Numerical Computation of Fractional Second-Order Sturm-Liouville Problems

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Abstract
In this article, we investigate the eigenvalues of nonsingular fractional second-order Sturm-Liouville problem. The fractional derivative in this paper is in the conformable fractional derivative sense. We implement the reproducing kernel Hilbert space method to approximate the eigenvalues. Convergence of the proposed method is discussed. The main properties of the Sturm-Liouville problem are investigated. Numerical results demonstrate the accuracy of the present algorithm. Comparisons with other methods are presented.

Keywords: Eigenvalues, Fractional second-order Sturm-Liouville problem, Reproducing kernel Hilbert space method, Conformable fractional derivative.

1 Introduction
The Sturm-Liouville eigenvalue problem has several applications in modeling many physical problems. The theory of the problem is well developed and many results have been obtained concerning the eigenvalues and corresponding eigenfunctions. It should be noted that since finding analytical solutions for this problem is an extremely difficult task, several numerical algorithms have been developed to seek approximate solutions.


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Variational methods and Inverse Laplace transform method applied in [14-15], respectively. Recently P. Antunes and R. Ferreira [16] constructed numerical schemes using radial basis functions while Jin and et [17] used Galerkin finite element method to solve the problem.

In this paper, we develop a numerical technique for approximating the eigenvalues of the following non-singular fractional Sturm-Liouville problem of the form

\[ D^a \{ p(x) y'(x) \} + q(x) y = -\lambda w(x) y(x), \quad x \in I = [0,1], 0 < \alpha \leq 1 \]

subject to

\[ a_0 y(0) + a_1 y'(0) = 0, \]
\[ a_2 y(1) + a_3 y'(1) = 0, \]

where \( p, p', q, \) and \( w(x) \) are continuous functions on \([0,1]\) with \( p(x), w(x) > 0 \) for all \( x \in [0,1] \) and \( \det \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} \neq 0. \)

Here \( a_j \) (for \( j = 0, \ldots, 3 \)) are real constants such that

\[ a_0^2 + a_1^2 > 0, \]
\[ a_2^2 + a_3^2 > 0. \]

If the domain is \([a, b]\), then we use the following change of variable to make it \([0,1]\)

\[ x = (b - a) t + a. \]

For this reason, we assume that the domain is \([0,1]\). The fractional derivative here is in the conformable fractional derivative sense. The numerical solution of eigenvalue problems has received considerable interest in recent years because they have large number of applications in different areas of physics and engineering. A few examples of such applications are pendulums, vibrating and rotating shafts, viscous flow between rotating cylinders, the thermal instability of fluid spheres and spherical shells, earth's seismic behavior and ring structures; for more details, see [18], [19], [21], [26], [29], [31]. Note that Equation (1.1) is often referred to as the circular ring structure with constraints which has rectangular cross-sections of constant width and parabolic variable thickness; see [27] and [32].

Historically, problem (1.1) had been studied theoretically for \( \alpha = 1 \) by [23] who showed that it has an infinite sequence of eigenvalues \( \{\lambda_0, \lambda_1, \lambda_2, \ldots\} \) with the following property

\[ \eta < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \]

where

\[ \lim_{n \to \infty} \lambda_n = \infty, \]

and \( \eta \) is a constant and each eigenvalue has multiplicity at most 3. However, the numerical treatment of such problems has always been far from trivial which, therefore, attracts several authors to initiate or apply different numerical methods to investigate their solution. For instance, Lesnic and Attili [28] used the Adomian decomposition method (ADM) whereas Greenberg and Marletta [24] developed their own code using Theta Matrices (SLEUTH). Recently, Syam and Siyyam [30] implemented the iterated variation method.

The present work is motivated by approximating the eigenvalues of problem (1.1) using the Reproducing kernel Hilbert space method (RKHSM) which has better accuracy level. The RKHSM which accurately computes the series solution is of great interest to applied sciences. This technique gives the solution in a rapidly convergent series with components that can be easily computed. This method is used for the investigation of several scientific applications, see [20], [25], and [33].

This paper is organized as follows. In section 2, we present some preliminaries which we will use in this paper. A description of the RKHSM for discretization of the fractional second-order Sturm-Liouville problem (1.1) is presented in section 3. In addition, the existence and the uniformly convergent of the eigenfunctions are given in this section. Several numerical examples and comparisons with other methods are presented in Section 4. Conclusions and closing remarks are given in Section 5.
2 Preliminaries

In this section, we review the definition and some preliminary results of the conformable fractional derivatives as well as the α- fractional integral and their properties.

