Generalized finite-time function projective synchronization of two fractional-order chaotic systems via a modified fractional nonsingular sliding mode surface

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Abstract

This paper addresses the problem of generalized finite-time function projective synchronization (GFFPS) of fractional-order chaotic systems. A modified fractional nonsingular terminal sliding mode surface and an appropriate robust fractional sliding mode control law are proposed, taking into account the effects of model uncertainties and of the external disturbances. An appropriate Lyapunov functional candidate is used to prove the finite-time existence of the sliding motion. Compared with the existing nonsingular sliding mode surface, our sliding mode surface permits to reduce the settling time of synchronization. Finally, some numerical simulations taking into consideration the Gaussian white noise produced by the electrical line are presented to demonstrate the effectiveness and applicability of the proposed technique.

Keywords: Fractional-order, Finite-time function projective synchronization, Nonsingular sliding mode surface, Simplest chaotic circuit, Hyperchaotic Lorenz system.

1 Introduction

For about five decades, the many centuries old fractional calculus emerged from pure mathematical sciences to become a reality for other fields of applications like in physics and engineering [1], [2]. Since then, it has been shown...
that fractional-order dynamics can be observed for instance in viscoelastic systems, in dielectric polarization, in electromagnetic waves, just to name some [3]-[5]. Meanwhile another prolific field of science, namely chaos theory, has acquired a new development to address the challenging problem of chaos synchronization due to its potential applications in physics, chemistry, telecommunication, biology, medicine and so on [6]-[18]. The streaming of research within this area aims at achieving master-slave synchronization between two chaotic systems by choosing various kinds of methods to extend and enrich the pioneering work of Pecora and Carroll [19]. Recently Mainieri and Rehacek introduced a new concept of synchronization called projective synchronization [20]. Later on, Li studied a more general form of it called modified projective synchronization where the chaotic systems can synchronize up to a constant scaling matrix [21]. More recently, the concept of function projective synchronization (FPS) was introduced by some other researchers [22]-[24], where the master and slave systems could be synchronized but rather up to a scaling function. The FPS could be used to get more discretion in applications to secure communication, because it is obvious that the unpredictability of the scaling function in FPS can additionally enhance the security of communication. All these forms of synchronization were first observed for the integer-order chaotic systems, then, were developed for fractional-order systems within the last decade. The publication of several papers on projective synchronization of fractional-order systems [25]-[32] and on the function projective synchronization of fractional-order systems [33]-[34] confirms this tendency. In ref. [33], an adaptive function projective synchronization scheme between two entirely different fractional-order chaotic systems with uncertain system parameters has been demonstrated, while Zhou and Zhu [34] have investigated the function projective synchronization for fractional-order chaotic systems based on the stability theory of fractional-order systems and tracking control. All of the works mentioned above show the interest of the scientific community on the projective synchronization as well as on the function projective synchronization. Regrettably, these works on the projective synchronization of fractional-order chaotic systems have not presented any stability discussion for the sliding mode motion. The sliding mode controller has some attractive advantages, including: fast dynamic responses and good transient performance, external disturbance rejection and insensitivity to parameter variations and model uncertainties [35]-[36]. Futhermore, the stability of the error system has not been performed on the basis of the newly introduced fractional-order Lyapunov stability theory, these authors use the technique based on the equivalent Routh-hurwitz method for the fractional-order derivative system. These studies have guaranteed the convergence of the slave system trajectories towards the master system trajectories with an infinite settling time. However, from a practical point of view, it is more valuable to stabilize fractional-order chaotic system in a given finite-time rather than merely asymptotically. Moreover, it has now been identified that finite-time stabilization of dynamics systems gives rise to a high-precision performance besides finite-time convergence to zero [37],[38]. The problem of finite-time chaotic synchronization using the nonsingular sliding mode surface was solved by [39]. Here, the authors proposed a nonsingular sliding mode on the form \( S(t) = D^{\alpha - 1}e + D^{\alpha - 2}(Ke + K|e|^\beta \text{sign}(e)) \) with \( e = x - y \) adapted for the synchronization of fractional-order chaotic system. The problem of finite-time hybrid projective synchronization was very recently resolved by authors of ref. [40], who proposed the following nonsingular surface for the Hybrid projective synchronization of fractional systems: \( S(t) = D^{\alpha - 1}e + D^{\alpha - 2}(Ke + K|ae|^\beta \text{sign}(e)) \), where \( e = x - ay \), with \( a \) the scaling factor. This two nonsingular sliding surfaces are identical because they use a constant scaling.

