On the approximation of inverse of some band matrices and their applications in local splines

T. Zhanlav¹, R. Mijiddorj²*, H. Behforooz³

(1) Institute of Mathematics, National University of Mongolia, Ulaanbaatar, Mongolia.
(2) Department of Informatics, Mongolian National University of Education, Ulaanbaatar, Mongolia.
(3) Department of Mathematics, Utica College, Utica, NY 13502, USA.

Abstract
In this paper, we obtain approximate inverses of popular tri-diagonal and penta-diagonal matrices which are used to construct local (or a discrete quasi-interpolant) interpolatory and integro splines.

Keywords: Tri-diagonal matrices; Penta-diagonal matrices; Inverses; Local construction; Splines.

1 Introduction

The band matrices often arise in a range of science and engineering applications such as numerical solutions of ordinary and partial differential equations, spline approximation, image and signal processing, and parallel computing, see [1, 5, 6, 11] and references therein. In many of these areas, inversion of the tri-diagonal matrix is required. In particular, in [11] Yamamoto obtained explicit formulas for the entries of the inverse of nonsingular tri-diagonal matrices. In [7], Jia and Li derived the numerical or symbolic algorithms for the inverses of \( k \)-diagonal matrices. Moreover, in [9] Smolarski discussed a particular type of banded matrix, namely a diagonally striped matrix, and the structure of its inverse. Bickel and Lindner in [3] proved that if an infinite matrix \( A \), which is invertible as a bounded operator on \( l^2 \), can be uniformly approximated by banded matrices then so can the inverse of \( A \).

Although there are explicit formulas for entries of the inverse of band matrices but most of the time, practically, they are not suitable for simple and hand calculations. In some cases, it suffices to find only approximate inverse of these matrices. On the other hand, for band matrices, it is well established that the entries of its inverse decay exponentially away from the main diagonal; see for example [4]. Therefore, we only need to find approximate entries \( a_{ij} \) of the main and its few adjacent diagonals i.e. we need

\[
a_{ij}, \quad \text{for } |i - j| \leq k, \quad \text{for } k = 1, 2, 3. \tag{1.1}
\]

For example, in constructing interpolatory splines and integro-splines with small degrees, it is often required to solve a system of linear equations

\[
A\mathbf{x} = \mathbf{f}, \tag{1.2}
\]

where \( A \) is a band matrix. In particular, we consider the following cases:
If we have approximate inverse $A^{-1} = (\alpha_{ij})$, $|i - j| \leq k$ then we obtain the approximate solution of (1.2) as follows:

$$\hat{x}_i = \sum_{|i-j| \leq k} \alpha_{ij} f_j, \quad (1.3)$$

The error of approximate solution given by (1.3) is estimated as

$$||x - \hat{x}|| = \max_{|i-j|>k} |\alpha_{ij}||f||. \quad (1.4)$$

From (1.4) it is clear that it is better to restrict $k$ by small values, because of the exponential decay of entries of inverse of band matrices [4]. To find approximate inverse $A^{-1} = (\alpha_{ij})$ we use the approximate solution of system (1.2) known in some cases. First we consider the system (1.2) with matrix $A =$Tri-diag$\{1, 4, 1\}$. Such system arises in constructing interpolatory cubic spline on the uniform partition $[a, b]$ with knots $x_i = a + ih, i = 0(1)n, h = \frac{b-a}{n}$.

## 2 Approximate inverse of special tri-diagonal matrices and its applications

Let $S_3(x)$ be a cubic $C^2$ spline satisfying the interpolation conditions

$$S_3(x_i) = f_i, \quad f_i = f(x_i), \quad i = 0(1)n. \quad (2.5)$$

By using the $B$-spline representation of $S_3(x) \in C^2$, we have:

$$S_3(x) = \sum_{i=1}^{n+1} c_i B_i(x), \quad (2.6)$$

where $B_i(x)$ are normalized cubic $B$-splines that constitute basis for $S_3 \in C^2[a,b]$ cubic splines space, see [12]. Then the interpolatory conditions (2.5) implies

$$c_{i-1} + 4c_i + c_{i+1} = 6f_i, \quad i = 0(1)n, \quad (2.7a)$$

or

$$c = 6A^{-1}f. \quad (2.7b)$$

Now we will find the entries of near of the main diagonal, using the approximate explicit formula given in [8, 12]

$$c_i = \frac{8f_i - f_{i-1} - f_{i+1}}{6}, \quad i = 1(1)n - 1, \quad (2.8)$$

with accuracy $O(h^4)$. Using (2.8), we write (2.7b) as

$$8f_i - f_{i-1} - f_{i+1} + O(h^4) = 36\{\cdots + \alpha_{i-2}f_{i-2} + \alpha_{i-1}f_{i-1} + \alpha_{i}f_i + \alpha_{i+1}f_{i+1} + \alpha_{i+2}f_{i+2}\cdots\}. \quad (2.9)$$