Definition 2.1.
Given a function $f : [0, \infty) \to \mathbb{R}$. Then the conformable fractional derivative of $f$ of order $\alpha$ is defined by

$$D^\alpha f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon}$$

for all $x > 0$, $0 < \alpha < 1$. If $f$ is $\alpha$-differentiable is some $(0, \alpha)$, $\alpha > 0$, and $\lim_{\alpha \to 0^+} D^\alpha f(x)$ exists, then define $D^\alpha f(0) = \lim_{\alpha \to 0^+} D^\alpha f(x)$.

Among the properties of the conformable fractional derivatives, we mention the following properties. Let $0 < \alpha < 1$ and $f$, $g$ be $\alpha$-differentiable at a point $x > 0$. Then,

- $D^\alpha[af + bg] = \alpha D^\alpha f(x) + b D^\alpha g(x)$, for all $a, b \in \mathbb{R}$.
- $D^\alpha x^p = p x^{p-\alpha} f$ for all $p \in \mathbb{R}$.
- $D^\alpha p = 0$ for all $p \in \mathbb{R}$.
- $D^\alpha[f g] = g D^\alpha f(x) + f D^\alpha g(x)$.
- $D^\alpha \left[\frac{f}{g}\right](x) = \frac{g(x)D^\alpha f(x) - f(x)D^\alpha g(x)}{g^2(x)}$ provided that $g(x) \neq 0$.
- $D^\alpha f(x) = x^{1-\alpha} f'(x)$.
- $D^1 f(x) = f'(x)$.

Next, we define the $\alpha$- fractional integral.

Definition 2.2.
The $\alpha$- fractional integral is defined by

$$I^\alpha f(x) = \int_0^x \frac{f(t)}{t^{1-\alpha}} \, dt,$$

where the integral is the Riemann improper integral and $\alpha \in (0, 1)$. For more details, see [22].

Among the properties of the $\alpha$- fractional integral, we mention the following property. If $f(x)$ is any continuous function in the domain of $I^\alpha$ and $x \geq 0$,

$$D^\alpha I^\alpha f(x) = f(x).$$

The reproducing kernel is given by this definition.

Definition 2.3.
Let $A$ be a nonempty set. A function $K : A \times A \to C$ is a reproducing kernel of the Hilbert space $H$ if and only if

- $K(., x) \in H$ for all $x \in A$,
- $(\varphi(., K(., x))) = \varphi(x)$ for all $x \in A$ and $\varphi \in H$.

The second condition is called the reproducing property and a Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space (RKHS); for more details, see [34].

3 Analysis of RKHSM for solving the eigenvalue problem

In this section, we discuss how to solve the following non-singular fractional Sturm-Liouville problem of the form
subject to
to

where \( p, p', q \), and \( w(x) \) are continuous functions on \([0,1]\) with \( p(x), w(x) > 0 \) for all \( x \in [0,1] \) and

Here \( a_j \) (for \( j = 0, \cdots, 3 \)) are real constants such that

If \( \alpha = 1 \), we get non-singular Sturm-Liouville problem of the form

The eigenvalues of the non-singular Sturm-Liouville problem (3.5) are well known. For this reason, we assume that \( 0 < \alpha < 1 \). Using the properties mentioned in the previous section, we can Equation (3.3) as

Assume that \( y(0) = \mu_1 \) and \( y'(0) = \mu_2 \). To homogenize these conditions, we assume that

Then, Equation (3.6) becomes

or

subject to

where

Since \( a_0^2 + a_1^2 > 0 \) and \( \mu_1 a_0 + \mu_2 a_1 = 0 \), we have the following two cases.

Using the first boundary condition, we can find the values of \( \mu_1 \) and \( \mu_2 \). We use the second boundary condition to find the eigenvalues of Equation (3.3). In order to solve problem (3.7)-(3.8), we construct the kernel Hilbert spaces \( W_2^1[0,1] \) and \( W_2^3[0,1] \) in which every function satisfy the boundary conditions (3.8).

Let

The inner product in \( W_2^3[0,1] \) is defined as

\[
(u(z), v(z))_{W_2^3[0,1]} = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + u'(1)v'(1) + \int_0^1 u^3(y)v^3(y)dy,
\]
and the norm $\|u\|_{W^3_z[0,1]}$ is given by

$$\|u\|_{W^3_z[0,1]} = \sqrt{(u(z), u(z))_{W^3_z[0,1]}}$$

where $u, v \in W^3_z[0,1]$.

