Certain new integral formulas involving the generalized $k$-Bessel function

Kottakkaran S. Nisar1,*, Gauhar Rahman2, Shahid Mubeen3, Muhammad Arshad2

1 Department of Mathematics, College of Arts and Science-Wadi Al-Dawaser, Prince Sattam bin Abdulaziz University, Saudi Arabia
2 Department of Mathematics, International Islamic University, Islamabad, Pakistan
3 Department of Mathematics, University of Sargodha, Sargodha, Pakistan

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Abstract
In this present paper, we investigate generalized integration formulas containing the generalized $k$-Bessel function $W^{k}_{v,c}(z)$ based on the well known Oberhettinger formula [12] and obtain the results in term of Wright-type function. Also, we establish certain special cases of our main result.

Keywords: Gamma function, $k$-gamma function, Generalized hypergeometric function, Oberhettinger formula, Wright function, Generalized $k$-Bessel function

1 Introduction

We begin with the generalized hypergeometric function $pF_q(z)$ is defined in [6] as:

$$pF_q(z) = pF_{q} \left[ \begin{array}{c} \left( \alpha_{1}, \alpha_{2}, \cdots \alpha_{p} \right) \\ \left( \beta_{1}, \beta_{2}, \cdots \beta_{q} \right) \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_{1})_{n}(\alpha_{2})_{n} \cdots (\alpha_{p})_{n}}{(\beta_{1})_{n}(\beta_{2})_{n} \cdots (\beta_{q})_{n}} \frac{z^{n}}{n!},$$

(1.1)

where $\alpha_i, \beta_j \in \mathbb{C}; \ i = 1, 2, \cdots, p, \ j = 1, 2, \cdots, q$ and $b_j \neq 0, -1, -2, \cdots$ and $(z)_n$ is the Pochhammer symbols. The familiar gamma function is defined as by the following formula:

$$\Gamma(\mu) = \int_{0}^{\infty} t^{\mu-1} e^{-t} dt, \mu \in \mathbb{C},$$

(1.2)

and beta function is defined as:

$$B(x, y) = \int_{0}^{1} t^{x-1}(1-t)^{y-1} dt.$$

(1.4)
The Wright type hypergeometric function is defined (see [16]-[18]) by the following series as:

$$\mathbf{W}_q(z) = \mathbf{W}_q \left[ \begin{array}{c} (\alpha_1,A_1)_{1,p} \\ (\beta_1,B_1)_{1,q} \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_q + B_q n)} \frac{z^n}{n!}, \quad \text{(1.5)}$$

where $\beta_j$ and $\alpha_p$ are real positive numbers such that

$$1 + \sum_{j=1}^{q} \beta_j - \sum_{r=1}^{p} \alpha_r > 0.$$

The generalized hypergeometric function $\mathbf{P}_q(z)$ is a special case of $\mathbf{W}_q(z)$ for $A_i = B_j = 1$, where $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$:

$$\frac{1}{\prod_{j=1}^{q} \Gamma(\beta_j)} \mathbf{P}_q \left[ \begin{array}{c} (\alpha_1,\cdots,\alpha_p) \\ (\beta_1,\cdots,\beta_q) \end{array} ; z \right] = \frac{1}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \mathbf{W}_q \left[ \begin{array}{c} (\alpha_1,1,p) \\ (\beta_j,1,q) \end{array} ; z \right]. \quad \text{(1.6)}$$

The generalized $k$-Bessel function defined in [11] as:

$$W_{v^k}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{G_k(nk + v + k)n!} \left( \frac{z}{2} \right)^{2n+\frac{k}{2}}, \quad \text{(1.7)}$$

where $k > 0, v > -1$, and $c \in \mathbb{R}$ and $G_k(z)$ is the $k$-gamma function defined in [5] as:

$$G_k(z) = \int_{0}^{\infty} e^{-\frac{z}{x}} x^{k-1} dt, z \in \mathbb{C}. \quad \text{(1.8)}$$

By inspection the following relation holds:

$$G_k(z + k) = zG_k(z) \quad \text{(1.9)}$$

$$G_k(z) = k^{z-1} \Gamma \left( \frac{z}{k} \right). \quad \text{(1.10)}$$

The Pochhammer $k$-symbols can be defined as:

$$(x)_{n,k} = x(x + k) \cdots (x + (n - 1)k), n \neq 0, n \in \mathbb{N}, (x)_{0,k} = 1. \quad \text{(1.11)}$$

The relation between Pochhammer $k$-symbols and $k$-gamma function is defined as:

$$(x)_{n,k} = \frac{G_k(x + nk)}{G_k(x)}. \quad \text{(1.12)}$$

If $k \to 1$ and $c = 1$, then the generalized $k$-Bessel function defined in (2.12) reduces to the well known classical Bessel function $J_{\alpha}$, defined in [7]. For further detail about $k$-Bessel function and its properties (see [8]-[10]).