Finally, to the best of our knowledge, there are still few works devoted to the generalized finite-time function projective synchronization for fractional-order chaotic systems using an appropriate fractional nonsingular terminal sliding mode surface. The above discussion inspired us to solve the GFFPS problem for the fractional-order chaotic systems in the presence of both model uncertainties and external disturbances. The modified nonsingular terminal sliding mode surface is introduced, its finite-time stability to zero is proved. The importance of such a surface is that, on one hand, it supports the robustness of synchronization in the presence of disturbances, uncertainties and Gaussian white noise; secondly, it is easy to be stabilized to zero and thirdly, it is very speedily converging to zero. So, on the basis of an appropriate Lyapunov function, a robust control law is designed to force the trajectories of the synchronization error system onto the sliding surface within a finite-time and ensure that it remains there forever. Numerical simulations demonstrate the applicability and efficiency of the nonlinear control law and verify the theoretical results of the paper.

In section 2, generalized finite-time function projective synchronization of two fractional chaotic systems scheme and the designed controller are presented. Section 3 is devoted to two examples used to verify the effectiveness of the proposed scheme. Finally, conclusions are drawn in section 4.
2 Generalized finite-time function projective synchronization of two fractional-order chaotic systems

2.1 Preliminaries

Consider the following two n-dimensional chaotic systems:

Driven system:

\[ D^\alpha x_i = g_i(t, x_i) \quad i = 1, \ldots, n \]  \hspace{1cm} (2.1)

Response system:

\[ D^\alpha y_i = f_i(t, y_i) + \Delta f_i(t, y_i) + d_i + u_i \quad i = 1, \ldots, n \]  \hspace{1cm} (2.2)

where \( u_i \in \mathbb{R}^n \) represent the nonlinear controllers, \( q_d \) and \( q_e \) are fractional-order satisfying \( 0 < q_d \leq 1, 0 < q_e \leq 1 \) respectively, \( \Delta f_i(t, y_i) \in R \) and \( d_i \in R \) represent unknown model uncertainty and external disturbances of the system.

Setting the error system as:

\[ e_i = y_i - \eta(t) x_i \quad i = 1, \ldots, n \]  \hspace{1cm} (2.3)

where \( \eta(t) \neq 0 \) is a scaling positive function, we can point out the following:

**Definition 2.1.** Drive system (2.1) and the response system (2.2) are GFFPS with respect to the scaling function \( \eta(t) \) if there exists a controller \( u_i(x, y, \eta(t)) \) such that:

\[ \lim_{t \to t_0} |e(t)| = \lim_{t \to t_0} \|y(t) - \eta(t)x(t)\| = 0; \text{ therefore the GFFPS between the system (2.1) and (2.2) is achieved.} \]

**Theorem 2.1.** [41] If \( x(t) \in C^1[0, T] \) for some \( T > 0 \), if \( D^\alpha \) is the Caputo fractional derivative, then

\[ D^\alpha D^\alpha x(t) = D^\alpha y_\alpha x(t), \quad t \in [0, T], \]

where \( q_1, q_2 \in R^+ \) and \( q_1 + q_2 \leq 1 \).