**Theorem 3.1.** The space $W^3_z[0,1]$ is a reproducing kernel Hilbert space, i.e.; there exists $K(s, z) \in W^3_z[0,1]$ such that for any $u \in W^3_z[0,1]$ and each fixed $z, x \in [0,1]$, we have

$$(u(z), K(x, z))_{W^3_z[0,1]} = u(x).$$

In this case, $K(x, z)$ is given by

$$K(x, z) = \begin{cases} 
\sum_{i=0}^{5} c_i(x) z^i, & z \leq x \\
\sum_{i=0}^{5} d_i(x) z^i, & z > x 
\end{cases}$$

where

$$c_0 = 0, c_1 = 0, c_2 = \frac{1}{120} (5z^4 - 111z^2 - 10z^3 - z^5),$$

$$c_3 = 0, c_4 = -\frac{z}{24}, c_5 = \frac{1}{120} (1 + z^2),$$

$$d^0 = \frac{z^5}{120}, d_1 = -\frac{z^4}{24}, d_2 = \frac{1}{120} (5z^4 - 111z^2 - z^5),$$

$$d^3 = -\frac{z^2}{12}, d^4 = 0, d^5 = \frac{z^2}{120}.$$  

**Proof.** Using the integration by parts, one can get

$$(u(z), K(x, z))_{W^3_z[0,1]} = u(0)K(x, 0) + u(1)K(x, 1) + u'(0)K_x(x, 0) + u'(1)K_x(x, 1)$$

$$+ u''(1) \frac{\partial^3 K}{\partial z^3}(x, 1) - u''(0) \frac{\partial^3 K}{\partial z^3}(x, 0)$$

$$- u'(1) \frac{\partial^4 K}{\partial z^4}(x, 1) + u'(0) \frac{\partial^4 K}{\partial z^4}(x, 0) + u(1) \frac{\partial^5 K}{\partial z^5}(x, 1) - u(0) \frac{\partial^5 K}{\partial z^5}(x, 0)$$

$$- \int_0^1 u(z) \frac{\partial^6 K}{\partial z^6}(x, z) dz.$$  

Since $u(z)$ and $K(x, z) \in W^3_z[0,1]$,  

$u(0) = u'(0) = 0$  

and  

$K(x, 0) = K_x(x, 0) = 0$.  

Thus,

$$(u(z), K(x, z))_{W^3_z[0,1]} = u(1)K(x, 1) + u'(1)K_x(x, 1) + u''(1) \frac{\partial^3 K}{\partial z^3}(x, 1) - u''(0) \frac{\partial^3 K}{\partial z^3}(x, 0)$$

$$- u'(1) \frac{\partial^4 K}{\partial z^4}(x, 1) + u'(0) \frac{\partial^4 K}{\partial z^4}(x, 0) + u(1) \frac{\partial^5 K}{\partial z^5}(x, 1) - u(0) \frac{\partial^5 K}{\partial z^5}(x, 0)$$

$$- \int_0^1 u(z) \frac{\partial^6 K}{\partial z^6}(x, z) dz.$$  

Since $K(x, z)$ is a reproducing kernel of $W^3_z[0,1]$,  

$$(u(z), K(x, z))_{W^3_z[0,1]} = u(x)$$

which implies that
\[ \frac{\partial^6 K}{\partial z^6}(x, z) = \delta(x - z) \]  
(3.10)

where \( \delta \) is the dirac-delta function and

\[ K(x, 1) + \frac{\partial^5 K}{\partial z^5}(x, 1) = 0, \]  
(3.11)
\[ K_z(x, 1) - \frac{\partial^4 K}{\partial z^4}(x, 1) = 0, \]  
(3.12)
\[ \frac{\partial^3 K}{\partial z^3}(x, 1) = 0, \]  
(3.13)
\[ \frac{\partial^2 K}{\partial z^2}(x, 0) = 0. \]  
(3.14)

Since the characteristic equation of \( \frac{\partial^6 K}{\partial z^6}(x, z) = \delta(x - z) \) is \( \lambda^6 = 0 \) and its characteristic value is \( \lambda = 0 \) with 6 multiplicity roots, we write \( K(x, z) \) as

\[
K(x, z) = \begin{cases} 
\sum_{i=0}^{5} c_i(x)z^i, & z \leq x \\
\sum_{i=0}^{5} d_i(x)z^i, & z > x 
\end{cases}
\]

Since \( \frac{\partial^6 K}{\partial z^6}(x, z) = \delta(x - z) \), we have

\[
\frac{\partial^m K}{\partial z^m}(x, x + 0) = \frac{\partial^m K}{\partial z^m}(x, x - 0), m = 0, 1, \ldots, 4.
\]  
(3.15)

