new method to evaluate relative efficiency measure in dynamic DEA

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Abstract
As known in data envelopment analysis literature, TDT measure has been used to get the relative efficiency measure of decision making units. Then, the aim of this paper is to extend TDT measure into the dynamic framework of data envelopment analysis to get the dynamic relative efficiency measure by which units’ productivity would be evaluated exactly on an assessment window. To do this, it is needed firstly to identify some factors named link factors bearing truly and exactly connectivity between time periods of an assessment window to develop an accurate dynamic framework of data envelopment analysis.

Keywords: DEA; Dynamic DEA; Relative Efficiency; TDT measure

1 Introduction

Data envelopment analysis (DEA, hereafter) and dynamic DEA (DDEA, hereafter) are two major subjects which deal with many problems related to Operations Research (OR). DEA is a linear programming approach which is pioneered by Charnes et al. [1]. Moreover, DEA is an optimization tool based on mathematical programming generalizing the Farrell’s model [5] from the single-output/multiple-input form to multiple-output/multiple-input form. DDEA is a very extended form of DEA in which assessments of a decision making units (DMU, hereafter) will be expanded over an assessment window and continue during several time periods. DDEA was originally developed by Färe et al. [4] to cope with a long time assessment incorporating concepts of quasi-fixed inputs or investment activities. In other words, DDEA is an approach dealing with assessing performance of a group of DMUs during several non-separate time periods. Actually, what discriminates between DDEA and the other existing models depending on time factor such as Window analysis [6] and the Malmquist indices [3] is the attention to factors called link factors, e.g. investment activities. In fact, the most important feature of DDEA can be considering the link factors arising the continuity between periods of assessment. This paper at first clarifies such link factors over periods of assessment. These developments can give DMs new insights relating to their judgment after assessments. As noted earlier, DDEA models unlike the separate-time models such as window analysis and time series are sensitive to link factors (such as quasi-fixed inputs, intermediate outputs and investment activities between two consecutive periods of assessment) and their effects on efficiency measure of DMUs. So far, several researchers have offered some DDEA models, e.g. Nemoto and Goto [7],[8], Sueyoshi and Sekitani [9], and many papers have been published into this context. But all of them consider different aspects of DDEA such as returns to scale (RTS), the cost efficiency, dynamic slack-based

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measure (DSBM), etc. However, none of the papers have developed relative efficiency measure at the end of the frequent assessments of an assessment window. The paper unfolds as follows. Section 2 provides a preliminary of DDEA. Section 3 reports the main idea with a numerical example, and section 4 concludes the paper.

2 Preliminary of DDEA

This section exhibits some definitions and expressions in DDEA. Assume there are n DMUs which are assessed in T non-separate time periods belonging to an assessment window, namely W. Nemoto and Goto’s (hereafter NG) defined $\phi^1_{C RS}$ as a dynamic production possibility set as follows:

$$\phi^1_{C RS} = \left\{(x_t,k_{t-1},y_t,k_t) \in \mathbb{R}^{m+1} \times \mathbb{R}^{s+1} | (x_t,k_{t-1}) \text{ can produce } (y_t,k_t) \right\}.$$  \hfill (2.1)

Suppose $\lambda_t = (\lambda_{t,1},\lambda_{t,2},...,\lambda_{t,n})^\tau$ as a weight vector in which $\tau$ represents the transpose of a vector, and $X_t = [x_{t,1},x_{t,2},...,x_{t,n}]$, $K_{t-1} = [k_{t-1,1},k_{t-1,2},...,k_{t-1,n}]$ and $Y_t = [y_{t,1},y_{t,2},...,y_{t,n}]$ as the matrices of inputs, quasi-fixed inputs and outputs, respectively. Then, the above-mentioned $\phi^1_{C RS}$ can be rewritten as follows:

$$\phi^1_{C RS} = \left\{(x_t,k_{t-1},y_t,k_t) \in \mathbb{R}^{m+1} \times \mathbb{R}^{s+1} | X_t \lambda_t \leq x_t, K_{t-1} \lambda_t \leq k_{t-1}, Y_t \lambda_t \geq y_t, K_t \lambda_t \geq k_t, \lambda_t \geq 0 \right\}.$$  \hfill (2.2)