**Theorem 2.2.** [42] Let \( x = 0 \) be an equilibrium point for the nonautonomous fractional-order system \( D^\alpha x = w(t, x) \) where \( w(t, x) \) satisfies the Lipschitz condition with constant \( l > 0 \) and \( \alpha \in (0, 1) \). Assume that there exists a Lyapunov function \( V(t, x) \) satisfying:

\[ \alpha_1 \|x\|^d \leq V(t, x) \leq \alpha_2 \|x\|, \]

\[ V(t, x) \leq -\alpha_3 \|x\|. \]

Here, \( \alpha_1, \alpha_2, \alpha_3 \) and \( d \) are positive constants. Then the equilibrium point of the system \( D^\alpha x = w(t, x) \) is Mittag-Leffler stable.

**Assumption 1:** The function \( \eta(t) \) is bounded by \( m \leq \eta(t) \leq M \), with \( m \) and \( M \) positive constants.

**Assumption 2:** The uncertainty terms \( \Delta f_i(t, y_i) \) and external disturbance \( d_i \) are bounded by \( |\Delta f_i(t, y_i)| \leq \delta_i \) and \( |d_i| \leq \beta_i, i = 1, \ldots, n \). Here also, \( \delta_i \) and \( \beta_i \) are positive constants.

**Lemma 2.1.** [43] Assume \( c \) and \( b \) and \( 0 < b < 1 \) are real numbers, then the following inequality holds:

\[ (|c| + |b|)^p \leq |c|^p + |b|^p. \]

**Remark 2.1.** Due to the fact that fractional order \( q_d \) may be different from fractional order \( q_e \), and the two continuous vector functions \( f_i(t, x_i) \) and \( g_i(t, x_i) \) may be also different.

In the following, we only consider the Caputo definition of the fractional-order derivative.

2.2 Main results

In this subsection, a robust finite-time sliding mode controller which includes the fractional-order terms is designed to realize GFFPS between two chaotic systems. Two main steps are used to design the proposed finite-time controller. Firstly, we select a modified nonsingular terminal sliding surface for the desired sliding motion and secondly, we design an Finite-time control law to guarantee the existence of the sliding motion in a given finite-time. Consequently, in this paper, a modified nonsingular terminal sliding surface is proposed as:

\[ S_i(t) = D^{\alpha-1} e_i + D^{\alpha-2} (K_i e_i + K_i |\eta(t) e_i|^p \text{sign}(e_i)) \quad i = 1, \ldots, n \]  \hspace{1cm} (2.4)
Therefore, the state trajectories of the error system (2.6) will converge to $e(t) = 0$ in a finite-time $t_{s_1}$, defined by the inequality

$$t_{s_1} \leq \frac{1}{K(1-\mu)} \left[ \ln \left( 1 + \frac{\|e(0)\|^{1-\mu}}{m^n} \right) \right],$$

with $K = \min \{K_i\}$.

**Proof.** Consider the following positive defined Lyapunov function candidate

$$V_1 = \sum_{i=1}^{n} |e_i| = \|e\|.$$  (2.8)

The time derivative of this Lyapunov function along the trajectories of (2.6) is:

$$\dot{V}_1 = \sum_{i=1}^{n} \text{sign}(e_i) \dot{e}_i$$

$$= \sum_{i=1}^{n} \text{sign}(e_i) (D^{1-\mu} (D^{\mu} e_i))$$

$$= \sum_{i=1}^{n} \text{sign}(e_i) [D^{1-\mu} (K_i e_i + K_i |\eta(t)| \|e_i\|^\mu \text{sign}(e_i))]$$

$$= -\sum_{i=1}^{n} \text{sign}(e_i) (K_i e_i + K_i |\eta(t)| \|e_i\|^\mu \text{sign}(e_i))$$

$$\leq -K \sum_{i=1}^{n} |e_i| + m^\mu \sum_{i=1}^{n} |e_i|^\mu.$$  (2.9)

Using the above defined lemma, one can obtain

$$\dot{V}_1 \leq -K (\|e\| + m^\mu \|e\|^{\mu}) \leq -K \|e\|.$$  (2.10)