If we use explicit formulas given in [11] for matrix $A =$Tri-diag$\{1, 4, 1\}$ then it is easy to show that

$$\alpha_{i} = \alpha_{ji}, \quad \text{and} \quad \alpha_{i,j} = \alpha_{i,-j} \text{ for } j = 1, 2, \cdots. \quad (2.9)$$
Therefore, the last expression can be rewritten as
\[ 8f_i - f_{i-1} - f_{i+1} + O(h^4) = 36\{\cdots + \alpha_{i-j-2}(f_{i-2} + f_{i+2}) + \alpha_{i-j-1}(f_{i-1} + f_{i+1}) + \alpha_i f_i\}. \] (2.10)

If we take into account the formula
\[ f_{i-2} - 4f_{i-1} + 6f_i - 4f_{i+1} + f_{i+2} = O(h^4), \]
which holds for \( f(x) \in C^4 \), then the expression (2.10) becomes
\[ 8f_i - f_{i-1} - f_{i+1} = 36\{\cdots + (4\alpha_{i-2} + \alpha_{i-1})(f_{i-1} + f_{i+1}) + (\alpha_i - 6\alpha_{i-2})f_i\} + O(h^4). \] (2.11)

It follows from (2.11) that
\[ 36(4\alpha_{i-2} + \alpha_{i-1}) = -1, \quad 36(\alpha_i - 6\alpha_{i-2}) = 8, \] (2.12)
where \( \alpha_j = O(h^4) \), \( |i - j| \geq 3 \). From (2.12) the unknowns \( \alpha_{i-1} \) and \( \alpha_i \) are expressed by \( \alpha_{i-2} \) as
\[ \alpha_{i-1} = -\frac{1}{36} - 4\alpha_{i-2}, \quad \alpha_i = \frac{2}{9} + 6\alpha_{i-2}. \] (2.13)

We know that
\[ (AA^{-1})_{ii} = 1, \] (2.14)
which leads to
\[ \alpha_{i-2} = \frac{1}{96}. \] (2.15a)

Hence, from (2.13) we find
\[ \alpha_{i-j} = -\frac{5}{72}, \quad \alpha_i = \frac{41}{144}, \] (2.15b)
and
\[ \alpha_j = O(h^4), \quad |i - j| \geq 3. \] (2.15c)

Thus, the entries of \( i \)-th row of \( A^{-1} \) are given by explicit formula (2.15). Further, if we use the notation \( M_i = S'\!_i(x_i) \) then we have the following system of equations [12]
\[ M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(f_{i-1} - 2f_i + f_{i+1}), \quad i = 1(1)n - 1. \] (2.16)

The matrix of system (2.16) is, as preceding case, \( A = \text{Tri-diag}\{1, 4, 1\} \). Then according to (2.13) we have
\[ M_i = \frac{6}{h^2}\{\cdots + \alpha_{i-2}(f_{i-3} - 2f_{i-2} + f_{i-1} - f_{i+1} - 2f_{i+2} + f_{i+3}) + (\frac{1}{36} - 4\alpha_{i-2})(f_{i-2} - 2f_{i-1} + 2f_i - 2f_{i+1} + f_{i+2}) + (\frac{2}{9} + 6\alpha_{i-2})(f_{i-1} - 2f_i + f_{i+1})\} = \frac{1}{6h^2}\{[-f_{i-2} + 10f_{i-1} - 18f_i + 10f_{i+1} - f_{i+2}] + 6\alpha_{i-2}\{f_{i-3} - 6f_{i-2} + 15f_{i-1} - 20f_i + 15f_{i+1} - 6f_{i+2} + f_{i+3}\} + O(h^2). \]

