The output estimation of a DMU to preserve and improvement of the relative efficiency

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Abstract

In this paper, we consider the inverse BCC model is used to estimate output levels of the Decision Making Units (DMUs), when the input levels are changed and maintain the efficiency index for all DMUs. Since the inverse BCC problem is in the form of a multi objective nonlinear programming model (MONLP), which is not easy to solve. Therefore, we propose a linear programming model, which gives a Pareto-efficient solution to the inverse BCC problem. So far, we propose a model for improvement of the current efficiency value for considered DMU. Numerical examples are, also, used to illustrate the proposed approaches.

Keywords: Data Envelopment Analysis, Inverse DEA model, Multi-objective programming problem.

1 Introduction

Data Envelopment Analysis (DEA) is a methodology used to estimate the relative efficiency of Decision Making Units (DMUs). DEA was originated by Charnes et al. [3] in 1978, and was called CCR [3]. Since, Banker et al. [2] in 1984 developed a variable returns to scale version of the CCR model that was called BCC model [2]. Nowadays, DEA has allocated a wide variety of research in operations research to itself. One of the concepts that have sparked considerable interest in the DEA is that of inverse DEA model. Wei et al. [9] proposed for the first time, an inverse DEA model for input and output estimation where an inverse DEA model was discussed to answer the following question: among a group of DMUs, if we increase certain inputs (or outputs) of a particular unit and assume that the DMU maintains its current efficiency value with respect to other units, how much more outputs(or inputs) could the unit produce?[9]. Yan et al. [10] discussed an inverse DEA problem with preference cone constraints to represent decision makers’ preferences which was useful in resource planning. Jahanshaloo et al. [5] estimated output levels of considered DMU when some or all of input values were increased and the efficiency value it’s needed to be improved by specified percentage of its current efficiency value. Jahanshaloo et al. [6] proposed other inversed DEA models where estimate inputs for a DMU when some or all outputs increased and the efficiency value of the DMU improved by specified percentage of its current efficiency value.

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Jahanshahloo et al. [7] proposed a modified inverse DEA model for sensitivity analysis of efficiency classifications of efficient and inefficient DMUs in which important policies over inputs, outputs and DMUs were represented by preference cones. Hadi-Vencheh and Foroughi [4] developed an inverse DEA model where an increase of some inputs (outputs) and a decrease due to some of the other inputs (outputs) are taken into account at the same time. Since show that the solution proposed by Wei et al. [9] does not guarantee the efficiency result for input estimating and show that the current models may fail in a special case whereas they model overcomes this flaw. Alinezhad et al. [1] proposed a methodology that uses an interactive MOLP for solving the inverse DEA problems. Lertworasirikul et al. [8] considered the inverse DEA model for the case of variable returns to scale where exists at least an optimal solution to them model if and only if the new output vector is in the set of current production possibility set (PPS).

In this paper, we show the inverse DEA model to the BCC model when input levels of a considered DMU are changed and we need estimated output levels where maintained its current efficiency value. This problem transformed into and solved as a multi-objective programming model (MOLP), where we propose a linear programming model, which gives a pareto-efficient solution to the this model. So far, there exists at least an optimal solution to our model if and only if the new input vector is in the set of proposed set. Later than we proposed a model for estimate output levels where input levels of a considered DMUs are changed and we want improved by specified percentage of its current efficiency value. This paper is organized as follows: In Section 2, presents our proposed model to determine the best possible values of outputs for the considered DMU to preserve relative efficiency values of all DMUs. In Section 3, contains the proposed model for improvement of efficiency value of the considered DMU. In Section 4, numerical examples are used to illustrate the proposed approaches.

2 The inverse BCC model

Assume that we have n DMUs (DMU$_j$: $j = 1, 2, \ldots, n$) use m inputs to produce s outputs, which is defined as follows:

\[
\begin{align*}
\text{x}_j &= (x_{1j}, x_{2j}, \ldots, x_{mj})^T, \quad x_j \geq 0, x_j \neq 0, \\
\text{y}_j &= (y_{1j}, y_{2j}, \ldots, y_{sj})^T, \quad y_j \geq 0, y_j \neq 0.
\end{align*}
\]

Consider the output oriented BCC model for evaluating the relative efficiency of these DMUs, where can be defined as follows:

Maximize \[ z \]
\[ \text{s.t.} \]
\[ \sum_{i=1}^{m} \lambda_i x_{ij} \leq x_{i0}, \quad i=1, 2, \ldots, m, \]  
\[ \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{r0}, \quad r=1, 2, \ldots, s, \]  
\[ \sum_{j=1}^{n} \lambda_j = 1, \]  
\[ \lambda_j \geq 0, \quad j=1, 2, \ldots, n. \]  

Suppose that for DMU$_0$, the inputs are changed from $x_0$ to $x_0 + \Delta x_0 \geq 0$, when the efficiency index to be maintained at $z_0^*$ and we need to estimate the corresponding output level $\beta$, where $\beta = y_0 + \Delta y_0 \geq 0$. 

