the point, let's assume that:

$Max\{a_1,a_2,\ldots,a_n\}=\alpha$

It is obvious that through this relation, we can obtain n relation with the following disjunction:

$$Max\{a_1, a_2, ..., a_n\} = \alpha \Rightarrow a_1 = \alpha \vee a_2 = \alpha \vee ... \vee a_n = \alpha \tag{2.8}$$

However, it is evident that if $a_1 = \alpha \vee a_2 = \alpha \vee ... \vee a_n = \alpha$ is true, then we can not conclude that $Max\{a_1,a_2,\ldots,a_n\}=\alpha$. For example, it is obvious that if $5 = 5 \vee 6 = 5 \vee 3 = 5$ is a true proposition, then one can not conclude that Max{5,6,3}=5. The relation (2.8) is recursion relation if the proposition $a_1 = \alpha \vee a_2 = \alpha \vee ... \vee a_n = \alpha$ is true, then the proposition $a_1, a_2, ..., a_n \leq \alpha$ is true too . Thus, we can write :

$$Max\{\alpha_1, ..., \alpha_n\} = \alpha \Leftrightarrow \left\{ \begin{array}{l} \alpha_1 = \alpha \\ \alpha_i \leq \alpha \\ i = 1, ..., n \end{array} \right. \vee ... \vee \left\{ \begin{array}{l} \alpha_n = \alpha \\ \alpha_i \leq \alpha \\ i = 1, ..., n \end{array} \right.$$

In order to the optimal value of problem (2.7), they belived that (2.7) can be reduced to the following n LP problems:

$$\begin{aligned} \theta_{oj}^E &= \underset{U}{Min}\, U^t Y_o \qquad (j = 1, ..., n) \\ S.t. \quad & V^t X_o = 1, \\ & U^t Y_j - V^t X_j = 0, \\ & U \geq 0, \quad V \geq 0. \end{aligned} \tag{2.9}$$

They considered the optimal solution of (2.9) as the lower limit of interval efficiency. Because when j is o, obviously $U^t Y_o = 1$. Thus, the lower limit of interval efficiency can be mathematically written as follows :

$$\theta_{o^*}^E = 1 \wedge \underset{j \neq 0}{Min}\, \theta_{oj}^E \tag{2.10}$$

where

$a \wedge b = Min\{a, b\}$

At last, they proved that the lower limit of interval efficiency can be obtained as follows:

$$\theta_{o^*}^E = \underset{p,r}{Min} \frac{\frac{Y_{op}}{X_{or}}}{\underset{j}{Max} \frac{Y_{jp}}{X_{jr}}} \tag{2.11}$$

The above model can be criticized because of the following drawbacks, to which Wang et al. (2007) have referred in their articale.

1. It is obvious that the models (2.5) and (2.7) can not be equivalent. In model (2.5), the denominator of the objective function can be equal to one for getting some optimal solutions only when the problem is in maximization type.

2. This model can specify only a single decision making unit with the smallest lower limit, thus the inefficiency This model can specify only a single decision making unit with the smallest lower limit, thus the inefficiency border will not be determined. The model utilizes only one decision making unit, a DMU_j which is a source collection for DMU_o . So the model has only two constraints. Therefore, the weights of only one input and output is not equal to zero and the weights of the other inputs and outputs are zero. Cosequently, the data of only one input and output are used.

3. The model (2.7) guarantees that:

$$\frac{U^t Y_j}{V^t X_j} = 1,$$

It means that it does not guarantee that

$$\forall j \qquad \frac{U^t Y_j}{V^t X_j} \leq 1 \tag{2.12}$$

is the maximum of all the decision making units. That is, it does not guarantee that the rest of the decision making units are in one side of the hyperplane which passes the designated unit. In other words, it does not guarantee that the defining hyperplane is on PPS. We represent this intuitivly:

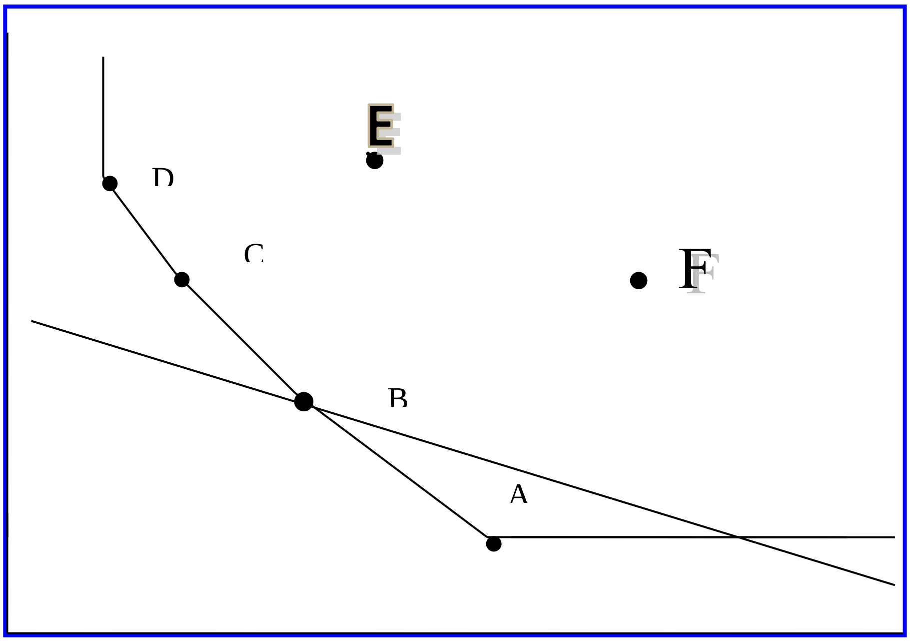

Figure 1: Counterexample

Assume that the unit (E) is under evaluation, and the constraint of model (2.7) is unit (B). We see that the hyperplane passing the point (B) does not guarantee that the other DMUs are in one side of the mentioned hyperplane. The unit (A) does not apply to the constraint $U_B Y - V_B X \leq 0$.

Wang et al.(2007) employ a virtual DMU which consumes the most inputs only to produce the least outputs and called it anti-ideal DMU (ADMU). They found the best relative efficiency and base on that presented a model from which upper and lower bounds on efficiency interval are obtained. They employed minimization of objective function to obtain the lower limit of efficiency interval. Since this virtual DMU may not exist in reality and in practical uses, a model is presents which lacks the virtual.

To solve these problems, we add constraint (2.12) to the set of constraints (2.9). In this case, the following linear model is obtained :

$$\begin{aligned}
&\varphi_{oj}^{E} = Min \quad U^t Y_o \\
&s.t \quad V^t X_o = 1, \\
&U^t Y_j - V^t X_j = 0, \\
&U^t Y_p - V^t X_p \le 0, \qquad p \ne j \\
&U \ge 0, V \ge 0.
\end{aligned} \tag{2.13}$$

To evaluate DMU_o , this model need to solve n-1 in return for all units. Note that this model may not be feasible for some $DMU_j \in intT_c$.

Lemma2.1. *If H is a support hyperplane on T_c in* $\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix}$*, then* $\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \in \partial T_C$.

Lemma 2.2. *If* $\begin{pmatrix} X_j \\ Y_j \end{pmatrix} \in \operatorname{int} T_c$ *then (2.13) infeasible.*

Proof. Assume that $\begin{pmatrix} X_j \\ Y_j \end{pmatrix} \in \operatorname{int} T_c$ and (2.13) is feasible. In this case there is $(\bar{U}, \bar{V}) \ge 0$ so that $\bar{U} Y_p - \bar{V} X_p \le 0$ is for all p and $\bar{U} Y_j - \bar{V} X_j = 0 \quad j \ne p$. Assume that H={$\begin{pmatrix} X \\ Y \end{pmatrix}$: $\bar{U} Y - \bar{V} X = 0$}

In this case $T_c \subseteq H^-$ and $\begin{pmatrix} X_j \\ Y_j \end{pmatrix} \in T_c \cap H$, then H on T_c in $\begin{pmatrix} X_j \\ Y_j \end{pmatrix}$ is support According to Lemma 2.1 $\begin{pmatrix} X_j \\ Y_j \end{pmatrix} \in \partial T_c$ and this is inconsistent with $\begin{pmatrix} X_j \\ Y_j \end{pmatrix} \in \operatorname{int} T_c$.

Proposition 2.1. *If there is an DMU_k that $\exists \lambda > 0$, $\forall j (j \ne k), \forall (U,V) \ge 0, \frac{U\lambda Y_k}{V\lambda X_k} > \frac{UY_j}{VX_j}$, then minimum and maximum efficiency DMU_k is equal to 1.*

Proof. if there is an DMU_k that $\forall j \begin{pmatrix} -\lambda X_k \\ \lambda Y_k \end{pmatrix} \ge \begin{pmatrix} -X_j \\ Y_j \end{pmatrix}$ then $\forall$ j: $\forall$ $U \ge 0$ $U\lambda Y_k \ge UY_j$ and $\forall$ j: $\forall$ $V \ge 0$ $V\lambda X_k \le VX_j$; therefore $\forall$ (U,V) $\left(\frac{U\lambda Y_k}{V\lambda X_k}\right) \ge \left(\frac{UY_j}{VX_j}\right)$. From this relation we have Max $\frac{UY_j}{VX_j} = \frac{UY_k}{VX_k}$, then maximum and minimum efficiency DMU_k is equal to 1.