The "0" is used as a vector whose all components are 0. Indeed, in the $t$th period of assessment ($t=1,...,T$), a $DMU_p$ utilizes an input vector $(x_t,k_{t-1}) \in \mathbb{R}^{m+1}$ to yield an output vector $(y_t,k_t) \in \mathbb{R}^{s+1}$. Note that $k_t$ produced in the $t$th period of assessment might be used as a vector of quasi-fixed inputs in the $t+1$th period of assessment. Here, $W = \{1,2,...,T\}$ refers to an assessment window whose periods of assessment are non-separate. That means several interconnected factors or link factors such as quasi-fixed inputs, intermediate outputs or investment activities might be between its periods of assessment.

3 Main idea

In this section, the works of previously mentioned researchers is extended by considering some specific factors, named link factors, arising continuity between time periods of an assessment window. Then consequently based on them, the relative efficiency measure of a DMU could be computed in dynamic DEA.

3.1 Link factors in DDEA

In this section, the input and output vectors used in DDEA framework is developed. First, suppose that $X_{t,p}$ is an input vector consumed by a DMU, namely $DMU_p$ at the beginning of the $t$th time period which $X_{t,p} = \left( \begin{array}{c} x_{t,p} \\ A_{t,p}k_{t-1,p} \end{array} \right)$. $k_{t-1,p}$ is a $s$-vector of investment inputs seen at the beginning of the time period $t$. $A_{t,p}$ is a $s \times s$-diagonal matrix whose the $i$th diagonal element named $a_{ii}$ ($i = 1,...,s$) is as a percentage of the consumption of the $i$th element of the input vector $k_{t-1,p}$ (indicated by $k_{t-1,p,i}$) reported by decision maker (DM) at the end of $t$th time period. $\tilde{A}_{t,p}$ is a $s \times s$-diagonal matrix whose the $i$th diagonal element as $\tilde{a}_{ii}$ is $1-a_{ii}$, i.e. $\tilde{a}_{ii}$ ($i = 1,...,s$) is as a percentage of the $k_{t-1,p,i}$ which is not consumed at the end of the $t$th time period, so $\tilde{A}_{t,p} k_{t-1,p}$ is mentioned as a part of inputs that have been not used at the end of the $t$th time period by $DMU_p$, and it could be positive or negative analyzed by DM. Note that in the last time period the $\tilde{A}_{t,p} k_{t-1,p}$ could be mentioned as a part of outputs, but in the internal time periods is mentioned as part of inputs used in the $t+1$th time period in terms of saving. $x_{t,p} \in \mathbb{R}^{m}$ is a vector of variable inputs; $Y_{t,p} \in \mathbb{R}^{s}$ is a vector of total outputs produced by the $DMU_p$ at the end of the $t$th time period. Moreover, $y_{t,p} \in \mathbb{R}^{t}$ is a vector of external outputs sent to markets. $h_{t,p} \in \mathbb{R}^{t}$ is a vector of internal outputs sent to the next time period as a vector of investment inputs. On the other hand, $y_{t,p} = B_{t,p}Y_{t,p}$ and $h_{t,p} = \tilde{B}_{t,p}Y_{t,p}$. $B_{t,p}$ is a $s \times s$-diagonal matrix whose the $i$th diagonal element as $b_{ii}$, ($i = 1,...,s$) is as a percentage of outputs sent to the markets. Also, $\tilde{B}_{t,p}$ is a $s \times s$-diagonal matrix whose the $i$th diagonal element as $\tilde{b}_{ii}$, ($i = 1,...,s$) is a percentage of outputs sent to the next time period instead of markets. To satisfy in DDEA framework, it should be expected that $Y_{t,p} = y_{t,p} + h_{t,p}$, in the
absence of waste. Moreover, $DMU_p$ has an input vector named quasi-fixed input vector, $\mathbf{x} \in \mathbb{R}_{+}^m$, whose components have constant value over all time periods of an assessment window and are fixed for all DMUs such as the acres of land in a farm. In order to keep continuing between two time periods $t$ and $t+1$ relative to vector $k_{t,p}$, the following constraint must be hold:

$$h_{t,i} + \tilde{A}_t k_{t-1,j} = k_{t,i} = A_{t+1,j} k_{t+1} + \tilde{A}_{t+1,j} k_{t,j}, \quad t = 1, ..., T - 1, \quad j = 1, ..., n. \quad (3.3)$$

Note the following figure illustrates the position of the link factors between two time periods $t$ and $t+1$.

![Link factors between two time periods](image)

Based on the figure above, Eq. (3.3) can be summarized as follows:

$$h_{t,j} + \tilde{A}_t k_{t-1,j} = A_{t+1,j} k_{t+1} + \tilde{A}_{t+1,j} k_{t,j}, \quad t = 1, ..., T - 1, \quad j = 1, ..., n. \quad (3.4)$$

### 3.2 Relative efficiency measure in DDEA

As known, the TDT measure [2] was developed to obtain relative efficiency measures (RE) of the $DMU_p$ and was introduced by the "ratio of ratios" model as follows:

$$
DRE = \max_{u, v, \omega, \sigma} \frac{\sum_{t=1}^{T} (u_t y_{t,p} + \sigma_t h_{t,p})}{\sum_{t=1}^{T} v_t x_{t,p}} = \max_{u, v, \omega, \sigma} \frac{\sum_{t=1}^{T} (u_t y_{t,p} + \sigma_t h_{t,p})}{\sum_{t=1}^{T} v_t x_{t,p}}
$$

where $\sum_{t=1}^{T} u_t y_{t,p}$ is maximum $j=1,2,...,n$ 

$$
\sum_{t=1}^{T} v_t x_{t,p} \cdot \sum_{t=1}^{T} \sigma_t x_{t,p} = \sum_{t=1}^{T} (v_t x_{t,p} + \mu_t A_t k_{t-1,p} + \sigma_t x_{t,p})
$$

Actually, the model (3.5) is a maximin model in DEA literature that can be seen as a non-normalized model maximizing the relative efficiency of $DMU_p$. Here, the dynamic relative efficiency measures of $DMU_p$ is developed based on the DDEA model considering the link factors. For this purpose, suppose that $DMU_p$ has been assessed over an assessment window involving $T$ non-separate time periods displayed by $W = \{1, 2, ..., T\}$. Also, assume that $DMU_p$ consumes the input vector $\mathbf{x}_{t,p}$ entirely to produce the output vector $\mathbf{y}_{t,p}$. Now to develop the dynamic relative efficiency measure, it should be extend the maximin model(3.5) into DDEA framework considering the link factors via the following model:

$$
DRE_p = \max_{u_t, v_t, \omega_t, \sigma_t} \frac{\sum_{t=1}^{T} (u_t y_{t,p} + \sigma_t h_{t,p})}{\sum_{t=1}^{T} (v_t x_{t,p} + \mu_t A_t k_{t-1,p} + \sigma_t x_{t,p})}
$$

s.t. $h_{t,j} + \tilde{A}_t k_{t-1,j} = A_{t+1,j} k_{t+1} + \tilde{A}_{t+1,j} k_{t,j}, \quad t = 1, ..., T - 1, \quad j = 1, ..., n, \quad u_t, v_t, \omega_t, \mu_t \in \mathbb{R}_{+}^n, \quad \sigma_t \in \mathbb{R}_{+}^n, \quad t = 1, ..., T, \quad (3.6)$