In this paper, we define a class of integral formulas which containing the generalized $k$-Bessel function as defined in (1.7). Also, we investigate some special cases as the corollaries. For this continuation of our study, we recall the following result of Oberhettinger [12].

$$\int_{0}^{\infty} z^{\alpha-1} \left( z + b + \sqrt{z^2 + 2bc} \right)^{-\beta} dz = 2\beta b^{-\beta} \left( \frac{b}{2} \right)^\alpha \frac{\Gamma(2\alpha)\Gamma(\beta - \alpha)}{\Gamma(1 + \alpha + \beta)}, \quad 0 < \Re(\alpha) < \Re(\beta). \quad \text{(1.11)}$$

For various other investigation containing special function, the reader may refer to the recent work of researchers (see [3], [4], [13], [14], [15]).
2 Main Result

In this section, we establish two generalized integral formulas containing \( k \)-Bessel function defined (1.7), which represented in terms of Wright-type function defined in (1.5) by inserting with the suitable argument defined in (1.11).

**Theorem 2.1.** For \( \alpha, \beta, v, c \in \mathbb{C} \) with \( \Re(\beta + \frac{v}{k}) > \Re(\alpha) > 0, \Re(v/k) > -1 \), then the following result holds:

\[
\begin{align*}
\int_0^\infty z^{\alpha-1} \left( z + a + \sqrt{z^2 + 2az} \right)^{-\beta} W_{\alpha,c} \left( \frac{y}{z+a+\sqrt{z^2+2az}} \right) dz \\
= \frac{(y)^\frac{\alpha}{2} \Gamma(2\alpha) a^{\alpha-\beta-\frac{v}{k}}}{2^{\frac{\alpha-1}{2} - \frac{v}{k}}} \\
\times \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk+v+k)n!} \left( \frac{y}{2(z+a+\sqrt{z^2+2az})} \right)^{2n+\frac{v}{k}} \left( \begin{array}{c}
2n+\frac{v}{k} \\
\frac{v}{k}+1
\end{array} \right), \tag{2.12}
\end{align*}
\]

**Proof.** Let \( \mathcal{L}_1 \) be the left hand side of (2.1) and applying (1.7) to the integrand of (2.12), we have

\[
\begin{align*}
\mathcal{L}_1 &= \int_0^\infty z^{\alpha-1} \left( z + a + \sqrt{z^2 + 2az} \right)^{-\beta} \\
&\times \sum_{n=0}^{\infty} \Gamma_k(nk+v+k)n! \left( \frac{y}{2(z+a+\sqrt{z^2+2az})} \right)^{2n+\frac{v}{k}} dz \\
&= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk+v+k)n!} \left( \frac{y}{2} \right)^{2n+\frac{v}{k}} \\
&\times \int_0^\infty z^{\alpha-1} \left( z + b + \sqrt{z^2 + 2bz} \right)^{-(\beta + \frac{v}{k}+2n)} dz. \tag{2.13}
\end{align*}
\]

By considering the assumption given in theorem 2.1, since \( \Re(\beta + \frac{v}{k}) > \Re(\alpha) > 0, k > 0 \) and applying (1.11) to (2.13), we obtain

