Communications in Numerical Analysis 2017 No.2 (2017) 217-232

Available online at www.ispacs.com/cna

Volume 2017, Issue 2, Year 2017 Article ID cna-00334, 16 Pages

doi:10.5899/2017/cna-00334

Research Article

A Numerical method for solving a class of fractional Sturm-Liouville eigenvalue problems

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Abstract

This article is devoted to both theoretical and numerical studies of eigenvalues of regular fractional 2α-order Sturm-Liouville problem where $\frac{1}{2} < \alpha \leq 1$. In this paper, we implement the reproducing kernel method RKM) to approximate the eigenvalues. To find the eigenvalues, we force the approximate solution produced by the RKM satisfy the boundary condition at $x = 1$. The fractional derivative is described in the Caputo sense. Numerical results demonstrate the accuracy of the present algorithm. In addition, we prove the existence of the eigenfunctions of the proposed problem. Uniformly convergence of the approximate eigenfunctions produced by the RKM to the exact eigenfunctions is proven.

Keywords: Eigenvalues, Fractional Sturm-Liouville problem, Reproducing kernel method, Shooting method.

1 Introduction

Fractional differential equations (FDEs) appear as generalizations to existing models with integer derivative and they also present new models for some physical problems [1]. In recent years, great interests were devoted to the analytical and numerical treatments of fractional differential equations. In general, fractional differential equations don't have exact solutions in closed forms, and therefore, numerical methods such as, the variational iteration [4], the homotopy analysis method [5], and the Adomian decomposition method [3, 7, 8], have been implemented for several types of fractional differential equations. Also, the maximum principle and the method of lower and upper solutions have been extended to deal with FDEs and obtain analytical and numerical results [6]. The Tau method, the Pseudo-spectral method, and the wavelet method based on the Legendre polynomials have been implemented for several types of FDEs [2].

The fractional Sturm-Liouville eigenvalue problem was studied earlier [9] and [10]. In [9], the existence of a solution to such boundary value problem was established. In [10], the aforementioned relation between eigenvalues and zeros of Mittag-Leffler function was shown. The Adomian decomposition method was established for estimating fractional second order eigenvalues [11, 12]. The Homotopy Analysis method has been used to numerically approximate the eigenvalues of the fractional Sturm-Liouville Problems [13]. In [14], fractional differential transform method used to approximate the eigenvalues of Sturm–Liouville problems of fractional order. Fourier series were used in [15], the method of Haar wavelet operational matrix was used in [16] and [17]. In [18]-[20], [21], extended some spectral

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properties of fractional Sturm-Liouville problem. Variational Methods and Inverse Laplace transform method applied in [23] and [24], respectively. In [25], it is presented a series method for solving higher eigenvalue Sturm–Liouville problems. Recently P. Antunes and R. Ferreira constructed numerical schemes using radial basis functions [26], B. Jin et al used Galerkin finite element method to solve fractional eigenvalue problems [27].
In this paper, we discuss the following regular fractional Sturm-Liouville problem of the form

$$D^{\alpha}[p(x)D^{\alpha}u(x)]+\lambda q(x)u(x)=\mu(x),\ 0\leq x\leq 1, \frac{1}{2}<\alpha\leq 1, \tag{1.1}$$

subject to

$$a_0u(0)+a_1\,D^{\alpha}u(0) = 0,\ a_0^2+a_1^2>0, \tag{1.2}$$
$$a_2u(1)+a_3\,D^{\alpha}u(1) = 0,\ a_2^2+a_3^2>0, \tag{1.3}$$

where a_0,a_1,a_2,a_3 are constants, $p(x),q(x)$, $r(x)$ are continuous functions with $p(x),q(x)>0$ for all $x\in[0,1]$, and D^{α} is the Caputo fractional derivative.
Historically, problem (1.1)-(1.2) had been studied theoretically for $\alpha=1$ by [28] and [29] who showed that it has an infinite sequence of eigenvalues $\{\lambda_0,\lambda_1,\lambda_2,...\}$ with the following property

$$\eta<\lambda_0\leq\lambda_1\leq\lambda_2\leq...$$

where

$$\lim_{n\to\infty}\lambda_n=\infty,$$

η is a constant and each eigenvalue has multiplicity at most 3. However, the numerical treatments of such problems have always been far from trivial which, therefore, attracts several authors to initiate or apply different numerical methods to investigate their solution. For instance, Lesnic and Attili [33] used the Adomian decomposition method (ADM) whereas Greenberg and Marletta [30]-[32] developed their own code using Theta Matrices (SLEUTH). Syam and Siyyam [34] implemented the iterated variation method. The present work is motivated by approximating the eigenvalues of problem (1.1)-(1.2) using reproducing kernel method (RKM).
This paper is organized as follows. In section 2, we present some preliminaries which we will use in this paper. A description of the RKM for discretization of the fractional 2α-order Sturm-Liouville problem (1.1)-(1.2) is presented in section 3. Several numerical examples and conclusions are discussed in Section 4. Conclusions and closing remarks are given in Section 5.

2 Preliminaries

In this section, we review the definition and some preliminary results of the Caputo fractional derivatives, as well as, the definition of the Rimann-Liouville fractional and their properties.