where $(u_t, v_t, \omega_t, \mu_t)$ for $(t=1, ..., T)$ is a sequence of weight vectors. Based on the theorem 3.1, the model (3.6) can be transformed as follows:

$$
DRE_p = \max_{u_t, v_t, \omega_t, \mu_t} \frac{\sum_{t=1}^{T} (u_t y_{t,p} + \sigma_t h_{t,p})}{\sum_{t=1}^{T} (v_t x_{t,p} + \mu_t A_t k_{t-1,p} + \sigma_t x_{t,p})}
$$

s.t. $h_{t,j} + \tilde{A}_t k_{t-1,j} = A_{t+1,j} k_{t+1} + \tilde{A}_{t+1,j} k_{t,j}, \quad t = 1, ..., T - 1, \quad j = 1, ..., n, \quad u_t, v_t, \omega_t, \mu_t \in \mathbb{R}_{+}^n, \quad \sigma_t \in \mathbb{R}_{+}^n, \quad t = 1, ..., T, \quad (3.7)$

$$
\sum_{t=1}^{T} (v_t x_{t,p} + \mu_t A_t k_{t-1,p} + \sigma_t x_{t,p}) \leq 1,
$$

$$
h_{t,j} + \tilde{A}_t k_{t-1,j} = A_{t+1,j} k_{t+1} + \tilde{A}_{t+1,j} k_{t,j}, \quad t = 1, ..., T - 1, \quad j = 1, ..., n, \quad (3.8)
$$
Theorem 3.1. At least one of the efficiency inequality in (3.8) satisfies as equality in all optimal solutions of (3.7).

Proof. On contrary, let there is an optimal solution to DMU\textsubscript{p}, e.g. \(\{u^*_t, v^*_t, \omega^*_t, \mu^*_t, \sigma^*_t\}^T\) which satisfies in (3.8) as strict inequality for all j.

Obviously, \(\forall j \exists \gamma_j > 1: \frac{\sum_{t=1}^{T}(a_t^u y_{t,j} + a_t^v h_{t,j})}{\sum_{t=1}^{T}(v_{t,j} + \mu^*_t \bar{A}_{t+1,p} k_{t-1,j} + \sigma^*_t \bar{g}_j)} \gamma_j = 1\). Let \(\gamma^* = \min_{j=1, ..., n} \{\gamma_j \} \geq 1\). Then, \(\frac{\sum_{t=1}^{T}(a_t^u y_{t,j} + a_t^v h_{t,j})}{\sum_{t=1}^{T}(v_{t,j} + \mu^*_t \bar{A}_{t+1,p} k_{t-1,j} + \sigma^*_t \bar{g}_j)} \gamma^* \leq 1, \quad j = 1, ..., n\). Considering (3.7) demonstrates that \(\{u^*_t, v^*_t, \omega^*_t, \mu^*_t, \sigma^*_t\}^T\) is a feasible solution to (3.6) and its objective value \(\gamma^* DRE\textsubscript{p}\). Since \(\gamma^* > 1\), then \(\gamma^* DRE\textsubscript{p} > DRE\textsubscript{p}\). This contradicts to being optimal the \(\{u^*_t, v^*_t, \omega^*_t, \mu^*_t, \sigma^*_t\}^T\) to model (3.7) and completes the proof.

The two models (3.6) and (3.7) are equivalent based on the theorem, and their optimal values are equal. As known in the literature of DEA, by using Charnes-Cooper transformation for the fractional model (3.7) yields an equivalent linear form as model (3.11) as follows.

\[
DRE\textsubscript{p} = \max_{u_t, \omega_t, \mu_t, \sigma_t} \sum_{j=1}^{T} (u_t^* y_{t,p} + \omega_t^* h_{t,p}), \tag{3.11}
\]

s.t. \(\sum_{j=1}^{T} (v_t^* x_{t,p} + \mu_t^* A_{t,p} k_{t-1,p} + \sigma_t^* g_p) = 1\),

\[
\sum_{j=1}^{T} (u_t^* y_{t,j} + \omega_t^* h_{t,j}) - \sum_{j=1}^{T} (v_t^* x_{t,j} + \mu_t^* A_{t,j} k_{t-1,j} + \sigma_t^* g_j) \leq 0,
\]