On the other hand, Integrating \( \frac{\partial^6 K}{\partial z^6}(x, z) = \delta(x - z) \) from \( x - \varepsilon \) to \( x + \varepsilon \) with expect to \( z \) and letting \( \varepsilon \to 0 \) to get

\[
\frac{\partial^5 K}{\partial z^5}(x, x + 0) - \frac{\partial^5 K}{\partial z^5}(x, x - 0) = -1.
\]  
(3.16)

Using the conditions (3.9), and (3.11)-(3.16), we get the following system of equations

\[ c_6(x) = 0, c_{1}(x) = 0, c_{3}(x) = 0, \]

\[ 6d_3(x) + 24d_4(x) + 60d_5(x) = 0, \]

\[ \sum_{i=1}^{5} i d_i(x) - 24d_4(x) - 12d_3(x) = 0, \]

\[ \sum_{i=0}^{5} c_i(x) x^i = \sum_{i=0}^{5} d_i(x) x^i, \]

\[ \sum_{i=1}^{5} ic_i(x) x^{i-1} = \sum_{i=1}^{5} id_i(x) x^{i-1}, \]

\[ \sum_{i=2}^{5} i(i - 1)c_i(x)x^{i-2} = \sum_{i=2}^{5} i(i - 1)d_i(x)x^{i-2}, \]

\[ \sum_{i=3}^{5} i(i - 1)(i - 2)c_i(x)x^{i-3} = \sum_{i=3}^{5} i(i - 1)(i - 2)d_i(x)x^{i-3}, \]

\[ \sum_{i=4}^{5} i(i - 1)(i - 2)(i - 3)c_i(x)x^{i-4} = \sum_{i=4}^{5} i(i - 1)(i - 2)(i - 3)d_i(x)x^{i-4}, \]

\[ 5!d_5(x) - 5!c_5(x) = -1. \]

We solved the last system using Mathematica to get

\[ c_0 = 0, c_1 = 0, c_2 = \frac{1}{120}(5z^4 - 111z^2 - 10z^3 - z^5), \]
\[ c_3 = 0, c_4 = -\frac{z}{24}, c_5 = \frac{1}{120}(1 + z^2), \]
\[ d^0 = \frac{z^5}{120}, d_1 = -\frac{z^4}{24}, d_2 = \frac{1}{120}(5z^4 - 111z^2 - z^5), \]
\[ d^3 = -\frac{z^2}{12}, d^4 = 0, d^5 = \frac{z^2}{120}, \]

which completes the proof of the theorem.

Next, we study the space \( W_2^1[0,1]. \) Let \( W_2^1[0,1] = \{ u(x): u \text{ are absolutely continuous real value functions, } u' \in L^2[0,1] \}. \)

The inner product in \( W_2^3[0,1] \) is defined as
\[
(u(z), v(z))_{W_2^3[0,1]} = u(0)v(0) + \int_0^1 u'(y)v'(y)dy,
\]
and the norm \( \|u\|_{W_2^3[0,1]} \) is given by
\[
\|u\|_{W_2^3[0,1]} = \sqrt{(u(z), u(z))_{W_2^3[0,1]}},
\]
where \( u, v \in W_2^3[0,1] \).

**Theorem 3.2.** The space \( W_2^1[0,1] \) is a reproducing kernel Hilbert space, i.e.; there exists \( R(s, z) \in W_2^1[0,1] \) such that for any \( u \in W_2^1[0,1] \) and each fixed \( z, x \in [0,1] \), we have
\[
(u(z), R(x, z))_{W_2^1[0,1]} = u(x).
\]

In this case, \( R(x, z) \) is given by
\[
R(x, z) = \begin{cases} 
1 + z & z \leq x \\
1 + x & z > x
\end{cases}
\]

**Proof.** Using the integration by parts, one can get
\[
(u(z), R(x, z))_{W_2^1[0,1]} = u(0)R(x, 0) + \int_0^1 u(z) \frac{\partial R}{\partial z}(x, z)dz
\]
\[
= u(0)R(x, 0) + u(1) \frac{\partial R}{\partial z}(x, 1) - u(0) \frac{\partial R}{\partial z}(x, 0) - \int_0^1 u(z) \frac{\partial^2 R}{\partial z^2}(x, z)dz.
\]