Definition 2.1. *A real function $f(t),t>0$,is said to be in the space $C_{\mu},\mu\in\mathbb{R}$ if there exists a real number $p>\mu$, such that $f(t)=t^pf_1(t)$, where $f_1(t)\in C[0,\infty)$, and it is said to be in the space C_{μ}^m if $f^{(m)}\in C_{\mu}$, $m\in\mathbb{N}$.*

Definition 2.2. *The left Riemann-Liouville fractional integral of order $\delta\geq 0$, of a function $f\in C_{\mu},\mu\geq -1$, is defined by*

$$I^{\delta}f(t)=\begin{cases}\frac{1}{\Gamma(\delta)}\int_0^t(t-s)^{\delta-1}f(s)ds, & \delta>0,\\ f(t), & \delta=0.\end{cases} \tag{2.4}$$

Definition 2.3. *For $\delta>0$, $m-1<\delta<m$, $m\in\mathbb{N},t>0$, and $f\in C_{-1}^m$, the left Caputo fractional derivative is defined by*

$$D^{\delta}f(t)=\begin{cases}\frac{1}{\Gamma(m-\delta)}\int_0^t(t-s)^{m-1-\delta}f^{(m)}(s)ds, & \delta>0,\\ f'(t), & \delta=0,\end{cases} \tag{2.5}$$

where Γ is the well-known Gamma function.

The Caputo derivative defined in (2.5) is related to the Riemann-Liouville fractional integral, I^{δ}, of order $\delta \in \mathbb{R}^{+}$, by $D^{\delta} f(t) = I^{m-\delta} f^{(m)}(t)$. The Caputo fractional derivative satisfy the following properties for $f \in C_{\mu},\ \mu \geq -1$ and $\alpha \geq 0$, see [36].

1. $D^{\alpha} I^{\alpha} f(t) = f(t)$,
2. $I^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}$,
3. $D^{\alpha} c = 0$, where c is constant,
4. $D^{\alpha} t^{\gamma} = \left\{ \begin{array}{cc} 0, & \gamma < \alpha, \gamma \in \{0,1,2,...\} \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & \text{otherwise} \end{array} \right\}$,
5. $D^{\alpha}\left(\sum_{k=0}^{m} c_i f_i(t)\right) = \sum_{k=0}^{m} c_i\, D^{\alpha} f_i(t)$, where $c_1, c_2, ..., c_m$ are constants.

3 Analysis of RKM for a class of fractional second-order Sturm-Liouville Problems

In this section, we discuss the numerical solution of the following class of fractional 2α-order Sturm-Liouville Problems using RKM:

$$D^{\alpha}[p(x) D^{\alpha} u(x)] + \lambda q(x) u(x) = \mu(x),\ 0 < x < 1, \frac{1}{2} < \alpha \leq 1, \tag{3.6}$$

subject to

$$a_0 u(0) + a_1\, D^{\alpha} u(0) \ = \ 0,\ a_0^2 + a_1^2 > 0, \tag{3.7}$$
$$a_2 u(1) + a_3\, D^{\alpha} u(1) \ = \ 0,\ a_2^2 + a_3^2 > 0, \tag{3.8}$$

where a_0, a_1, a_2, a_3 are constants, $p(x), q(x)$, and $r(x)$ are continuous with $p(x), r(x) > 0$ for all $x \in [0,1]$. Assume that

$$u(0) = \tau_0, u(1) = \tau_1, y(x) = u(x) + (-\tau_1 + \tau_0)x - \tau_0.$$

Then, Problem (3.1)-(3.3) becomes

$$D^{\alpha}[p(x) D^{\alpha} y(x)] + \lambda q(x) y(x) = r(x),\ 0 < x < 1, \frac{1}{2} < \alpha \leq 1, \tag{3.9}$$

subject to

$$y(0) = 0,\ y(1) = 0, \tag{3.10}$$

where $r(x) = \mu(x) - \frac{(-\tau_1+\tau_0)}{\Gamma(2-\alpha)} D^{\alpha}[x^{1-\alpha} p(x)] - \lambda[(-\tau_1 + \tau_0)x - \tau_0] q(x)$. Applying the operator I^{α} to both sides of Equation (3.4) to get

$$p(x) D^{\alpha} y(x) - p(0)\left(D^{\alpha} y(x)\right)|_{x=0} + \lambda I^{\alpha}[q(x) y(x)] = I^{\alpha}[r(x)]$$

or

$$D^{\alpha} y(x) + \frac{\lambda}{p(x)} I^{\alpha}[q(x) y(x)] = \frac{I^{\alpha}[r(x)] + p(0)\left(D^{\alpha} y(x)\right)|_{x=0}}{p(x)}. \tag{3.11}$$

Applying the operator I^{α} to both sides of Equation (3.6) yields

$$y(x) + I^{\alpha}\left[\frac{\lambda}{p(x)} I^{\alpha}[q(x) y(x)]\right] = f(x) \tag{3.12}$$

where $f(x) = I^{\alpha}\left(\frac{I^{\alpha}[r(x)] + p(0)(D^{\alpha} y(x))|_{x=0}}{p(x)}\right)$.

In order to solve problem (3.5) and (3.7), we construct kernel Hilbert space $W_2^3[0,1]$ in which every function satisfy the boundary conditions (3.4). First, we define the reproducing kernel.