Let \( f \in C^6[a,b] \). The Taylor expansions of \( f(x_i + kh) \) give us
\[ f_{i+k} = f_i + khf_i' + \frac{(kh)^2}{2}f_i'' + \frac{(kh)^3}{6}f_i^{(3)} + \frac{(kh)^4}{24}f_i^{(4)} + \frac{(kh)^5}{120}f_i^{(5)} + O(h^6), \quad k = \pm 1,\pm 2,\pm 3, \]
from these we have
\[ f_{i-3} - 6f_{i-2} + 15f_{i-1} - 20f_i + 15f_{i+1} - 6f_{i+2} + f_{i+3} = O(h^6). \]
Using the last formulas and (2.15a), we have

\[ M_i = \frac{1}{6h^2} \{ -f_{i-2} + 10f_{i-1} - 18f_i + 10f_{i+1} - f_{i+2} \} + O(h^2) = \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} + O(h^2). \]

Thus, we find the solution of (2.16) with accuracy \( O(h^2) \) without solving it. One can write system for \( m_i = S_i^3(x_i) \)

\[ m_{i-1} + 4m_i + m_{i+1} = \frac{3}{h} (f_{i+1} - f_{i-1}), \quad i = 1(1)n - 1, \]

which has the same matrix \( A \) as (2.16). Consequently, using the same technique as above, we find

\[ m_i = \frac{1}{12h} \left( f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2} \right) + O(h^4). \]

(2.17)

Note that the system (1.2) with matrix \( A \) = Tri-diag \{1, 4, 1\} arises also in constructing integro splines.

3 Application of approximate inverse matrices on constructing integro splines

In a uniform partition case the integro quadratic spline \( S_2(x) \) satisfies relations [10]

\[ S_2(x_{i-1}) + 4S_2(x_i) + S_2(x_{i+1}) = \frac{3}{h} (I_i + I_{i-1}), \quad i = 1(1)n - 1, \]

(3.18)

where

\[ \int_{x_i}^{x_{i+1}} S_2(x) dx = \int_{x_i}^{x_{i+1}} y(x) dx = I_i, \quad i = 0(1)n - 1, \]

(3.19)

i.e. the integral values \( I_i \) of function \( y(x) \) are known on the subintervals \([x_i, x_{i+1}], h = (b - a)/n\). Obviously, one can use the \( B \)-spline representation of \( S_2(x) \):

\[ S_2(x) = \sum_{i=1}^{n} b_i B_i(x), \]

(3.20)

where \( B_i(x) \) are a normalized quadratic \( B \)-splines that forms a basis for \( C^1 \) quadratic splines space. For convenience, we present here \( B_i \) as:

\[ B_i(x) = \frac{1}{2h^2} \begin{cases} \frac{(x-x_{i-1})^2}{h^2}, & \left[ x_{i-1}, x_i \right], \\ \frac{(x-x_{i-1})^2 - 3(x-x_i)^2}{h^2}, & \left[ x_i, x_{i+1} \right], \\ 0, & \left[ x_{i+1}, x_{i+2} \right], \end{cases} \]

(3.21)

The values of \( B_i(x) \) and \( B_i'(x) \) at the knots are given in Table 1.

<table>
<thead>
<tr>
<th>x_{i-1}</th>
<th>x_i</th>
<th>x_{i+1}</th>
<th>x_{i+2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>B(x)</td>
<td>0</td>
<td>\frac{1}{2}</td>
<td>\frac{1}{2}</td>
</tr>
<tr>
<td>B_i(x)</td>
<td>0</td>
<td>\frac{1}{h}</td>
<td>-\frac{1}{h}</td>
</tr>
</tbody>
</table>

From (3.20) and using the properties of \( B \)-spline in Table 1, we obtain

\[ S_2(x_i) = \frac{b_{i-1} + b_i}{2}, \quad S_2'(x_i) = \frac{b_i - b_{i-1}}{h}, \quad i = 0(1)n. \]

(3.22)
Taking into account (3.22), the relations (3.18) can be written in term of coefficients $b_i$ as:

$$b_{i-2} + 5b_{i-1} + 5b_i + b_{i+1} = \frac{6}{h}(I_i + I_{i-1}), \quad i = 1(1)n - 1,$$  \hfill (3.23a)

or

$$z_{i-1} + z_i = \frac{6}{h}(I_i + I_{i-1}), \quad i = 1(1)n - 1,$$  \hfill (3.23b)

where

$$z_i = b_{i-1} + 4b_i + b_{i+1}.$$  \hfill (3.24)