To get the DMU_s efficiency, in addition to the source unit of $DMU_{j,}$ the aforementioned model, uses other decision making units. So it guarantees that the other decision making units are in one side of the hyperplane passing from DMU_o. In other words, define hyperplane on PPS will be guaranteed. An example is shown to compare models.

3 Numerical example

Let us assume an example which has 10 DMU, with one input and two outputs. The data are shown in Table 1.

DMU	Input	Output1	Output2
A	10	1	8
B	1	299	399
C	18	2	6
D	15	3	3
E	123	3	7
F	154	4	2
G	176	4	5
H	175	5	2
I	1678	6	2
J	16678	7	1

Table 1.Value of input and outputs

When we obtain the lower limit of interval efficiency according to formula (2.10), the information with five decimal digits are represented in Table 2.

DMU	$\theta_{o^*}^E$
A	0.00033
B	1.00000
C	0.00037
D	0.00051
E	0.00008
F	0.00003
G	0.00007
H	0.00003
I	0.00000
J	0.00000

Table 2. The lower limit of interval efficiency according to Entani formula (2.11)

Now, we find the optimal solution according to model (2.9). The data are shown in Table 3.

DMU	$\theta_{o^*}^E$	V^*	$U1^*$	$U2^*$
A	0.00100	0.10000	0.00020	0.00010
B	1.00000	1.00000	0.00010	0.00249
C	0.41679	0.05556	0.00010	0.06943
D	0.25026	0.06667	0.00010	0.08332
E	0.07135	0.00813	0.00010	0.01015
F	0.01661	0.00649	0.00010	0.00810
G	0.03585	0.00568	0.00010	0.00709
H	0.01479	0.00571	0.00010	0.00713
I	0.00207	0.00600	0.00010	0.00073
J	infeasible			

Table 3: The optimal value of $\theta_{o^*}^E$ and the optimal values of V^* ، $U1^*$ and $U2^*$.

By studying table 3, we see that the condition $\forall j \quad UY_j - VX_j \leq 0$ does not apply to every j. For DMU C in optimal solution for the second index, we have $U^* Y_2 - V^* X_2 = 26.98337$.
For DMU D in optimal solution for the second index, we have $U^* Y_2 - V^* X_2 = 32.37504$.
These conditions exist for DMUs of I,H,G,F,E; that is, their values, in the second index are also positive. Now, on the basis of model (2.13), we obtain the optimal solution and optimal value. Table 4 demonstrates these pieces of information.

DMU	$\varphi_{O^*}^E$	V^*	$U1^*$	$U2^*$
A	0.00040	0.10000	0.00032	0.00001
B	1.00000	1.00000	0.00010	0.00256
C	0.00041	0.05556	0.00017	0.00001
D	0.00052	0.06667	0.00001	0.00016
E	0.00011	0.00813	0.00001	0.00001
F	0.00006	0.00649	0.00001	0.00001
G	infeasible			
H	infeasible			
I	infeasible			
J	infeasible			

Table 4: The optimal value $\varphi_{O^*}^E$ and optimal values of V^* , $U1^*$, $U2^*$

By comparing the tables 2 and 3, we see that the optimal solutions obtain through formula (2.9) and (2.11) are different from each other, which is inconsistent with the theorem represented by Entani et al. paper. By studying Table 3 and according to the optimal value of weight vectors, we see that these values of DMU ,C to I, do not apply to other constraints (second constraint), but our model lacks this defect. By studying Table 4, it is shown that the optimal values of weight vectors for each DMU apply to all of the other constraints. In addition, according to the optimal solution represented in Table 4, we see that the DMUs of G,H,I do not have the feasible solution, where according to Table 3, these have the optimal solution.

4 Conclusion

The previous DEA models calculate the efficiency from an optimistic viewpoint. In other words, if θ^* is the solution of the CCR model and RE is a real efficiency of a unit, then it will always be $RE \leq \theta^*$. Therefore, the scientists` are looking for an (a) that $a \leq RE$ exsist. So, finding the lower limit of interval efficiency is one of the scientists efforts. Entani et al. have represented a model which had some obscurities. Their model cannot do this important problem properly. We have presented a model which lacks these defects. Knowing how to classify the efficient and inefficient units helps us to remove inefficient ones. To prove that our
Model is correct, an example has been presented and the results have been compared with each other.

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