Definition 3.1. *Let A be a nonempty abstract set. A function $K : A\times A \to C$ is a reproducing kernel of the Hilbert space H if*

- $K(.,x) \in H$ *for all* $x \in A$,
- $(y(.),K(.,x)) = y(x)$ *for all* $x \in A$ *and* $y \in H$.

The second condition which possesses a reproducing kernel is called a reproducing kernel Hilbert space (RKHS). For more details, see [35].
Let

$$\begin{aligned} W_2^3[0,1] &= \{y(x):\ y, y',\ \text{and } y'' \text{ are absolutely continuous real-valued functions,} \\ y''' &\in L^2[0,1],\ y(0)=0, y(1)=0\}. \end{aligned}$$

The inner product in $W_2^3[0,1]$ *is defined as*

$$(y(t),u(t))_{W_2^3[0,1]} = y(0)u(0) + y'(0)u'(0) + y(1)u(1) + \int_0^1 y'''(t)u'''(t)dt,$$

and the norm $\|y\|_{W_2^3[0,1]}$ *is given by*

$$\|y\|_{W_2^3[0,1]} = \sqrt{(y(t),y(t))_{W_2^3[0,1]}}$$

where $y,u \in W_2^3[0,1]$.

Theorem 3.1. *The space* $W_2^3[0,1]$ *is a reproducing kernel Hilbert space, i.e.; there exists* $K(x,t) \in W_2^3[0,1]$ *such that for any* $u \in W_2^3[0,1]$ *and each fixed* $x,t \in [0,1]$, *we have*

$$(y(x),K(x,t))_{W_2^3[0,1]} = y(x).$$

In this case, $K(x,t)$ *is given by*

$$K(x,t) = \left\{ \begin{array}{ll} \sum_{i=0}^{5} c_i(x)t^i, & t \le x \\ \sum_{i=0}^{5} d_i(x)t^i, & t > x \end{array} \right\}.$$

where

$$\begin{aligned} c_0 &= 0,\ c_1 = x+x^2,\ c_2 = -\frac{120x+107x^2+10x^3-5x^4+x^5}{120}, \\ c_3 &= 0, c_4 = -\frac{x+x^2}{24},\ c_5 = \frac{1+2x^2}{120}, \\ d_0 &= \frac{x^5}{120},\ d_1 = \frac{24x+24x^2-x^4}{24},\ d_2 = -\frac{120x+107x^2-5x^4+x^5}{120}, \\ d_3 &= -\frac{x^2}{12},\ d_4 = -\frac{x^2}{24},\ d_5 = \frac{x^2}{60}. \end{aligned}$$

Proof. Using the integration by parts, we have

$$\begin{aligned} (y(t),K(x,t))_{W_2^3[0,1]} &= y(0)K(x,0) + y'(0)K_t(x,0) + y(1)K(x,1) + y''(1)\frac{\partial^3 K}{\partial t^3}(x,1) \\ &-y''(0)\frac{\partial^3 K}{\partial t^3}(x,0) - y'(1)\frac{\partial^{iv} K}{\partial t^{iv}}(x,1) + y'(0)\frac{\partial^{iv} K}{\partial t^{iv}}(x,0) + y(1)\frac{\partial^{v} K}{\partial t^{v}}(x,1) \\ &-y(0)\frac{\partial^{v} K}{\partial t^{v}}(x,0) - \int_0^1 y(t)\frac{\partial^{vi} K}{\partial t^{vi}}(x,t). \end{aligned}$$

Since $y(t)$ and $K(x,t) \in W_2^3[0,1]$,

$$y(0) = y(1) = 0$$

and

$$K(x,0)=K(x,1)=0. \tag{3.13}$$

Thus,

$$\begin{aligned}(y(t),K(x,t))_{W_2^3[0,1]} &= y'(0)K_t(x,0)+y''(1)\frac{\partial^3 K}{\partial t^3}(x,1)\\ &-y''(0)\frac{\partial^3 K}{\partial t^3}(x,0)-y'(1)\frac{\partial^{iv} K}{\partial t^{iv}}(x,1)+y'(0)\frac{\partial^{iv} K}{\partial t^{iv}}(x,0)\\ &-\int_0^1 y(t)\frac{\partial^{vi} K}{\partial t^{vi}}(x,t).\end{aligned}$$

Since $K(x,t)$ is a reproducing kernel of $W_2^3[0,1]$,

$$(y(t),K(x,t))_{W_2^3[0,1]}=u(x)$$

which implies that

$$\frac{\partial^{vi} K}{\partial t^{vi}}(x,t)=\delta(x-t) \tag{3.14}$$

where δ is the Dirac-delta function and

$$K_t(x,0)+\frac{\partial^{iv} K}{\partial t^{iv}}(x,0) = 0, \tag{3.15}$$

$$\frac{\partial^3 K}{\partial t^3}(x,1) = 0, \tag{3.16}$$

$$\frac{\partial^3 K}{\partial t^3}(x,0) = 0, \tag{3.17}$$

$$\frac{\partial^{iv} K}{\partial t^{iv}}(x,1) = 0. \tag{3.18}$$

Since the characteristic equation of $\frac{\partial^{vi} K}{\partial t^{vi}}(x,t)=\delta(x-t)$ is $\xi^6=0$ and its characteristic value is $\xi=0$ with 6 multiplicity roots, we write $K(\eta,y)$ as

$$K(x,t)=\left\{\begin{array}{ll}\sum_{i=0}^5 c_i(x)t^i, & t\le x\\ \sum_{i=0}^5 d_i(x)t^i, & t>x\end{array}\right\}.$$