Note that, we consider DMU$_{n+1}$ respesents DMU$_0$after changing its inputs and outputs. The primal form of the inverse BCC model is as follows:

Maximize \[ \Delta y_{ro} \]
\[ \text{s.t.} \]
\[ \sum_{j=1}^{n} \lambda_j x_{ij} + \lambda_{n+1} (x_{i0} + \Delta x_{i0}) \leq x_{i0} + \Delta x_{i0}, \quad i=1, 2, \ldots, m, \]  
\[ \sum_{j=1}^{n} \lambda_j y_{rj} + \lambda_{n+1} (y_{r0} + \Delta y_{r0}) \geq z_0^* (y_{r0} + \Delta y_{r0}), \quad r=1, 2, \ldots, s, \]  
\[ \sum_{j=1}^{n} \lambda_j + \lambda_{n+1} =1, \]  
\[ \lambda_j \geq 0, \quad j=1, 2, \ldots, n+1. \]
Where \( y_{ro} + \Delta y_{ro} \neq 0 \) and \( z_0^* \) is the optimal value of problem (1).

By following theorem, we show the relative efficiency values of all DMUs are unchanged.

**Theorem 2.1.**

Suppose the optimal value of problem (1) is \( z_0^* \) and the inputs of DMU\(_o\) are changed from \( x_{o} \to x_{o} + \Delta x_{o} \geq 0 \), \( \Delta x_{o} \neq 0 \), we need find maximum \( \Delta y_{ro} \) of the DMU\(_{n+1}\) which the relative efficiency values of all DMUs are unchanged, can be obtained by solving the following model.

Maximize \( \Delta y_{ro} \)

\[
\begin{align*}
\text{s.t.} & \quad \sum_{i=1}^{n} \lambda_i x_{ij} \leq x_{i0} + \Delta x_{i0}, \quad i=1, 2, \ldots, m, \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq z_0^*(y_{ro} + \Delta y_{ro}), \quad r=1, 2, \ldots, s, \\
& \quad \sum_{i=1}^{n} \lambda_i = 1, \\
& \quad \lambda_j \geq 0, \quad j=1, 2, \ldots, n.
\end{align*}
\]

**Proof.** We assume that the following model for evaluate the relative efficiency of DMU\(_{n+1}\):

Maximize \( z \)

\[
\begin{align*}
\text{s.t.} & \quad \sum_{i=1}^{n} \lambda_i x_{ij} + \lambda_{n+1}(x_{i0} + \Delta x_{i0}) \leq x_{i0} + \Delta x_{i0}, \quad i=1, 2, \ldots, m, \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} + \lambda_{n+1}(y_{ro} + \Delta y_{ro}) \geq z_0^*(y_{ro} + \Delta y_{ro}), \quad r=1, 2, \ldots, s, \\
& \quad \sum_{i=1}^{n} \lambda_i + \lambda_{n+1} = 1, \\
& \quad \lambda_j \geq 0, \quad j=1, 2, \ldots, n+1.
\end{align*}
\]

**Case 1: \( z_0^* > 1 \)**

From the set of constraints in (4), if \( \lambda_{n+1} = 1 \) then \( \lambda_j = 0 \) for \( j=1, 2, \ldots, n \) and \( z_{n+1} = 1 \). This solution is not the optimal to the problem (4), but we have the better solution for \( \lambda_{n+1} = 0 \) when \( z_0^* > 1 \).

When \( \lambda_{n+1} = 0 \) the constraints of problem (4) are in the same form as the constraints in the model (3). So the objective value is \( z_0^* \), which is later that 1.