Since \( R(x, z) \) is a reproducing kernel of \( W_2^1[0,1] \),
\[
(u(z), R(x, z))_{W_2^1[0,1]} = u(x)
\]
which implies that
\[
-\frac{\partial^2 R}{\partial z^2}(x, z) = \delta(z - x)
\]
(3.17)

and
\[
R(x, 0) - \frac{\partial R}{\partial z}(x, 0) = 0,
\]
(3.18)
\[
\frac{\partial R}{\partial z}(x, 1) = 0.
\]
(3.19)

since the characteristic equation of \(- \frac{\partial^2 R}{\partial z^2}(x, z) = \delta(z - x) \) is \( \lambda^2 = 0 \) and its characteristic value is \( \lambda = 0 \) with 2 multiplicity roots, we write \( R(x, z) \) as
\[
R(x, z) = \begin{cases} 
1 + z & z \leq x \\
1 + x & z > x
\end{cases}
\]
Since $\frac{\partial^2 R}{\partial x^2}(x, z) = -\delta(z - x)$, we have

$$R(x, x + 0) - R(x, x + 0) = 0$$

$$\frac{\partial R}{\partial x}(x, x + 0) - \frac{\partial R}{\partial x}(x, x + 0) = -1$$

Using the conditions (3.18)-(3.21), we get the following system of equations

$$c_0(x) - c_1(x) = 0,$$  \hspace{1cm} (3.22)

$$d_0(x) = 0,$$

$$c_0(x) + c_1(x) x = d_0(x) + d_1(x) x,$$

$$d_1(x) + c_1(x) = -1,$$

which implies that

$$c_0(x) = 1, c_1(x) = 1, d_0(x) = 1 + x, d_1(x) = 0,$$

which completes the proof of the theorem.

Now, we present how to solve Problem (3.7)-(3.8) using the reproducing kernel method. Let

$$\sigma_i(x) = R(x_i, x)$$

for $i = 1, 2, \ldots$. It is clear that $L: W^3_z[0,1] \rightarrow W^4_z[0,1]$ is bounded linear operator. Let

$$\psi_i(x) = L^* \sigma_i(x)$$

where $L(\sigma_i(x)) = g^0(x)\sigma_i(x) + g^1(x)\sigma_i'(x) + g^2(x)\sigma_i(x)$ and $L^*$ is the adjoint operator of $L$. Using Gram-Schmidt orthonormalization to generate orthonormal set of functions $\{\psi_i(x) : i = 1, 2, 3, \ldots\}$ where

$$\psi_i(x) = \sum_{j=1}^{i} \alpha_{ij} \psi_j(x)$$

and $\alpha_{ij}$ are coefficients of Gram-Schmidt orthonormalization. In the next theorem, we show the existence of the solution of Problem (3.7)-(3.8).

**Theorem 3.3.** If $\{x_i: i = 1, 2, 3, \ldots\}$ is dense on $[0,1]$, then

$$f(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij} h(x_j) \psi_i(x).$$

(3.23)

**Proof.** First, we want to prove that $\{\psi_i(x) : i = 1, 2, 3, \ldots\}$ is the complete system of $W^3_z[0,1]$ and $\psi_i(x) = L(K(x, x_i))$. It is clear that $\psi_i(x) \in W^3_z[0,1]$ for $i = 1, 2, \ldots$ simple calculations implies that

$$\psi_i(x) = L^* \sigma_i(x) = (L^* \sigma_i(x), K(x, z))_{W^3_z[0,1]}$$

$$= (\sigma_i(x), L(K(x, z)))_{W^3_z[0,1]} = L(K(x, x_i)).$$

For each fixed $f(x) \in W^3_z[0,1]$, let

$$\left(f(x), \psi_i(x)\right)_{W^3_z[0,1]} = 0, i = 1, 2, \ldots.$$  

Then

$$\left(f(x), \psi_i(x)\right)_{W^3_z[0,1]} = \left(f(x), L^* \sigma_i(x)\right)_{W^3_z[0,1]}$$

$$= \left(Lf(x), \sigma_i(x)\right)_{W^3_z[0,1]}$$

$$= \left(Lf(x_i), 0\right) = 0.$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, $Lf(x) = 0$. Since $L^{-1}$ exists, $u(x) = 0$. Thus, $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W^3_z[0,1]$. Second, we prove equation (3.23). Simple calculations implies that

$$f(x) = \sum_{i=1}^{\infty} \left(f(x), \psi_i(x)\right)_{W^3_z[0,1]} \psi_i(x)$$
\[ = \sum_{i=1}^{\infty} \sum_{j=1}^{i} a_{ij} \left( f(x), L^* \left( K(x, x_j) \right) \right)_{W_2^2[0,1]} \tilde{\psi}_i(x) \]

\[ = \sum_{i=1}^{\infty} \alpha_i \left( Lf(x), K(x, x_j) \right)_{W_2^2[0,1]} \tilde{\psi}_i(x) \]

\[ = \sum_{i=1}^{\infty} \sum_{j=1}^{i} a_{ij} \left( h(x), K(x, x_j) \right)_{W_2^2[0,1]} \tilde{\psi}_i(x) \]

\[ = \sum_{i=1}^{\infty} \sum_{j=1}^{i} a_{ij} h(x_j) \tilde{\psi}_i(x) \]

and the proof is complete.