Since $\frac{\partial^{vi} K}{\partial t^{vi}}(x,t)=\delta(x-t)$, we have

$$\frac{\partial^m K}{\partial t^m}(x,x+0)=\frac{\partial^m K}{\partial t^m}(x,x-0),\ m=0,1,2,3,4. \tag{3.19}$$

On the other hand, Integrating $\frac{\partial^{vi} K}{\partial t^{vi}}(x,t)=\delta(x-t)$ from $x-\varepsilon$ to $x+\varepsilon$ with respect to t and letting $\varepsilon\to 0$ to get

$$\frac{\partial^v K}{\partial t^v}(x,x+0)-\frac{\partial^v K}{\partial t^v}(x,x-0)=-1. \tag{3.20}$$

Using the conditions (3.8), (3.10)-(3.15), we get the following system of equations

$$\begin{aligned} c_0(x) &= 0,\ \sum_{i=0}^{5} d_i(x) = 0, \\ c_1(x)+24c_4(x) &= 0, 6d_3(x)+12d_4(x)+60d_5(x) = 0, 6c_3(x) = 0, \\ 24d_4(x)+60d_5(x) &= 0, \sum_{i=0}^{5} c_i(x)x^i = \sum_{i=0}^{5} d_i(x)x^i, \\ \sum_{i=1}^{5} ic_i(x)x^{i-1} &= \sum_{i=0}^{5} id_i(x)x^{i-1}, \sum_{i=2}^{5} i(i-1)c_i(x)x^{i-2} = \sum_{i=2}^{5} i(i-1)d_i(x)x^{i-2}, \\ \sum_{i=3}^{5} i(i-1)(i-2)c_i(x)x^{i-3} &= \sum_{i=3}^{5} i(i-1)(i-2)d_i(x)x^{i-3}, \\ \sum_{i=4}^{5} i(i-1)(i-2)(i-3)c_i(x)x^{i-4} &= \sum_{i=4}^{5} i(i-1)(i-2)(i-3)d_i(x)x^{i-4}, \\ 5!d_5(x)-5!c_5(x) &= -1. \end{aligned} \tag{3.21}$$

We solved system (3.16) using Mathematica to get

$$\begin{aligned} c_0 &= 0,\ c_1 = x+x^2,\ c_2 = -\frac{120x+107x^2+10x^3-5x^4+x^5}{120}, \\ c_3 &= 0, c_4 = -\frac{x+x^2}{24},\ c_5 = \frac{1+2x^2}{120}, \\ d_0 &= \frac{x^5}{120},\ d_1 = \frac{24x+24x^2-x^4}{24},\ d_2 = -\frac{120x+107x^2-5x^4+x^5}{120}, \\ d_3 &= -\frac{x^2}{12},\ d_4 = -\frac{x^2}{24},\ d_5 = \frac{x^2}{60}. \end{aligned}$$

Next, we study the space $W_2^1[0,1]$. Let

$$W_2^1[0,1] = \{y(x):\ y \text{ are absolutely continuous real-valued functions, } y' \in L^2[0,1]\}.$$

The inner product in $W_2^1[0,1]$ is defined as

$$(y(t),u(t))_{W_2^1[0,1]} = y(0)u(0)\ + \int_0^1 y'(t)u'(t)dt$$

and the norm $\|y\|_{W_2^1[0,1]}$ is given by

$$\|y\|_{W_2^1[0,1]} = \sqrt{(y(t),y(t))_{W_2^1[0,1]}}$$

where $y,u \in W_2^1[0,1]$. □

Theorem 3.2. *The space $W_2^1[0,1]$ is a reproducing kernel Hilbert space, i.e.; there exists $R(x,t) \in W_2^1[0,1]$ such that for any $y \in W_2^1[0,1]$ and each fixed $x,t \in [0,1]$, we have*

$$(y(t),R(x,t))_{W_2^1[0,1]} = y(x).$$

In this case, $R(x,t)$ is given by

$$R(x,t) = \left\{ \begin{array}{ll} 1+t, & t \le x \\ 1+x, & t > x \end{array} \right\}.$$

Proof. Using integration by parts, we have

$$\begin{aligned} (y(t), R(x,t))_{W_2^1[0,1]} &= y(0)R(x,0) + \int_0^1 y'(t)\frac{\partial R}{\partial t}(x,t)dy \\ &= y(0)R(x,0) + y(1)\frac{\partial R}{\partial t}(x,1) - y(0)\frac{\partial R}{\partial t}(x,0) - \int_0^1 y(y)\frac{\partial^2 R}{\partial t^2}(x,t)dt. \end{aligned}$$

Since $R(x,t)$ is a reproducing kernel of $W_2^1[0,1]$, we have

$$(y(t), R(x,t))_{W_2^1[0,1]} = y(x)$$

which implies that

$$-\frac{\partial^2 R}{\partial t^2}(x,t) = \delta(t-x) \tag{3.22}$$

and

$$R(x,0) - \frac{\partial R}{\partial t}(x,0) = 0, \tag{3.23}$$

$$\frac{\partial R}{\partial t}(x,1) = 0. \tag{3.24}$$

Since the characteristic equation of $-\frac{\partial^2 R}{\partial t^2}(x,t) = \delta(t-x)$ is $\zeta^2 = 0$ and its characteristic value is $\zeta = 0$ with 2 multiplicity roots, we write $R(x,t)$ as

$$R(x,t) = \left\{ \begin{array}{ll} c_0(x) + c_1(x)t, & t \leq x \\ d_0(x) + d_1(x)t, & t > x \end{array} \right\}.$$