\(h_{t,j} + A_{t,j} k_{t-1,j} = A_{t+1,j} k_{t,j} + \bar{A}_{t+1,j} k_{t,j}, \quad t = 1, ..., T - 1, \quad j = 1, ..., n\),

\(u_t, \omega_t, \mu_t \in R^+_+, \quad v_t \in R^{w+}, \quad \sigma_t \in R^{w+}, \quad t = 1, ..., T\).

Theorem 3.2. The model (3.11) maximizes relative efficiency measure of a DMU\textsubscript{p} in DDEA.

Proof. Theorem 1, showed that the two models (3.6) and (3.7) were equivalent. Since using the Charnes-Cooper transformation brought an equivalent linear form as (3.11) from (3.7), then the two models (3.6) and (3.11) are equivalent and the model (3.11) like the model (3.6) maximizes dynamic relative efficiency measure. This completes the proof.

3.3 Numerical example

See Table (1). It shows some data of 5 DMUs where the first, second, third, fourth and fifth columns display respectively, variable inputs, investment inputs, external outputs, internal outputs and quasi-fixed inputs, respectively. Note that in this example for 5 DMUs, the vectors of all data have two elements except the vector of quasi-fixed inputs. Dynamic relative efficiency measures of the 5 DMUs after assessing during two periods of assessment are gathered into Table (2).
Table 1: The data of 5 DMUs in two time periods

<table>
<thead>
<tr>
<th>DMU</th>
<th>Assessment window $w$</th>
<th>Assessment window $w$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Time period $t$</td>
<td>Time period $t+1$</td>
</tr>
<tr>
<td>$x_{t,j}$</td>
<td>$A_{t,j}$</td>
<td>$k_{t-1,j}$</td>
</tr>
<tr>
<td>$DMU_1$</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>6</td>
</tr>
<tr>
<td>$DMU_2$</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>5</td>
</tr>
<tr>
<td>$DMU_3$</td>
<td>8</td>
<td>6.8</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>$DMU_4$</td>
<td>12</td>
<td>14.25</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>3.5</td>
</tr>
<tr>
<td>$DMU_5$</td>
<td>15</td>
<td>8.4</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>5.28</td>
</tr>
</tbody>
</table>

Table 2: DMUs’ relative efficiency measure

<table>
<thead>
<tr>
<th>DRE$_p$</th>
<th>$DMU_1$</th>
<th>$DMU_2$</th>
<th>$DMU_3$</th>
<th>$DMU_4$</th>
<th>$DMU_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DRE$_p$</td>
<td>0.8678</td>
<td>0.8875</td>
<td>1.0000</td>
<td>0.9184</td>
<td>0.7234</td>
</tr>
</tbody>
</table>

Obviously from Table 2, only $DMU_3$ is evaluated as a relative efficient DMU among all existing DMUs.
4 Conclusions

In this paper a new method was proposed by which a DMU’s dynamic relative efficiency measure could be measured. To do this, some factors, named link factors in this study, bearing connectivity between time periods and have basic role in DDEA were defined. Also based on the link factors, input and output vectors define precisely and narrowly in DDEA. So far, input and output in dynamic framework of DEA had been defined incompletely because such specific matrices, shown here $A_{t,j}$, $A_{t,j}$, $B_{t,j}$, $B_{t,j}$, had not been already mentioned in definition of input and output vectors and in the associated models in the literature of DDEA. In other words, in this study established a new dynamic framework of DEA where dynamic input and output vectors were developed with accuracy and detail. Finally in this work, based on the dynamic input and output vectors TDT model was extended into DDEA, and then based on it, the dynamic relative efficiency measure of units was developed.

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References