Let the approximate solution of Problem (3.7)-(3.8) be given by

\[ f_N(x) = \sum_{i=1}^{N} \sum_{j=1}^{i} a_{ij} h(x_j) \tilde{\psi}_i(x). \]

(3.24)

In the next theorem, we show the uniformly convergent of the\[ \frac{d^m f_N(x)}{dx^m} \]

Theorem 3.4. If \( f(x) \) and \( f_N(x) \) are given as in (3.23) and (3.24), then \( \frac{d^m f_N(x)}{dx^m} \) converges uniformly to \( \frac{d^m f(x)}{dx^m} \) for \( m = 0, 1, 2 \).

**Proof.** First, we prove the theorem for \( m = 0 \). For any \( x \in [0,1] \),

\[ \| f(x) - f_N(x) \|_{W_2^2[0,1]}^2 = \left( f(x) - f_N(x), f(x) - f_N(x) \right)_{W_2^2[0,1]} \]

\[ = \sum_{i=N+1}^{\infty} (f(x), \psi_i(x))_{W_2^2[0,1]} \psi_i(x) \]

\[ = \sum_{i=N+1}^{\infty} (f(x), \tilde{\psi}_i(x))^2_{W_2^2[0,1]} \]

Thus,

\[ \text{Sup}_{x \in [0,1]} \| f(x) - f_N(x) \|_{W_2^2[0,1]}^2 = \sum_{i=N+1}^{\infty} (f(x), \tilde{\psi}_i(x))^2_{W_2^2[0,1]} \]

From Theorem (3.3), one can see that \( \sum_{i=1}^{\infty} (f(x), \tilde{\psi}_i(x))^2_{W_2^2[0,1]} \) converges uniformly to \( f(x) \). Thus,

\[ \lim_{N \to \infty} \text{Sup}_{x \in [0,1]} \| f(x) - f_N(x) \|_{W_2^2[0,1]} = 0 \]

which implies that \( f_N(x) \) converges uniformly to \( f_N(x) \).

Second, we prove the uniformly convergence for \( m = 1, 2 \). Since \( (d^m (K(x, z)))/(dx^m) \) is bounded function on \([0,1] \times [0,1], \)

\[ \frac{d^m K(x, z)}{dx^m} \leq \chi_m, m = 1, 2. \]

Thus, for any \( x \in [0,1] \),

\[ \| f^{(m)}(x) - f_N^{(m)}(x) \|_{W_2^2[0,1]} \]

\[ = \left( f(x) - f_N(x), \frac{d^m K(x, z)}{dx^m} \right)_{W_2^2[0,1]} \]
\begin{align*}
\leq & \|f(x) - f_N(x)\|_{W^2[0,1]} \left\| \frac{d^m K(x, z)}{dx^m} \right\|_{W^2[0,1]} \\
\leq & \chi_m \|f(x) - f_N(x)\|_{W^2[0,1]} \\
\leq & \chi_m \sup_{x \in [0,1]} \|f(x) - f_N(x)\|_{W^2[0,1]}.
\end{align*}

Hence,
\begin{align*}
\sup_{x \in [0,1]} \left\| f^{(m)}(x) - f^{(m)}_N(x) \right\|_{W^2[0,1]} & \leq \chi_m \sup_{x \in [0,1]} \|f(x) - f_N(x)\|_{W^2[0,1]} \\
which implies that
\lim_{N \to \infty} \sup_{x \in [0,1]} \left\| f^{(m)}(x) - f^{(m)}_N(x) \right\|_{W^2[0,1]} = 0
\end{align*}

Therefore, \( \left\{ \frac{d^m f_N(x)}{dx^m} \right\}_{N=1}^{\infty} \) converges uniformly to \( \frac{d^m f(x)}{dx^m} \) for \( m = 0, 1, 2 \).