Since $\frac{\partial^2 R}{\partial t^2}(x,t) = -\delta(t-x)$, we have

$$R(x,x+0) - R(x,x+0) = 0 \tag{3.25}$$

$$\frac{\partial R}{\partial y}(x,x+0) - \frac{\partial R}{\partial y}(x,x+0) = -1 \tag{3.26}$$

Using the conditions (3.18)-(3.21), we get the following system of equations

$$\begin{aligned} c_0(x) - c_1(x) &= 0, \\ d_1(x) &= 0, \\ c_0(x) + c_1(x)\,x &= d_0(x) + d_1(x)\,x, \\ d_1(x) - c_1(x) &= -1 \end{aligned} \tag{3.27}$$

which implies that

$$c_0(x) = 1,\ c_1(x) = 1, d_0(x) = 1 + x,\ d_1(x) = 0.$$

and the proof is completed. □

Now, we will present how to solve Problem (3.5) and (3.7) using the reproducing kernel method. Let

$$\sigma_i(x) = R(x_i, x)$$

for $i = 1, 2, ...$ It is clear that $L : W_2^2[0,1] \to W_2^1[0,1]$ is bounded linear operator where

$$L[y](x) = y(x) + I^{\alpha}\left[\frac{\lambda}{p(x)} I^{\alpha}[q(x)y(x)]\right] = f(x). \tag{3.28}$$

Let

$$\psi_i(x) = L^*\sigma_i(x)$$

where L^* is the adjoint operator of L. Using Gram-Schmidt orthonormalization to generate orthonormal set of functions $\left\{\bar{\psi}_i(x)\right\}_{i=1}^{\infty}$ where

$$\bar{\psi}_i(x) = \sum_{j=1}^{i} \alpha_{ij}\psi_j(x) \tag{3.29}$$

and α_{ij} are the coefficients of Gram-Schmidt orthonormalization.

Theorem 3.3. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$, *then*

$$y(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{i} \alpha_{ij} f(x_i)\bar{\psi}_i(x). \tag{3.30}$$

Proof. First, we want to prove that $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,1]$ and $\psi_i(x) = L(K(x,x_i))$. It is clear that $\psi_i(x) \in W_2^3[0,1]$ for $i = 1,2,\ldots$. Simple calculations implies that

$$\begin{aligned} \psi_i(x) &= L^*\sigma_i(x) = (L^*\sigma_i(x), K(x,t))_{W_2^3[0,1]} \\ &= (\sigma_i(x), L(K(x,t)))_{W_2^3[0,1]} = L(K(x,x_i)). \end{aligned}$$

For each fixed $y(x) \in W_2^3[0,1]$, let

$$(y(x), \psi_i(x))_{W_2^3[0,1]} = 0,\ i = 1,2,\ldots.$$

Then

$$\begin{aligned} (y(x), \psi_i(x))_{W_2^3[0,1]} &= (y(x), L^*\sigma_i(x))_{W_2^3[0,1]} \\ &= (Ly(x), \sigma_i(x))_{W_2^3[0,1]} \\ &= Ly(x_i) = 0. \end{aligned}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, $Ly(x) = 0$. Since L^{-1} exists, $y(x) = 0$. Thus, $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,1]$. Second, we prove that equation (3.25) holds true. Simple calculations implies that

$$\begin{aligned} y(x) &= \sum_{i=1}^{\infty} (y(x), \bar{\psi}_i(x))_{W_2^3[0,1]} \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{i} \alpha_{ij}(y(x), L^*(K(x,x_i)))_{W_2^3[0,1]} \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{i} \alpha_{ij}(Ly(x), K(x,x_i))_{W_2^3[0,1]} \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{i} \alpha_{ij}(f(x), K(x,x_i))_{W_2^3[0,1]} \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{i} \alpha_{ij} f(x_i)\bar{\psi}_i(x) \end{aligned}$$

and the proof is completed. □

Let the approximate solution of Problem (3.7) be given by

$$y_N(x) = \sum_{i=1}^{N}\sum_{j=1}^{i} \alpha_{ij} f(x_i)\bar{\psi}_i(x). \tag{3.31}$$

In the next theorem, we want to show that $\{y_N(x)\}_{N=1}^{\infty}$ is uniformly convergent to $y(x)$.