From (3.23b) we deduce

$$z_i = \frac{6}{h}I_i,$$

or

$$b_{i-1} + 4b_i + b_{i+1} = \frac{6}{h}I_i, \quad i = 0(1)n - 1.$$  \hfill (3.25)

Analogously, using (3.22) and the relations

$$b_i = S_2(x_i) + \frac{h}{2}S'_2(x_i),$$  \hfill (3.26)

one can obtain

$$S'_2(x_{i-1}) + 4S'_2(x_i) + S'_2(x_{i+1}) = \frac{6}{h^2}(I_i - I_{i-1}), \quad i = 1(1)n - 1.$$  \hfill (3.27)

Thus, we have the systems (3.18), (3.25), and (3.27) with the same matrix but different right-hand sides. Since the matrix of these system is $A=\text{Tri-diag}\{1, 4, 1\}$, we can use the above computed approximate inverse of this matrix. Using (2.15), from (3.18) we find

$$S_2(x_i) = \frac{3}{h} \left\{ \frac{1}{96}(I_{i-3} + I_{i-2} + I_{i+1} + I_{i+2}) - \frac{5}{72}(I_{i-2} + I_{i-1} + I_i + I_{i+1}) + \frac{41}{144}(I_i + I_{i-1}) \right\}$$

$$+ O(h^4) = \frac{1}{96h} \{3I_{i-3} - 17I_{i-2} + 62I_{i-1} - 62I_i - 17I_{i+1} + 3I_{i+2} \} + O(h^4).$$  \hfill (3.28)

For the values of $I_i$ and $y \in C^4$, the following property holds

$$I_{i-2} - 4I_{i-1} + 6I_i - 4I_{i+1} + I_{i+2} = O(h^5).$$  \hfill (3.29)

We can simplify (3.28) by using (3.29). As a result we have

$$S_2(x_i) = \frac{1}{12h} \{ -I_{i-2} + 7I_{i-1} + 7I_i - I_{i+1} \} + O(h^4), \quad i = 2(1)n - 2.$$  \hfill (3.30)

Using the same technique, as preceding case, in (3.25) and (3.27) we obtain

$$b_i = \frac{8I_i - I_{i-1} - I_{i+1}}{6h} + O(h^4), \quad i = 1(1)n - 2,$$  \hfill (3.31)

and

$$S'_2(x_i) = \frac{1}{6h^2} \{ -I_{i-2} - 9I_{i-1} + 9I_i - I_{i+1} \} + O(h^3), \quad i = 2(1)n - 2.$$  \hfill (3.32)

Thus, we first obtain approximate explicit formulas for $S_2(x_i)$, $b_i$, and $S'_2(x_i)$. In [10], we have the following estimation

$$S_2(x_i) = y_i + O(h^4),$$

but no estimation for the first derivative is given. Due to the explicit formula (3.32) one can obtain

$$S'_2(x_i) = y'_i + O(h^5), \quad i = 2(1)n - 3.$$  \hfill (3.33)
Another application of approximate inverse of matrix $A = \text{Tri-diag}\{1, 4, 1\}$ is the well-known relations in [15]:

$$n_i = \frac{6}{h^4} (-I_{i-3} + 3I_{i-2} - 3I_{i-1} + I_i) + O(h^3), \quad i = 1(1)n - 3,$$

(3.34)

where $n_i = S^{6''}_i(x_i)$. Such system appears in constructing quintic integro spline. As above, from (3.34) it follows that

$$n_i = \frac{6}{h^4} \left\{ \cdots + a_{i-2}(-I_{i-3} + 3I_{i-2} - 3I_{i-1} + I_i) + a_{i-1}(-I_{i-2} + 3I_{i-1} - 3I_i + I_{i+1}) + a_i(-I_{i-1} + 3I_{i} - 3I_{i+1} + I_{i+2}) + a_{i+1}(-I_i + 3I_{i+1} - 3I_{i+2} + I_{i+3}) + a_{i+2}(-I_{i+1} + 3I_{i+2} - 3I_{i+3} + I_{i+4}) + \cdots \right\} + O(h^3).$$