Hence, according to theorem 2.1, the error system (2.6) will converges asymptotically to zero. In order to show that the sliding motion occurs in finite-time, the convergence time can be determined as follows: From equation (2.8) and inequality (2.10),

$$\frac{d \|e\|}{dt} \leq -K (\|e\| + m^\mu \|e\|^{\mu}),$$  (2.11)

what leads to

$$dt \leq -\frac{1}{K(1-\mu)} \frac{d \|e\|^{1-\mu}}{\|e\|^{1-\mu} + m^\mu}. $$  (2.12)

Taking integral of both sides of (2.12) from 0 to $t_{s_1}$, and letting $e(t_{s_1}) = 0$, it can be deduced that $t_{s_1} \leq \frac{1}{K(1-\mu)} \left[ \ln \left( 1 + \frac{\|e(0)\|^{1-\mu}}{m^n} \right) \right].$

Therefore, the state trajectories of the error system (2.6) will converge to $e(t) = 0$ in the finite-time $t_{s_1} \leq \frac{1}{K(1-\mu)} \left[ \ln \left( 1 + \frac{\|e(0)\|^{1-\mu}}{m^n} \right) \right];$ this completes the proof.
Remark 2.2. It is worth noticing that the conventional quadratic form Lyapunov function candidates such as $V(t) = \|e(t)\|^2$ do not satisfy the stability conditions of Theorem 2.1. However, a Lyapunov function in the form of $V(t) = \|e(t)\|_1$ meets the conditions of Theorem 2.2. On the other hand, as mentioned in [45], non-smooth Lyapunov functions can be used to prove the finite-time convergence of a system. Therefore, selecting the non-smooth Lyapunov functions $V(t) = \|e(t)\|_1$ is practical and common in the literature.

Remark 2.3. In ref. [44], the authors proposed a nonsingular sliding mode in the form of $S_i(t) = D^{\mu - 1}e_i + D^{\mu - 2}(K_ie_i + K_i|e_i|^\mu \text{sign}(e_i))$, adapted for the synchronization of fractional-order chaotic system where $e_i = y_i - \eta(t)x_i$. Normally, the nonsingular sliding mode adapted for the GFFPS should be written in the same form but with $e_i = y_i - \eta(t)x_i$. In order to have a dynamic error converging more rapidly to zero over time, we introduce the supplementary scaling function such that the sliding surface become unpredictable and in the form: $S_i(t) = D^{\mu - 1}e_i + D^{\mu - 2}(K_ie_i + K_i|\eta(t)e_i|^\mu \text{sign}(e_i))$. Here, a good choice of the supplementary scaling function $\eta(t)$ permits to reduce the synchronization time.

Remark 2.4. We choose $\eta(t)$ such that $|\eta(t)|^\mu > 1$.

Remark 2.5. If we select the nonsingular sliding mode in the form of $S_i(t) = D^{\mu - 1}e_i + D^{\mu - 2}(K_ie_i + K_i|e_i|^\mu \text{sign}(e_i))$, with $e_i = y_i - \eta(t)x_i$, the settling time of synchronization becomes:

$$T \leq \frac{1}{\lambda(1-\mu)} \ln \left( 1 + \|e(0)\|^{-\mu} \right)$$

which is greater than $t_s$.

The next step consist to define a controller out of the additional compensation controller $u_\alpha$ and the controller $u_\beta$, leading to:

$$u_i = u_{ci} + u_{ai}. \quad (2.13)$$

with

$$u_{ci} = D^\mu (\eta(t)x_i) - f_i(t, \eta(t)x_i) - \Delta f_i(t, \eta(t)x_i). \quad (2.14)$$

Eqs.(2.2) can be rewritten as:

$$D^\mu y_i = f_i(t, y_i) + \Delta f_i(t, y_i) + d_i + D^\mu (\eta(t)x_i) - f_i(t, \eta(t)x_i) - \Delta f_i(t, \eta(t)x_i) + u_{ai}. \quad (2.15)$$

Consequently, the following set of equations redefines the synchronization errors

$$D^\mu e_i = f_i(t, y_i) - f_i(t, \eta(t)x_i) + \Delta f_i(t, y_i) - \Delta f_i(t, \eta(t)x_i) + d_i + u_{ai}. \quad (2.16)$$