Theorem 3.4. *If $y(x)$ and $y_N(x)$ are given as in (3.25) and (3.26), then $\{y_N(x)\}_{N=1}^{\infty}$ converges uniformly to $y(x)$.*

Proof. For any $x \in [0,1]$,

$$\begin{aligned} \|y(x)-y_N(x)\|^2_{W_2^3[0,1]} &= (y(x)-y_N(x), y(x)-y_N(x))_{W_2^3[0,1]} \\ &= \sum_{i=N+1}^{\infty} ((y(x),\bar{\psi}_i(x))_{W_2^3[0,1]}\bar{\psi}_i(x), (y(x),\bar{\psi}_i(x))_{W_2^3[0,1]}\bar{\psi}_i(x))_{W_2^3[0,1]} \\ &= \sum_{i=N+1}^{\infty} (y(x),\bar{\psi}_i(x))^2_{W_2^3[0,1]}. \end{aligned}$$

Thus,

$$\underset{x\in[0,1]}{Sup} \|y(x)-y_N(x)\|^2_{W_2^3[0,1]} = \underset{x\in[0,1]}{Sup} \sum_{i=N+1}^{\infty} (y(x),\bar{\psi}_i(x))^2_{W_2^3[0,1]}.$$

From Theorem (3.3), one can see that $\sum_{i=1}^{\infty}(y(x),\bar{\psi}_i(x))_{W_2^3[0,1]}\bar{\psi}_i(x)$ converges uniformly to $y(x)$. Thus,

$$\underset{N\to\infty}{Lim}\ \underset{x\in[0,1]}{Sup} \|y(x)-y_N(x)\|_{W_2^3[0,1]} = 0$$

which implies that $\{y_N(x)\}_{N=1}^{\infty}$ converges uniformly to $y(x)$.
To find the τ_0 and τ_1, we set

$$a_0\tau_0 + a_1\, D^{\alpha}y_N(0) = 0, a_2\ (\tau_1 + y_N(1)) + a_3\left(D^{\alpha}y_N(1) + \frac{(\tau_1-\tau_0)}{\Gamma(2-\alpha)}\right) = 0. \tag{3.32}$$

if $\frac{1}{2} < \alpha < 1$ and

$$a_0\tau_0 + a_1\left(y_N'(0) + \tau_1 - \tau_0\right) = 0, a_2\ (\tau_1 + y_N(1)) + a_3\left(y_N'(1) + \tau_1 - \tau_0\right) = 0. \tag{3.33}$$

if $\alpha = 1$. In this case, $y_N(x) = 0$ is a function of x and λ. To find the eigenvalues of Problem (3.1)-(3.3), we use the simple shooting method by setting $y_N(1) = 0$. In the next section, we sketch the graph of $y_N(1)$. When the graph intersects the $\lambda-$axis, this means we have an eigenvalue to problem (3.1)-(3.3). □

4 Numerical Results

In this section, we apply the RKM outlined in the previous section to solve numerically the following examples. Note that the maximum number of terms in the approximate series solution is taken as $N = 12$ for all examples considered in this paper. In this paper, we will focus only one the eigenvalues.

Example 4.1. *Consider the following fractional Sturm-Liouville problem*

$$D^{\alpha}[p(x)D^{\alpha}u(x)] + \lambda q(x)u(x) = r(x),\ 0 \le x \le 1, \frac{1}{2} < \alpha \le 1,$$

subject to

$$u(0) = 0,\ u(1) = 0,$$

where $p(x) = q(x) = 1$, and $r(x) = 0$.

Using the procedure described in the previous section, we scan the function $y_N(1)$ for λ on the interval $[0,\chi]$ where $y_N(1)$ approaches to infinity when λ approaches to χ. Figure 1 shows the graph of the the function $y_N(1)$ against the parameter λ. When the graph intersects the $\lambda-$axis, this means we have an eigenvalue. Then, we use the FindRoot command in Mathematica to find this root. The available results for λ obtained by the present method are summarized in Table 1.