4 Numerical examples

In this section, we apply the RKHSM outlined in the previous sections to solve numerically the following two examples. Note that the maximum number of terms in the series solution is taken as \( N = 12 \) for all examples considered in this paper.

Example 4.1. Consider the following regular fractional eigenvalue problem

\[ D^{0.5} y'(x) = -\lambda y(x), \quad 0 < x < 1, \]

subject to

\[ y'(0) = 0, \ y(1) = 0. \]

Al-Mdallal [3] solved this problem using the Adomian decomposition method in the Caputo fractional derivative sense and he found the first three eigenvalues only for \( N = 25 \). These eigenvalues are

\[ \lambda_1 = 2.11027708, \ \lambda_2 = 13.76538223, \ \lambda_3 = 24.24328676. \]

Using the conformable fractional derivative sense, the corresponding problem to Equations (3.6) is

\[ \sqrt{x} y''(x) = -\lambda y(x) \]

subject to

\[ y(0) = \mu_1, \ y'(0) = 0. \]

Using the change of variable \( f(x) = y(x) - \mu_1 \), we get

\[ \sqrt{x} f''(x) + \lambda f(x) = -\lambda \mu_1 \]

subject to

\[ f(0) = 0, \ f'(0) = 0. \]

We report the first five eigenvalues in Table 1. The eigenfunctions corresponding to these eigenvalues are shown in Figure (1).
Let $\delta_{i,j} = \left| \int_0^1 y_i(x)y_j(x)w(x)dx \right|$. In Table 2 we report the values of $\delta_{i,j}$ for $i,j=1,2,\ldots,5$ with $i \neq j$.

Example 4.2. Consider the following regular fractional eigenvalue problem

$$D^{0.5}y'(x) + \frac{1}{x}y(x) = -\lambda y(x), 0 < x < 1,$$
subject to 
\( y(0) = 0, y'(1) = 0. \)

Al-Mdallal [3] solved this problem using the Adomian decomposition method in the Caputo fractional derivative sense and he found the first three eigenvalues only using \( N = 25. \) These eigenvalues are 
\[ \lambda_1 = 1.66091840, \lambda_2 = 13.55041793, \lambda_3 = 20.51445520. \]

Using the conformable fractional derivative sense, the corresponding problem to Equations (3.6) is
\[ \sqrt{x}y''(x) + \frac{1}{x}y(x) = -\lambda y(x) \]
subject to 
\( y(0) = 0, y'(0) = \mu_2. \)

Using the change of variable \( f(x) = y(x) - \mu_2 x, \) we get
\[ \sqrt{x}f''(x) + \left( \frac{1}{x} + \lambda \right) f(x) = -(1 + \lambda x)\mu_2 \]
subject to 
\( f(0) = 0, f'(0) = 0. \)

We report the first five eigenvalues in Table 3. The eigenfunctions corresponding to these eigenvalues are shown in Figure (2).

**Table 3: Eigenvalues of Example (4.2)**

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \lambda_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11.965752085315401</td>
</tr>
<tr>
<td>2</td>
<td>56.321834802145510</td>
</tr>
<tr>
<td>3</td>
<td>131.68924375027683</td>
</tr>
<tr>
<td>4</td>
<td>237.96483801829456</td>
</tr>
<tr>
<td>5</td>
<td>374.93901733455990</td>
</tr>
</tbody>
</table>

**Figure 2:** The first five eigenfunctions of Example (4.2)

It worth mention that when we take \( n = 40, \) we find the following eigenvalues 

We notice that these eigenvalues satisfy the property 
\[ 11.965752085315401 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots. \]

Let
$$\delta_{i,j} = \int_0^1 y_i(x)y_j(x)\,w(x)\,dx.$$ 

In Table 4 we report the values of $\delta_{i,j}$ for $i, j = 1, 2, \ldots, 5$ with $i \neq j$.