Using (2.15) and $I_{i-3} - 6I_{i-2} + 15I_{i-1} - 20I_i + 15I_{i+1} - 6I_{i+2} + I_{i+3} = O(h^7)$

(3.35)

into the last formula, we obtain

$$n_i = \frac{1}{6h^4} (-I_{i-2} - 11I_{i-1} + 28I_i - 28I_{i+1} + 11I_{i+2} - I_{i+3}) + O(h^3).$$

(3.36)

Analogously, we can find approximate inverse of matrix $A = \text{Tri-diag}\{1, 10, 1\}$. Let $S_i(x_i)$ be an integro-quartic spline satisfying the conditions (3.19) and $m_i = S^{4''}_i(x_i), T_i = S^{4}_i(x_i)$. Then we have the following relations [14]

$$m_{i-1} + 10m_i + m_{i+1} = \frac{12}{h^4} (I_i - I_{i-1}) + O(h^4),$$

(3.37)

and

$$T_i - 10T_{i-1} + T_{i+1} = \frac{12}{h^4} (-I_{i-2} + 3I_{i-1} - 3I_i + I_{i+1}) + O(h^2), \quad i = 2(1)n - 2.$$ 

(3.38)

In [14], we obtained the approximate formula

$$m_i = \frac{1}{12h^4} (I_{i-2} - 15I_{i-1} + 15I_i - I_{i+1}) + O(h^4), \quad i = 2(1)n - 2.$$ 

(3.39)

As above, we denote the entries of inverse matrix $A^{-1}$ by $\alpha_{ij}$. Then from (3.37) and (3.39) we get

$$\frac{1}{144} (I_{i-2} - 15I_{i-1} + 15I_i - I_{i+1}) \equiv \cdots + a_{i-2}(I_{i-2} - I_{i-3}) + a_{i-1}(I_{i-1} - I_{i-2}) + a_i(I_i - I_{i-1}) + a_{i+1}(I_{i+1} - I_i) + a_{i+2}(I_{i+2} - I_{i+1}) + \cdots + O(h^4).$$

(3.40)

Using symmetry of $A^{-1}$ and matching the coefficients of $I_i$ on both sides (3.40) we obtain

$$4\alpha_{i-2} + \alpha_{i-1} - \alpha_i = -\frac{1}{144},$$

$$10\alpha_{i-2} + \alpha_{i-1} - \alpha_i = -\frac{15}{144},$$

(3.41)

$$\alpha_j = O(h^4), \quad |i - j| \geq 3.$$

To derive the last formulas, we have used the relation (3.29). In addition to (3.41) we require that

$$2\alpha_{i-1} + 10\alpha_i = 1,$$

(3.42)

which follows from (2.14). From (3.41), (3.42) we find that

$$\alpha_i = \frac{191}{1872}, \quad \alpha_{i-1} = \alpha_{i+1} = -\frac{19}{1872}, \quad \alpha_{i-2} = \alpha_{i+2} = \frac{1}{1248}, \quad \alpha_j = O(h^4), \quad |i - j| \geq 3.$$ 

(3.43)
Thus we find the entries of \( i \)-th row of \( A^{-1} \) by formulas (3.43). Now we can use (3.43) to determine \( T_i \) from (3.38). As above, we get

\[
T_i = \frac{12}{h^5} (\cdots + \alpha_{i-2}(-I_{i-3} + 3I_{i-2} - 3I_{i-1} + I_i) + \alpha_{i-1}(-I_{i-2} + 3I_{i-1} - 3I_i + I_{i+1}) \\
+ \alpha_i(-I_{i-1} + 3I_i - 3I_{i+1} + I_{i+2}) + \alpha_{i+1}(-I_{i} + 3I_{i+1} - 3I_{i+2} + I_{i+3}) \\
+ \alpha_{i+2}(-I_{i+1} + 3I_{i+2} - 3I_{i+3} + I_{i+4}) + \cdots )
\]

Using (3.29) and (3.43) into the last formula, we obtain the well-known explicit formula that was derived first in [14]

\[
T_i = \frac{1}{h^5} (-I_{i-2} + 3I_{i-1} - 3I_i + I_{i+1}) + O(h^2).
\] (3.44)