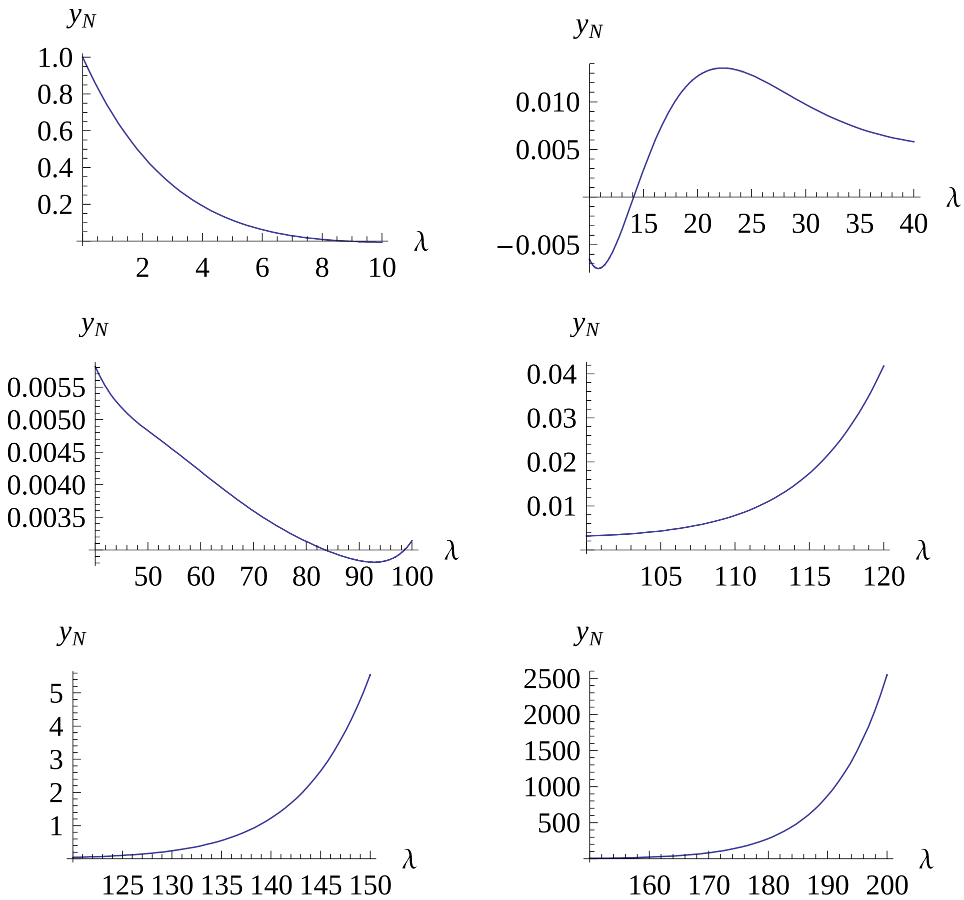

Figure 1: Graph of $y_N(1)$ for $\alpha = 0.75$

Figure 2 shows the graph of the eigenfunctions for $\alpha = 0.75$ and λ_1 and λ_2.

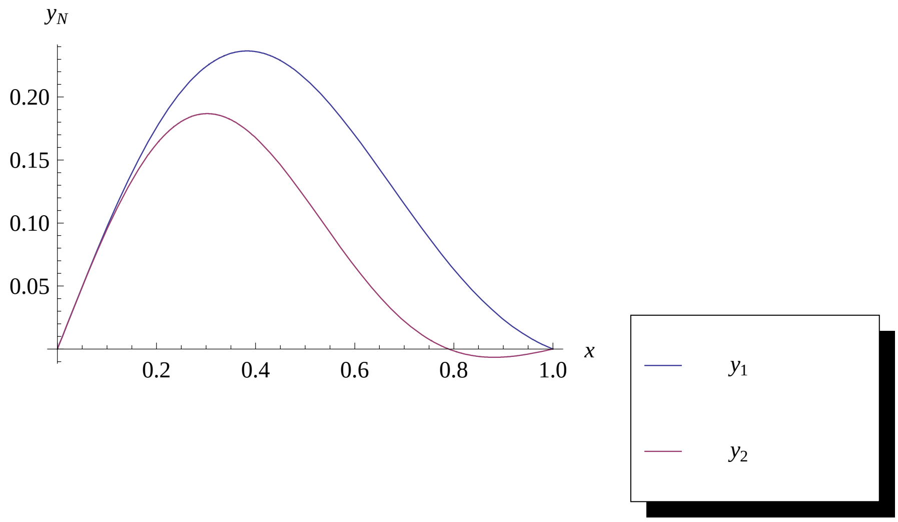

Figure 2: Graph of the eigenfunctions for $\alpha = 0.75$ and λ_1 and λ_2

Table 1: Eigenvalues for different values of α.

$\alpha = 0.501$	$\alpha = 0.75$	$\alpha = 0.95$	$\alpha = 0.99$
	8.782590560595702	8.271182644902353	9.663525870579704
	14.084450353939522	58.99016315983617	38.04408057881685
		96.67365275907841	84.97143809592463
		148.29535024361329	150.1337217020462
		199.57140250668647	233.5986357282636
		277.10713564792286	335.0977772309072
		295.4501496153064	454.7644065538165
			590.9308951941567

For $\alpha = 1$, the exact eigenvalues are well-known and they are given by

$$\lambda_n = n^2\pi^2, \;\; n = 1,2,3,....$$

It is worth mentioning that the eigenvalues of the problem in this example approaches to $n^2\pi^2$ when α approaches to 1. We noticed that the eigenvalue problem in Example (4.1) does not have any eigenvalue for $\alpha = 0.501$. For this reason, we look for the numerical value of α^* such that the eigenvalue problem of this example does not have any eigenvalue for $\frac{1}{2} < \alpha < \alpha^*$. We noticed that $\alpha^* = 0.7355$. Let

$$\delta_{i,j} = \left| \int_0^1 y_i(x)\; y_j(x)\; q(x)dx \right|.$$

For $\alpha = 0.75$, $\delta_{1,2} = 5.7 * 10^{-16}$. Sample of these values for $\alpha = 0.95$ are given as

$$\delta_{1,2} = 5.7 * 10^{-16}, \;\; \delta_{4,6} = 2.6 * 10^{-16}, \delta_{1,6} = 8.3 * 10^{-16}.$$

Similarly for $\alpha = 0.99$,

$$\delta_{1,2} = 3.1 * 10^{-16}, \;\; \delta_{4,6} = 4.2 * 10^{-16}, \delta_{1,7} = 2.0 * 10^{-16}.$$

This means, the orthogonality relation holds. We notice that the eigenvalues satisfy the property

$$\lambda_1 \leq \lambda_2 \leq ....$$

Example 4.2. *Consider the following fractional Sturm-Liouville problem*

$$D^{\alpha}[p(x)D^{\alpha}u(x)] + \lambda q(x)u(x) = r(x),\ 0 \leq x \leq 1, \frac{1}{2} < \alpha \leq 1,$$

subject to

$$u(0) = 0,\ u(1) = 0,$$

where $p(x) = 1, q(x) = 1 + x^{\alpha}$, *and* $r(x) = 0$.