Therefore, a control law which forces the error trajectories to go onto the sliding surface within a finite-time and remain on it forever is designed:

$$u_{ai} = f_i(t, \eta(t)x_i) - f_i(t, \eta(t)x_i) - \Delta f_i(t, y_i) - \Delta f_i(t, \eta(t)x_i) + d_i + u_{ai}. \quad (2.17)$$

Here, $i = 1, \ldots, n$, the $K_i$ are the sliding surface parameters to be introduced later. $\eta(t)$ denotes the scaling function contained in the synchronization errors, $\delta_i$ and $\beta_i$ are positive constants.

Theorem 2.4. If the error system (2.16) is controlled with the control law (2.17), then the states of the system will move towards the sliding surface and will approach the sliding surface $S_i(t) = 0$ in a finite-time $t_{s2}$ given by:

$$t_{s2} \leq \frac{1}{\epsilon(1-\eta)} \ln \left( \|S(0)\|^{\frac{1}{\epsilon}} + 1 \right). \quad (2.18)$$

Proof. Choosing a Lyapunov function in the form of

$$V_2 = \|S(t)\| = \sum_{i=1}^{n} |S_i|, \quad (2.19)$$
its time derivative is
\[ \dot{V}_2 = \sum_{i=1}^{n} \text{sign}(S_i) \dot{S}_i. \] (2.20)

Inserting (2.5) into (2.20) leads to
\[ V_2 = \sum_{i=1}^{n} \text{sign}(S_i) \left[ D^{\mu} e_i + D^{\mu-1} (K e_i + K_1 \eta(t) e_i |^\mu \text{sign}(e_i)) \right]. \] (2.21)

Inserting (2.16) into (2.21), \( \dot{V}_2 \) becomes:
\[ \dot{V}_2 = \sum_{i=1}^{n} \text{sign}(S_i) \left[ f_i(t, y_i) + \Delta f_i (t, y_i) + d_i - f_i(t, \eta(t) x_i) + u_{ai} + D^{\mu-1} (K e_i + K_1 |\eta(t) e_i|^\mu \text{sign}(e_i)) \right] + \]
\[ + \sum_{i=1}^{n} (|\Delta f_i (t, y_i)| + |d_i|) \text{sign}^2(S_i). \] (2.22)

Since the expression \( \text{sign}^2(S_i) = 1 \), eq.(2.22) can be converted into eq.(2.23) using eq.(2.18) as follows:
\[ V_2 \leq \sum_{i=1}^{n} \text{sign}(S_i) \left[ f_i(t, y_i) + u_{ai} - f_i(t, \eta(t) x_i) + D^{\mu-1} (K e_i + K_1 |\eta(t) e_i|^\mu \text{sign}(e_i)) \right] + \]
\[ + \sum_{i=1}^{n} (|\Delta f_i (t, y_i)| + |d_i|) \text{sign}^2(S_i). \] (2.23)

On the basis of Assumptions 1 and 2, it can be deduced that
\[ V_2 \leq \sum_{i=1}^{n} \text{sign}(S_i) \left[ f_i(t, y_i) + u_{ai} - f_i(t, \eta(t) x_i) + D^{\mu-1} (K e_i + K_1 |\eta(t) e_i|^\mu \text{sign}(e_i)) \right] + \]
\[ + \sum_{i=1}^{n} (\delta_i + \beta_i) \text{sign}^2(S_i). \] (2.24)

In other words,
\[ V_2 \leq \sum_{i=1}^{n} \text{sign}(S_i) \left[ f_i(t, y_i) + u_{ai} - f_i(t, \eta(t) x_i) + (\delta_i + \beta_i) \text{sign}(S_i) + D^{\mu-1} (K e_i + K_1 |\eta(t) e_i|^\mu \text{sign}(e_i)) \right]. \] (2.25)