We divide all constraints in the problem (4) by \( (1 - \lambda_{n+1}) \), where \( (1 - \lambda_{n+1}) \neq 0 \), and set \( \bar{Z}_{n+1} = \frac{z_{n+1} - \lambda_{n+1}}{1 - \lambda_{n+1}} \) and \( \bar{X}_{j} = \frac{\lambda_j}{1 - \lambda_{n+1}} \) for \( j=1, 2, \ldots, n \). Then the problem (4) becomes:

Maximize \( z_{n+1} \)

\[
\begin{align*}
\text{s.t.} & \quad \sum_{i=1}^{n} \bar{X}_{ij} \bar{X}_{ij} \leq x_{i0} + \Delta x_{i0}, \quad i=1, 2, \ldots, m, \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq \bar{Z}_{n+1}(y_{ro} + \Delta y_{ro}), \quad r=1, 2, \ldots, s, \\
& \quad \sum_{i=1}^{n} \bar{X}_i + \frac{\lambda_{n+1}}{1 - \lambda_{n+1}} = 1, \\
& \quad \lambda_j \geq 0, \quad j=1, 2, \ldots, n+1.
\end{align*}
\]

We find that \( \lambda_{n+1} = \frac{1 - \sum_{i=1}^{n} \bar{X}_i}{2 - \sum_{i=1}^{n} \bar{X}_i} \). Thus by substituting \( \lambda_{n+1} \) in to \( \bar{Z}_{n+1} \). Therefore, the problem (5) becomes:

Maximize \( \bar{Z}_{n+1} \)

\[
\begin{align*}
\text{s.t.} & \quad \sum_{i=1}^{n} \bar{X}_{ij} \bar{X}_{ij} \leq x_{i0} + \Delta x_{i0}, \quad i=1, 2, \ldots, m, \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq \bar{Z}_{n+1}(y_{ro} + \Delta y_{ro}), \quad r=1, 2, \ldots, s, \\
& \quad \lambda_j \geq 0, \quad j=1, 2, \ldots, n.
\end{align*}
\]

We set the denominator of the model (6) equal to 1. Then, equivalent problem of (6) in the following model:
The optimal solution of the model (7) is also optimal for the model (6) since the above transformation is reversible. Note that the constraints in the model (7) are in the same form of the constraints in the model (3). Using the maximum $\Delta y_o$ obtained from solving the model (3), all constraints in the model (7) are satisfied and the objective function is maximized at $\bar{z}_{n+1} = z_o^*$. Otherwise $\Delta y_o$ is not optimal for the model (3).

Case 2: $z_o^* = 1$

Note that if $\lambda_{n+1} \neq 1$ in the problem (4), we can prove that optimal solution of the problem (4) is equal to $z_o^* = 1$ by using the same way for the proof of case 1. And if $\lambda_{n+1} = 1$, then $\lambda_j = 0$ for $j = 1, 2, ..., n$ and $z_{n+1} = z_o^* = 1$.

Now let $\lambda_{n+1} = 1$ and $\lambda_j = 0$ for $j = 1, 2, ..., n$ and $z_{n+1} = 1, x_o + \Delta x_o = 0$. Where this solution is not an optimal solution for the inverse Bcc model. Therefore, the maximum $\Delta y_o$ of DMU$_{n+1}$, which changes the input values of DMU$_0$ and we desire the efficiency index $z_{n+1}$ to remain unchanged.

For other DMUs, dual form of the inverse Bcc model as follows:

Minimize $V^T x_k + v_0$

s.t. $U^T y_k = 1,$

$-V^T x_j + U^T y_j - v_0 \leq 0, j = 1, 2, ..., n,$

$-V^T (x_o + \Delta x_o) + U^T (y_o + \Delta y_o) - v_0 \leq 0,$

$U, V \geq 0, v_0$ is free.

Where $U^T = [u_1, u_2, ..., u_r], V^T = [v_1, v_2, ..., v_m]$ and $x_j = (x_{1j}, x_{2j}, ..., x_{mj})^T, x_j \geq 0, x_j \neq 0$ and $y_j = (y_{1j}, y_{2j}, ..., y_{sj})^T, y_j \geq 0, y_j \neq 0$.

As well $(x_o + \Delta x_o, y_o + \Delta y_o) \in P$ where $P$ is production possibility set of all DMUs. If $(x_o + \Delta x_o, y_o + \Delta y_o) \in P$, we obtain:

$-V^T (x_o + \Delta x_o) + U^T (y_o + \Delta y_o) - v_0 \leq -V^T (X\lambda) + U^T (Y\lambda) - v_0 \leq -\sum_{j=1}^n V^T \lambda_j x_j + \sum_{j=1}^n U^T \lambda_j y_j - v_0 \leq \sum_{j=1}^n (-V^T x_j + U^T y_j) \lambda_j - v_0.$

From model (8), $-V^T x_j + U^T y_j - v_0 \leq 0$ for $j = 1, 2, ..., n.$

Thus, $-V^T (x_o + \Delta x_o) + U^T (y_o + \Delta y_o) - v_0 \leq 0.$

So we can be dropped out this constraint of model (8) without changing problem. Therefore, the relative efficiency values of all DMUs remain unchanged.