Following the same procedure described in Example (4.1), we scan the function $y_N(1)$ for λ on the interval $[0,\chi]$ where $y_N(1)$ approaches to infinity when λ approaches to χ. The available results for λ obtained by the present are summarized in Table 2.

Table 2: Eigenvalues for different values of α.

$\alpha = 0.501$	$\alpha = 0.75$	$\alpha = 0.95$
3.7449684702770414	4.9059650882184220	5.8271192640206051
5.5935960731481420	9.9542383457076281	21.863097799385514
25.475119275691081	14.246865721715528	100.86879521196327
151.84584993600748	25.879708407281821	234.22568214592145
	124.47513819737418	439.20091275462871
		721.00934458721311
		984.12478134099354

Let

$$\delta_{i,j} = \left| \int_0^1 y_i(x)\ y_j(x)\ q(x)dx \right|.$$

For $\alpha = 0.502$, $\delta_{1,2} = 3.3 * 10^{-16}$ and $\delta_{2,4} = 4.9 * 10^{-16}$. Sample of these values for $\alpha = 0.75$ are given as

$$\delta_{1,2} = 2.2 * 10^{-16},\ \delta_{4,5} = 4.1 * 10^{-16}, \delta_{1,5} = 6.9 * 10^{-16}.$$

Similarly for $\alpha = 0.95$,

$$\delta_{1,2} = 1.2 * 10^{-16},\ \delta_{4,6} = 2.1 * 10^{-16}, \delta_{1,7} = 4.6 * 10^{-16}.$$

This means, the orthogonality relation holds. We notice that the eigenvalues satisfy the property

$$\lambda_1 \leq \lambda_2 \leq ....$$

Example 4.3. *Consider the following fractional Sturm-Liouville problem*

$$D^{\alpha}[p(x)D^{\alpha}u(x)] + \lambda q(x)u(x) = r(x),\ 0 \leq x \leq 1, \frac{1}{2} < \alpha \leq 1,$$

subject to

$$u(0) - D^{\alpha}y(0) = 0,\ u(1) + D^{\alpha}y(1) = 0,$$

where $p(x) = q(x) = 1$, *and* $r(x) = 0$.

Using the procedure described in Example (4.1), we scan the function $y_N(1)$ for λ on the interval $[0,\chi]$ where $y_N(1)$ approaches to infinity when λ approaches to χ. The available results for λ obtained by the present method are summarized in Table 3.

Table 3: Eigenvalues for different values of α.

$\alpha = 0.75$	$\alpha = 0.8$	$\alpha = 0.99$
1.5022830817370621	1.5178860852061056	1.6932293259045417
12.051003754885354	10.342640766463138	13.198585789687352
14.503581321319643	21.081583671397144	41.804367761190115
	44.238907623409132	88.779509288297120
		153.95456884426585
		237.42247998297339
		338.92086545541130
		458.58377852541501
		596.14001070105352
		752.04906711411381
		918.71319420262032

It worth mention that, there are eigenvalues for all $\frac{1}{2} < \alpha \leq 1$. For example,the first eigenvalue for $\alpha = 0.5001$ is 1.68861. Let

$$\delta_{i,j} = \left| \int_0^1 y_i(x)\, y_j(x)\, q(x) dx \right|.$$

For $\alpha = 0.75$, $\delta_{1,2} = 3.3 * 10^{-16}$ and $\delta_{2,3} = 4.9 * 10^{-16}$. Sample of these values for $\alpha = 0.8$ are given as

$$\delta_{1,2} = 1.6 * 10^{-16},\ \delta_{2,4} = 1.9 * 10^{-16}, \delta_{3,4} = 2.8 * 10^{-16}.$$

Similarly for $\alpha = 0.99$,

$$\delta_{1,2} = 3.2 * 10^{-16},\ \delta_{4,6} = 4.5 * 10^{-16}, \delta_{1,7} = 4.1 * 10^{-16}.$$

This means, the orthogonality relation holds. We notice that the eigenvalues satisfy the property

$$\lambda_1 \leq \lambda_2 \leq ....$$

5 Conclusion

In this paper, we study the eigenvalues of regular 2α-order fractional Sturm-Liouville problem for $\frac{1}{2} < \alpha \leq 1$. We used the RKM to approximate the eigenvalues. We present three examples. From these examples, we notice that our technique is very efficient for computing the eigenvalues of the fractional second order problems. We end this section by the following remarks.

- From Examples (4.1)-(4.3), we can find the eigenvalues with the following property

$$\lambda_1 < \lambda_2 < \lambda_3 < ... < \lambda_n < ....$$

- From Examples (4.1)-(4.3), the orthogonality property

$$\int_0^1 y_i(x)\, y_j(x)\, q(x) = 0,\ i \neq j$$

holds.

- The results in this paper confirm that RKM is a powerful and can be used in different fields of sciences and engineering.

- RKM is excellent tool due to rapid convergent.
- The existence and uniformly convergent are proven in Theorems (3.3) and (3.4).
- We do not compare our results with others because we are the first who discuss this class of eigenvalues.

Future work

- We state the following conjecture for the future work:

Conjecture

The eigenvalue problem in Example (4.1) does not have any eigenvalue for $\alpha < 0.7355$.

- Generalize the proposed method for higher order fractional Sturm-Liouville problems.

Acknowledgment

The authors would like to express their appreciation for the valuable comments of the reviewers.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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