Introducing (2.17) into (2.25), \( \dot{V}_2 \) can be reduced to
\[ \dot{V}_2 \leq \sum_{i=1}^{n} \text{sign}(S_i) \left[ -\varepsilon_i S_i - \varepsilon_i |S_i|^\eta \text{sign}(S_i) \right], \] (2.26)
and therefore.
\[ \dot{V}_2 \leq \sum_{i=1}^{n} [ -\varepsilon_i |S_i| - \varepsilon_i |S_i|^\eta]. \] (2.27)

which can be rewritten using the previous lemma as
\[ \dot{V}_2 \leq -\varepsilon \left( |S(t)| + |S(t)|^\eta \right), \] (2.28)
with \( \varepsilon = \min \{ \varepsilon_i \}. \)
Hence, according to theorem 2.1, the error system (2.16) will converge to \( S(t) = 0 \), asymptotically. In order to show that the sliding motion occurs in finite-time, the convergence time can be deduced from the inequality (2.28) as
\[ dt \leq - \frac{d |S(t)|}{\varepsilon \left( |S(t)| + |S(t)|^\eta \right) }; \] (2.29)
otherwise,
\[ dt \leq - \frac{1}{\varepsilon(1 - \eta)} \frac{d \|S(t)\|^{1-\eta}}{\|S(t)\|^{1-\eta} + 1}. \] (2.30)

The expression of the convergence time can be obtained if the integral function is applied on both sides of the inequality (2.30) from 0 to \( t_{s_2} \) and letting \( e(t_{s_2}) = 0 \): \( t_{s_2} \leq \frac{1}{\varepsilon(1 - \eta)} \ln \left( \|S(0)\|^{1-\eta} + 1 \right). \)

**Remark 2.6.** According to theorems 2.2 and 2.3, the sliding mode control law (2.17) and the sliding surface (2.4) can make the response system (2.2) reach the driven system (2.1) in the finite-time \( T_s = t_{s_1} + t_{s_2} \).

**Remark 2.7.** The sliding mode controller \( u_i \) (2.13) can easily be implemented in practice, using for example the Field Programmable Gate Arrays (FPGA) modules.

### 3 Numerical results

In this section, two examples of application of the defined GFFPS to two fractional chaotic systems via a nonsingular sliding mode surface are choosing. Their numerical simulations are performed for the purpose of illustration, that the proposed synchronization scheme is feasible. The numerical approach for solving fractional-order differential equations is that of Adams-Bashforth-Moulton predictor-corrector scheme, the detailed descriptions of this algorithm are available in [37].

In the first example, two identical systems with the same fractional-order are chosen. In the second one, two different systems with different fractional-orders are chosen. In both cases, the Gaussian white noise \( G = \sqrt{-2\ln(\text{rand})}\cos(2\pi\text{rand}) \) is used to test the robustness of the proposed synchronization scheme.

#### 3.1 Function projective synchronization of two identical fractional-order two-component circuits

It is not very long that the first two-component high frequency chaotic circuit made solely of simple and nonidealized components was introduced by one of us [46]. The present example is made around the fractional-order version of that very simple chaotic circuit. Two fractional-order such circuits with fractional-order \( q_d \) for the drive system, and \( q_r \) for the response system are considered. The mathematical expressions of these chaotic systems are described by the following nonlinear equations:

First drive (Two-component circuit):
\[
g_1(t,x) = \begin{pmatrix} g_1(t,x_1) \\ g_2(t,x_2) \\ g_3(t,x_3) \\ g_4(t,x_4) \end{pmatrix} = \begin{pmatrix} a_1(x_4 - x_3) - a_2[\exp(x_1) - 1] - a_3 g(x_1,x_2) \\ -\alpha(a_1x_4 - a_3g(x_1,x_2)) \\ b_1x_1 \\ b_2(e - x_1 + x_2) \end{pmatrix}. \quad (3.31)
\]