Lemma 2.1.

Suppose the optimal value of problem (1) is $z_o^*$. Also, suppose the inputs of DMU$_0$ are changed from $x_o$ to $x_o + \Delta x_o \geq 0, \Delta x_o \neq 0$. There exists at least an optimal solution to problem (3) if and only if $x_o + \Delta x_o \in P_m$. Where $P_m = \left \{ x | x \leq \sum_{j=1}^n \lambda_j x_j, \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0 \right \}, x_j = (x_{1j}, x_{2j}, ..., x_{mj})^T.$

Proof. If $x_o + \Delta x_o \in P_m$, then we have the following constraints:
\[
\begin{align*}
\Sigma_{j=1}^{n} \lambda_{j} x_{ij} & \leq x_{i0} + \Delta x_{i0}, & i=1,2,...,m, \\
\Sigma_{j=1}^{n} \lambda_{j} &= 1, \\
\lambda_{j} &\geq 0, & j=1,2,...,n.
\end{align*}
\]

Where satisfied in the problem (3). As well, we know \( x_{i0} + \Delta x_{i0} \geq 0 \) for i=1,2,...,m. Whit consider objective of model (3), we obtained where there exists at least an optimal solution to the model(3).

If there exists at least an optimal solution to the model (3), thus, satisfied constraints of model (3), then \( x_{o} + \Delta x_{o} \in P_{in} \).

As seen in Lemma 2.1, if all elements of \( x_{o} + \Delta x_{o} \) is more than or equal to all elements of the inputs of at least one DMU in the extreme DMUs set. Then \( x_{o} + \Delta x_{o} \) is in \( P_{in} \).

**Theorem 2.2.**

Consider the following LP problem:

Maximize \( W^T \Delta y_{r0} \)

s.t. \( \Sigma_{j=1}^{n} \lambda_{j} x_{ij} \leq x_{i0} + \Delta x_{i0}, \) \( i=1,2,...,m, \) (9)

\( \Sigma_{j=1}^{n} \lambda_{j} y_{rj} \geq z_{o}^{*}(y_{ro} + \Delta y_{ro}), \) \( r=1,2,...,s, \)

\( \Sigma_{j=1}^{n} \lambda_{j} = 1, \)

\( \lambda_{j} \geq 0, \) \( j=1,2,...,n. \)

Where \( z_{o}^{*} \) is optimal value of problem (1), any optimal solution of problem (9) is a Pareto solution for the problem (3).

**Proof.** Suppose \((\Delta x_{o}^{*}, \lambda^{*})\) are the optimal solution of model (9) but they are not Pareto solution to the model (3). There should be a possible \((\Delta \bar{x}_{o}, \bar{\lambda})\) from the model (3) where \( \Delta \bar{x}_{o} \leq \Delta x_{o}^{*} \), and thus, \( W^T \Delta \bar{x}_{o} \leq W^T \Delta x_{o}^{*}, W^T > 0 \).

Consider the constraints of model (3) and model (9), thus obvious \((\Delta \bar{x}_{o}, \bar{\lambda})\) are the solution of model (3). Where this leads to a contradiction; Therefore, \((\Delta x_{o}^{*}, \lambda^{*})\) is a Pareto solution of model (3).

**3 Improvement of the efficiency**

In this section, we have obtained the model for estimate the output levels of \( DMU_{o} \) when input levels are changed from \( x_{o} \) to \( x_{o} + \Delta x_{o} \geq 0, \Delta x_{o} \neq 0 \), where increases of some inputs and decreases of the other inputs of the considered DMU can be taken into account simultaneously.

Now we want improvement its efficiency to \( \eta \)-present of \( z_{o}^{*} \). Therefore, consider the following model:

Maximize \( \Delta y_{r0} \)

s.t. \( \Sigma_{j=1}^{n} \lambda_{j} x_{ij} \leq x_{i0} + \Delta x_{i0}, \) \( i=1,2,...,m, \) (10)

\( \Sigma_{j=1}^{n} \lambda_{j} y_{rj} \geq (1 - \frac{\eta}{100})z_{o}^{*}(y_{ro} + \Delta y_{ro}), \) \( r=1,2,...,s, \)

\( \Sigma_{j=1}^{n} \lambda_{j} = 1, \)

\( \lambda_{j} \geq 0, \) \( j=1,2,...,n. \)

where, \( 0 \leq \eta \leq 100(z_{o}^{*} - 1)/ z_{o}^{*} \).