Now we consider the matrix \( A = \text{Tri-diag}\{1, d, 1\} \) with \(|d| > 2\). Obviously, the above two cases are particular cases of this matrix with \( d = 4 \) and \( d = 10 \). From Theorem 2.1 in [7], we get the following explicit formula for \( A^{-1} \) of \( A = \text{Tri-diag}\{1, d, 1\} \),

\[
\alpha_{ij} = (-1)^{i-j} \frac{1}{p_j - q_j} \prod_{s=i}^{j-1} p_s, \quad i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor; \quad j = i, i+1, \ldots, n-i+1,
\] (3.45)

where

\[
p_1 = d, \quad p_i = d - \frac{1}{p_{i-1}}, \quad i = 2, 3, \ldots, n,
\]

\[
q_n = 0, \quad q_i = \frac{1}{d - q_{i+1}}, \quad i = n-1, n-2, \ldots, 1, \quad \text{and} \quad \alpha_{ij} = \alpha_{ji}.
\]

From (2.15) and (3.43) we obtain the approximate inverse of \( A = \text{Tri-diag}\{1, d, 1\} \) with \(|d| > 2\). Indeed from (3.45) it follows that

\[
\alpha_{ii} = \frac{1}{d - \frac{1}{2} \frac{8d^2 + 2}{d^3 + 8d^2 - 4d - 16}}, \quad \alpha_{ij} = \frac{1 - d\alpha_{ii}}{2}, \quad |i-j| \geq 3.
\] (3.46)

Using the general formulas (3.46) one can easily get (2.15) and (3.43).

### 4 Approximate inverse of penta-diagonal matrices

Let \( S_4(x) \) be an integro-quartic spline satisfying conditions (3.19) with its \( B \)-spline representation

\[
S_4(x) = \sum_{i=-2}^{n+1} c_i B_i(x).
\] (4.47)

Here \( B_i(x) \) are quartic \( B \)-spline which forms basis for \( S_4 \in C^3[a,b] \) spaces. Then with the uniform partition, the conditions (3.19) imply

\[
c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2} = \frac{120}{h^5} I_i, \quad i = 0(1)n - 1.
\] (4.48)

In [14] we obtained approximate and explicit formula

\[
c_i = \frac{13I_{i-3} - 39I_{i-2} - 94I_{i-1} + 746I_i - 159I_{i+1} + 13I_{i+2}}{480h} + O(h^2), \quad i = 3(1)n - 3.
\]
It is easy to show that by using expression (3.29), the $c_i$ can be rewritten in more symmetric form

$$c_i = \frac{13l_{i-2} - 112l_{i-1} + 438l_i - 112l_{i+1} + 13l_{i+2}}{240h} + O(h^5).$$

(4.49)

As before, we denote the entries of inverse of matrix $A$ = Penta-diag\{1, 26, 66, 26, 1\} of system (4.48) by $\alpha_{ij}$. Then, from (4.48) and (4.49) we get

$$\frac{1}{28800} \begin{align*}
13(l_{i-2} + l_{i+2}) - 112(l_{i-1} + l_{i+1}) + 438l_i + O(h^5) = \\
\cdots + \alpha_{i,-3}(l_{i-3} + l_{i+3}) + \alpha_{i,-2}(l_{i-2} + l_{i+2}) + \alpha_{i,-1}(l_{i-1} + l_{i+1}) + \alpha_{i}l_i
\end{align*}

(4.50)

in which we have used symmetry of $\alpha_{ij}$ and $\alpha_{i,j} = \alpha_{i,j}$, $j = 1, 2, 3$.

Using (3.35), from (4.50) we have

$$\frac{1}{28800} \begin{align*}
13(l_{i-2} + l_{i+2}) - 112(l_{i-1} + l_{i+1}) + 438l_i + O(h^5) = (20\alpha_{i,-3} + \alpha_{i})l_i \\
+ (-15\alpha_{i,-3} + \alpha_{i,-1})(l_{i-1} + l_{i+1}) + (6\alpha_{j,-3} + \alpha_{j,-2})(l_{i-2} + l_{i+2}) + \cdots
\end{align*}

(4.51)

Equating the coefficients of $l_{i-2} + l_{i+1}$ for $j = 0, 1, 2$ in both sides of last expression we get

$$6\alpha_{i,-3} + \alpha_{i,-2} = \frac{13}{28800},$$

$$-15\alpha_{i,-3} + \alpha_{i,-1} = \frac{-112}{28800},$$

$$20\alpha_{i,-3} + \alpha_{i} = \frac{438}{28800},$$

$$\alpha_{i,-j} = O(h^5), |i-j| \geq 4.$$  

(4.52)