First slave (Two-component circuit):
\[
f_1(t,y) = \begin{pmatrix} f_1(t,y_1) \\ f_2(t,y_2) \\ f_3(t,y_3) \\ f_4(t,y_4) \end{pmatrix} = \begin{pmatrix} a_1(y_4 - y_3) - a_2[\exp(y_1) - 1] - a_3 g(y_1,y_2) + G \\ -\alpha(a_1y_4 - a_3g(y_1,y_2)) + G \\ b_1y_1 + G \\ b_2(e - y_1 + y_2) + G \end{pmatrix}. \quad (3.32)
\]

Here,
\[
g(u,v) = \begin{cases} 0 & \text{if } u \leq x_m \\ (u - x_m)^2 & \text{if } u < x_m, v < x_m \\ (u - v)(u + v - 2x_m) & \text{if } v \geq x_m. \end{cases} \quad (3.33)
\]
The uncertainties and external disturbance in eq.(2.2) are selected to be:
\[
\begin{align*}
\Delta f_1(t, y_1) &= d_1 = 0.1 \sin(0.1\pi y_1) + 0.1 \cos(0.1t), \\
\Delta f_2(t, y_2) &= d_2 = 0.1 \cos(0.1\pi y_2) + 0.1 \sin(0.1t), \\
\Delta f_3(t, y_1) &= d_3 = 0.1 \sin(0.1\pi y_3) + 0.1 \cos(0.1t), \\
\Delta f_4(t, y_2) &= d_4 = 0.1 \cos(0.1\pi y_4) + 0.1 \sin(0.1t).
\end{align*}
\]
(3.34)

The modified sliding nonsingular surface is described by:
\[
S_i(t) = \begin{pmatrix}
S_1(t) \\
S_2(t) \\
S_3(t) \\
S_4(t)
\end{pmatrix} = \begin{pmatrix}
D^{n-1}e_1 + D^{n-2}(K_1e_1 + K_1\eta(t)e_1^{\mu}\text{sign}(e_1)) \\
D^{n-1}e_2 + D^{n-2}(K_2e_2 + K_2\eta(t)e_2^{\mu}\text{sign}(e_2)) \\
D^{n-1}e_3 + D^{n-2}(K_3e_3 + K_3\eta(t)e_3^{\mu}\text{sign}(e_3)) \\
D^{n-1}e_4 + D^{n-2}(K_4e_4 + K_4\eta(t)e_4^{\mu}\text{sign}(e_4))
\end{pmatrix}.
\]
(3.35)

Fig.1a and fig.1b represent respectively, the phase portrait and corresponding Poincaré map of the driven system, for \(q = 0.96, a_1 = 53.5332, a_2 = 0.0000535, a_3 = 0.0587, \alpha = 1.1152, b_1 = 0.0051, b_2 = 0.0313, x_m = -56.36, e = 100\) and the initial conditions: \(x_1(0) = 0.0, x_2(0) = 0.0, x_3(0) = 0.05, \) and \(x_4(0) = 0.03\). This Poincaré map contains many points, hence denotes the chaotic behavior of the system.

![Figure 1](image1.png)

Figure 1: (a) Phase portrait of the fractional-order simplest two-component chaotic circuit [46] and (b) corresponding Poincaré section in the plane \((x_1, x_2)\) for \(q = 0.96, a_1 = 53.5332, a_2 = 0.0000535, a_3 = 0.0587, \alpha = 1.1152, b_1 = 0.0051, b_2 = 0.0313, x_m = -56.36, \) and \(e = 100\).