**4 Numerical examples**

In this section, we proposed numerical examples to illustrate the proposed approaches via a case study of a motorcycle-part company. The data is taken from the article Lertworasirikul et al. [8], where consist of 25 DMU with seven inputs and three outputs.
If we compare the performance of all DMUs based on inputs only, the set of extreme DMUs includes $DMU_6, DMU_{11}, DMU_{12}, DMU_{14}, DMU_{16}, DMU_{18}, DMU_{19}, DMU_{22}$. By evaluating $DMU_4$ using model (3), we have $z_4^* = 1.03$. Assume that the input vector of $DMU_4$ is changed from $(24, 14, 163, 260, 15, 210, 0)^T$ to $(17, 5, 88, 89, 14, 120, 0)^T$ and let $W^T = (1, 1, 1)$ for output weights. Then by using a solver in Microsoft Excel 2007 for $DMU_4$, we find that there is no feasible solution to model (9), because the new input values of $DMU_4$ is not in $P_{in}$. Let us consider $DMU_4$ again but now let the input vector of $DMU_4$ is changed from $(24, 14, 163, 260, 15, 210, 0)^T$ to $(21, 7, 79, 150, 20, 110, 0)^T$ where $x_4 + \Delta x_4$ is the inputs values of the same $DMU_6$ and we know $DMU_6$ is an extreme DMU, then $x_4 + \Delta x_4 \in P_{in}$, and let $W^T = (1, 1, 1)$ for output weights. Now by solving model (9) and we find that $\Delta y_4 = (252.682, -0.024, 0.725)$. The new output values is $y_4 + \Delta y_4 = (924.682, 1.246, 95.275)$. Using the new input and output values for $DMU_4$, the relative efficiency values of all DMUs still remain. Now we want improvement the relative efficiency of $DMU_4$ to $\eta$-present of $z_4^*$. Assume that the input values its be changed from $(24, 14, 163, 260, 15, 210, 0)^T$ to $(30, 16, 187, 290, 20, 260, 3)^T$ and $\eta = 2$, by solving model (10) we find that $y_4 + \Delta y_4 = (979.790, 1.258, 98.078)$. Now by solving model (3) we find that optimal solution is $z_4^* = 1.009$. Let us consider $DMU_4$ again but now we want improve the relative efficiency of $DMU_4$ to $\eta = 1$ of $z_4^*$. Assume that the input values its be changed from $(24, 14, 163, 260, 15, 210, 0)^T$ to $(18, 7, 165, 90, 20, 120, 1)^T$, by solving model (10) we find that $y_4 + \Delta y_4 = (659.017, 1.245, 94.145)$. Now by solving model (3) we find that optimal solution is $z_4^* = 1.01$.

5 Conclusion

The traditional inverse DEA model is used to determine the best possible values of inputs (outputs) for given values of outputs (inputs) of a considered DMU such that relative efficiency value of a considered DMU with respect to other DMUs remain unchanged. In this paper, we propose an inverse BCC model where in this model is to identify how to adjust the changes in output levels when the input levels are changed and its efficiency index is unchanged. However, the proposed inverse BCC model is in the form of a MONLP, which is not easy to solve. To find the optimal solution to the inverse BCC model, we propose a linear programming model, which gives a Pareto-efficient solution to the inverse BCC problem. However, there exists at least an optimal solution to this model if and only if the new input vector is in the set of current production possibility set. Then, we proposed a model which improvement the efficiency index to $\eta$-present of $z_0^*$ for $DMU_0$ (considered DMU). A relevant but different question is: suppose there are fuzzy inputs/outputs which can be given to DMUs, and if we want interval inputs/outputs, how should the interval inputs/outputs be used. But the inverse DEA model proposed in this paper still can provide some useful ideas to solve such a problem.

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References


http://dx.doi.org/10.1287/mnsc.30.9.1078

http://dx.doi.org/10.1016/0377-2217(78)90138-8

http://dx.doi.org/10.1016/j.mcm.2005.08.005

http://dx.doi.org/10.1016/S0096-3003(02)00734-8

http://dx.doi.org/10.1016/j.amc.2003.08.001

http://dx.doi.org/10.1016/j.amc.2004.09.093

http://dx.doi.org/10.1016/j.cie.2011.06.014

http://dx.doi.org/10.1016/S0377-2217(99)00007-7

http://dx.doi.org/10.1016/S0377-2217(01)00046-7