In addition to (4.52), we require that

$$2\alpha_{i,-2} + 52\alpha_{i,-1} + 66\alpha_{i} = 1$$

(4.53)

which follows from (2.14). The solutions of (4.52) and (4.53) are given by

$$\alpha_{i} = \frac{4447}{1987200}, \quad \alpha_{i,-1} = -\frac{24529}{2649600},$$

$$\alpha_{i,-2} = \frac{3443}{1324800}, \quad \alpha_{i,-3} = -\frac{569}{1589760},$$

$$\alpha_{i,-j} = O(h^5), |i-j| \geq 4.$$  

(4.54)

Now we consider the following systems

$$m_{i-2} + 56m_{i-1} + 246m_i + 56m_{i+1} + m_{i+2} = b_i,$$

$$n_{i-2} + 56n_{i-1} + 246n_i + 56n_{i+1} + n_{i+2} = d_i, \quad i = 2(1)n - 2,$$

(4.55)

(4.56)

where $m_i = S'_x(x_i)$ and $n_i = S''_x(x_i)$ and

$$b_i = \frac{30}{h^2} (-l_{i-1} - 9l_i + 9l_{i+1} + l_{i+2}), \quad d_i = \frac{360}{h^2} (-l_{i-1} + 3l_i - 3l_{i+1} + l_{i+2}).$$

(4.57)

These systems arise in constructing integro quintic spline [2]. An explicit and approximate solution of system (4.55) can be found in [15] as

$$m_i = \frac{1}{180h^2} (-2l_{i-2} + 25l_{i-1} - 245l_i + 245l_{i+1} - 25l_{i+2} + 2l_{i+3}) + O(h^5), \quad i = 2(1)n - 4.$$  

(4.58)
In this case, using analogous technique, as above, we find that

\[
\begin{align*}
81\alpha_{i-3} + 16\alpha_{i-2} + \alpha_{i-1} &= \frac{1}{2700}, \\
295\alpha_{i-3} + 30\alpha_{i-2} - 9\alpha_{i-1} - \alpha_{i2} &= \frac{25}{5400}, \\
504\alpha_{i-3} + 34\alpha_{i-2} - 8\alpha_{i-1} + 9\alpha_{i1} &= \frac{245}{5400}, \\
2\alpha_{i-2} + 112\alpha_{i-1} + 246\alpha_{i} &= 1, \\
\alpha_{i-j} &= O(h^5), |i-j| \geq 4.
\end{align*}
\]

The solutions of (4.59) are

\[
\begin{align*}
\alpha_{i2} &= \frac{9979}{2195 \ 100}, & \alpha_{i-1} &= -\frac{8273}{7804 \ 800}, \\
\alpha_{i-2} &= \frac{297}{577}, & \alpha_{i-3} &= -\frac{299}{14 \ 048 \ 640}, \\
\alpha_{i-j} &= O(h^5), |i-j| \geq 4.
\end{align*}
\]

Using (3.35), (4.56), (4.57) and (4.60), we obtain (3.36). Note that, in [13] Z-folding algorithm was proposed for solving the penta-diagonal system of linear equations, which allows us to reduce the system by solving two tri-diagonal systems sequentially. We can find the approximate inverse of penta-diagonal matrices by using the Z-folding algorithm and (3.46), but this approach is not suitable to obtain explicit formulas as (3.36) and (4.58).

5 Conclusion

For some application cases it is not necessary to find all entries of the inverse matrices of band matrices. The main advantage of our approach are simple and explicit formulas for only main diagonal and its few adjacent diagonals entries of the inverse matrices.

Acknowledgments

The authors acknowledge the many helpful suggestions of the referees during the preparation of the paper. The work was partially supported by Foundation of Science and Technology of Mongolia (No. SST_007/2015).

References

https://doi.org/10.1016/S0024-3795(01)00462-1

https://doi.org/10.1016/j.camwa.2015.10.018


https://doi.org/10.1016/j.cam.2005.02.012

https://doi.org/10.1016/j.apm.2014.11.015

https://doi.org/10.1081/NFA-100105108


https://doi.org/10.1016/j.amc.2015.07.077

https://doi.org/10.5899/2016/cna-00267