<table>
<thead>
<tr>
<th>j</th>
<th>$\delta_{1,j}$</th>
<th>$\delta_{2,j}$</th>
<th>$\delta_{3,j}$</th>
<th>$\delta_{4,j}$</th>
<th>$\delta_{5,j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1.62796\times10^{-16}$</td>
<td>$1.46920\times10^{-15}$</td>
<td>$8.44321\times10^{-15}$</td>
<td>$1.04267\times10^{-14}$</td>
<td>$1.04267\times10^{-14}$</td>
</tr>
<tr>
<td>2</td>
<td>$1.62796\times10^{-16}$</td>
<td>$9.48483\times10^{-15}$</td>
<td>$1.17947\times10^{-14}$</td>
<td>$1.32435\times10^{-14}$</td>
<td>$1.32435\times10^{-14}$</td>
</tr>
<tr>
<td>3</td>
<td>$1.46920\times10^{-15}$</td>
<td>$9.48483\times10^{-15}$</td>
<td>$2.26513\times10^{-14}$</td>
<td>$4.83291\times10^{-14}$</td>
<td>$4.83291\times10^{-14}$</td>
</tr>
<tr>
<td>4</td>
<td>$8.44321\times10^{-15}$</td>
<td>$1.17947\times10^{-14}$</td>
<td>$2.26513\times10^{-14}$</td>
<td>$3.20322\times10^{-14}$</td>
<td>$3.20322\times10^{-14}$</td>
</tr>
<tr>
<td>5</td>
<td>$1.04267\times10^{-14}$</td>
<td>$1.32435\times10^{-14}$</td>
<td>$4.83291\times10^{-14}$</td>
<td>$3.20322\times10^{-14}$</td>
<td>$3.20322\times10^{-14}$</td>
</tr>
</tbody>
</table>

We take $n = 60$ for Al-Mdallal technique [3] but we do not get any new eigenvalue.

5 Conclusions

In this paper, we have developed a numerical technique to find the eigenvalues of non-singular second-order fractional Sturm-Liouville problem. The method of solution is based on RKHSM. The numerical results for the examples demonstrate the efficiency and accuracy of the present method. From the two examples which we mentioned in the previous section, we notice that our technique is very efficient for computing the eigenvalues of the fractional second order problems. It is competes the method in [3] and gives better and faster results. We end this section by the following remarks.

- From Examples (4.1) and (4.2), we can find as much eigenvalues as the model requires with the following property
  $$\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_{\{n\}} < \ldots$$
  while in <cite>q1</cite> only three eigenvalue can be found.
- From Examples (4.1) and (4.2), the orthogonality property
  $$\delta_{i,j} = \int_0^1 y_i(x)y_j(x)\,w(x)\,dx \approx 0, i \neq j$$
  holds while in [3], we get $\delta_{1,2} = 0.0011366, \delta_{1,3} = 0.00904938, \delta_{2,3} = 0.0270058$.
- From Figures (1) and (2), we see that corresponding to each eigenvalue $\lambda_i$ is a unique (up to a normalization constant) eigenfunction $y_i(x)$ which has exactly $i - 1$ zeros in (0,1).
- We notice that the conformable fractional derivative sense is more suitable to study the fractional second-order Sturm-Liouville problems than the Caputo fractional derivative sense.
- The results in this paper confirm that RKHSM is a powerful and efficient method for solving fractional second-order Sturm-Liouville problems in different fields of sciences and engineering.
- RKHSM is excellent tool due to rapid convergent.
- The existence and uniformly convergent are proven in Theorems (3.3) and (3.4).order to assess the advantages and the accuracy of the numerical scheme for solving multi-order linear fractional differential equations, we have applied the method to three different examples. All the results are calculated by using the Matlab.

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References


   https://doi.org/10.1016/j.chaos.2007.07.041

   https://doi.org/10.1007/s11075-009-9351-7

   https://doi.org/10.3390/mca16030712

   https://doi.org/10.2478/s13540-012-0010-7


   https://doi.org/10.1016/j.apm.2011.11.024


    https://arxiv.org/abs/1303.2839

    http://dx.doi.org/10.1155/2013/915830

    https://doi.org/10.1016/j.jcp.2013.06.031

https://doi.org/10.1016/j.jmaa.2014.02.009

http://cmde.tabrizu.ac.ir/article_2498_277.html

https://doi.org/10.1137/140954209

https://doi.org/10.1016/j.cam.2015.02.058

https://doi.org/10.1016/0377-0427(91)90202-U

https://doi.org/10.1016/0377-0427(93)90088-S

https://doi.org/10.1080/00207160802610843

https://doi.org/10.1006/jcph.1994.1073

https://doi.org/10.1016/j.cam.2014.01.002

https://doi.org/10.1137/0522067
https://doi.org/10.1145/279232.279231

https://doi.org/10.1016/j.jmaa.2006.05.011

https://doi.org/10.1016/0022-247X(88)90149-7

https://doi.org/10.1006/jsvi.1995.0396


https://doi.org/10.1016/0022-247X(91)90363-5

https://doi.org/10.1016/j.chaos.2007.01.105

https://doi.org/10.1016/S0377-0427(03)00541-7

https://doi.org/10.1002/nme.102

https://doi.org/10.1016/j.amc.2006.07.157