The scaling function is \(\eta(t) = 1.15 + 0.1\text{sign}(\sin(0.7\pi t))\); the fractional-order of the driven and slave system are fixed at \(q_d = q_s = 0.96\). The initial conditions for the drive are \(x_1(0)\) to \(x_4(0)\) given righ above and \(y_1(0) = -12, y_2(0) = -13, y_3(0) = 12, y_4(0) = 12\) for the response system, while the parameters of the controller are selected to be \(k_1 = k_2 = 5, k_3 = k_4 = 0.5, e_1 = e_2 = 500, e_3 = e_4 = 25, \) \(\eta = 0.95\), and \(\mu = 0.75\). The analytic time of synchronization is found to be \(T_n = 10.37\). The corresponding numerical results are shown in fig.2, fig.3 and fig.4 respectively. On these figures, it can be noted that the numerical time of synchronization is \(T_n = 10.98\), which respects the finite-time condition \(T_n \leq T_s\) [41]. Fig.2 depicts the graph of four sliding mode dynamics namely \(S_1\) in blue, \(S_2\) in red, \(S_3\) in green, and \(S_4\) in black. The sliding surfaces \(S_1\) and \(S_2\) converge to zero faster than \(S_3\), and \(S_4\) same behavior for dynamic errors, meaning that \(e_1\) and \(e_2\) converge to zero faster than \(e_3\) and \(e_4\) (see fig.3). We see that the recovered signal (blue curve) contains some noise which is due to the presence of several frequencies into a square signal. This noise can be reduced by using a suitable low pass filter (see fig.4), so that the recovered signal begins to follow the scaling function from \(t = T_s\), giving thus the possibility to appreciate the precision of the used technique. It can be concluded that for the function projective synchronization of the two identical fractional-order two-component systems using the square function, the proposed controller guarantees the convergence of the error system in a finite-time.
Figure 2: Time responses of the modified nonsingular sliding surface (3.35) in the case of example 1. On this graph it can be seen that the sliding surface converges very speedily to zero.

Figure 3: Time responses of the errors of synchronization in the case of example 1. Two identical fractional-order simplest two-component chaotic circuits synchronize at finite-time $T_n = 0.98$. 
The scaling function is now defined as $g_i(t,x)$ and of the slave function $f_i(t,x)$ are described by:

Second drive (Two-component circuit):

$$
\begin{align*}
\dot{g}_1(t,x) &= \frac{a_1(x_4 - x_3) - a_2[\exp(x_1) - 1] - a_3g(x_1,x_2)}{b_1x_1} \\
\dot{g}_2(t,x) &= \frac{\alpha(a_1x_4 - a_3g(x_1,x_2))}{b_2(e-x_1+x_2)} \\
\dot{g}_3(t,x) &= \frac{10(y_2 - y_1) + G}{28y_1 - y_2 - y_1y_3 + G} \\
\dot{g}_4(t,x) &= \frac{y_1y_2 - 8/3y_3 + G}{-y_2y_3 - y_4 + G}.
\end{align*}
$$

The analytic time of synchronization is $T_a = 0.0117$. The corresponding numerical result is depicted by fig.5, fig.6 and fig.7 respectively. These figures show that the numerical time of synchronization is $T_n = 0.098$, which confirms the theoretical prediction. On the graph of fig.7 it can be seen that the noise does not exist anymore because the scaling function is a sinusoidal function and as such, has only one frequency. Moreover it can be seen that the recovered signal (in blue) meets the initial scaling function after a short time $t = T_n$, meaning that the controller is indeed appropriate for this synchronization.
Figure 5: Time responses of the modified nonsingular sliding surface (3.35) in the case of example 2.

Figure 6: Time responses of the errors of synchronization in the case of example 2. On this graph it can be seen that two different fractional-order circuits synchronize via a sinus function at finite-time $T_n = 0.098$. 
4 Conclusion

In this paper the GFFPS via a modified nonsingular sliding mode surface was investigated in the presence of both model uncertainties and external disturbances. A modified nonsingular terminal fractional sliding surface was introduced and its finite-time convergence to zero was mathematically proven. On the basis of fractional Lyapunov stability theory we determined with success the analytic finite-time synchronization of two similar simplest two-component fractional-order chaotic circuits [46], then of one such simplest circuit with a fractional-order hyperchaotic Lorenz system [30], both systems having different fractional-order derivatives and different types of nonlinearities. The numerical investigation validated the analytic results as well as the robustness of the GFFPS technique. We believe that this controller can also be used for the finite-time synchronization of two integer-order chaotic systems. The practical implementation of the controller $u_i$ is our next prospect, also under the perspective of some other recent prototypes and models [48]-[51].

References


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