\[
\begin{align*}
\mathcal{L}_1 &= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk+v+k)n!} \left( \frac{y}{2} \right)^{2n+\frac{v}{k}} 2(\beta + \frac{v}{k}+2n) \left( b \right)^{\alpha} \\
&\times \Gamma(2\alpha+\Gamma(\beta + \frac{v}{k}+2n-\alpha)) \Gamma(1+\beta + \frac{v}{k}+\alpha+2n). \\
&= \frac{(y)^\frac{\alpha}{2} \Gamma(2\alpha) b^{\alpha-\beta-\frac{v}{k}}}{2^{\frac{\alpha-1}{2} - \frac{v}{k}}} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk+v+k)n!} \left( \frac{y^{2n}}{4^n k^n b^{2n}} \right) \\
&\times \Gamma(\beta + \frac{v}{k}+2n+1) \Gamma(\beta + \frac{v}{k}+2n-\alpha) \Gamma(\alpha + \beta + \frac{v}{k}+2n+1) \\
\end{align*}
\]

which upon using (1.5), we get the required result.
Theorem 2.2. For \( \alpha, \beta, v, c \in \mathbb{C} \) with \( \Re(\beta + \frac{v}{k}) > \Re(\alpha + \frac{v}{k}) > 0, \Re(v/k) > -1 \) and \( z > 0 \), then the following result holds:

\[
\begin{align*}
\int_{0}^{\infty} \frac{z^{a-1} (z + b + \sqrt{z^2 + 2bc})^{-\beta}}{2^\alpha + \alpha - 1} \frac{W^{k}_{\nu,c} \left( \frac{yz}{z + b + \sqrt{z^2 + 2bc}} \right)}{z} \, dz & = \frac{\Gamma(\beta - \alpha) b^{\alpha - \beta}}{2^\alpha + \alpha - 1} \times 2^\Psi_3 \left[ \begin{array}{c}
(\beta + \frac{v}{k} + 1; 2), (2\beta + 2\frac{v}{k}, 4);
(\beta + \frac{v}{k}, 2), (\beta + 1, 1), (\alpha + \beta + \frac{v}{k} + 1, 4)
\end{array} \right] \left( - \frac{c^2}{16} \right).
\end{align*}
\]

(2.14)

Proof. Let \( \mathcal{L}_2 \) be the left hand side of (2.13) and applying (1.7) to the integrand of (2.14), we have

\[
\mathcal{L}_2 = \int_{0}^{\infty} \frac{z^{a-1} (z + b + \sqrt{z^2 + 2bc})^{-\beta}}{2^\alpha + \alpha - 1} \frac{(-c)^n \Gamma(nk + v + k) n!}{2(z + b + \sqrt{z^2 + 2bc})^{2n+\frac{v}{k}}} \, dz.
\]

By interchanging the order of integration and summation, which is verified by the uniform convergence of the series under the given assumption of theorem 2.13, we have

\[
\mathcal{L}_2 = \sum_{n=0}^{\infty} \frac{(-c)^n \Gamma(nk + v + k) n!}{2(z + b + \sqrt{z^2 + 2bc})^{2n+\frac{v}{k}}} \frac{\Gamma(\beta + \frac{v}{k} + 2n + 1) \Gamma(2\alpha + 2\frac{v}{k} + 4n) \Gamma(\beta - \alpha)}{\Gamma(1 + \beta + \frac{v}{k} + \alpha + 4n)}.
\]

(2.15)

By considering the assumption given in theorem 2.13, since \( \Re(\beta + \frac{v}{k}) > \Re(\alpha + \frac{v}{k}) > 0, k > 0 \) and applying (1.11) to (2.15), we obtain

\[
\mathcal{L}_2 = \sum_{n=0}^{\infty} \frac{(-c)^n \Gamma(2\alpha + 4n) \Gamma(\beta - \alpha)}{\Gamma(1 + \beta + \frac{v}{k} + \alpha + 4n)} \frac{1}{2^\alpha + \alpha - 1} \times \frac{2(\beta + \frac{v}{k} + 2n) b^{\alpha - \beta}}{2^{\alpha + \frac{v}{k} + 2n}}.
\]

Applying (1.10) and (1.3), we get

\[
\mathcal{L}_2 = \frac{\Gamma(2\alpha) b^{\alpha - \beta}}{2^{\alpha + \alpha - 1} k^\Psi} \sum_{n=0}^{\infty} \frac{(-c)^n \Gamma(2\alpha + 4n) \Gamma(\beta - \alpha)}{\Gamma(1 + \beta + \frac{v}{k} + \alpha + 4n)} \left( \frac{y^2}{16} \right) \frac{1}{16} \frac{\gamma \delta \varepsilon}{\chi \psi}.
\]

which upon using (1.5), we get the required result.

\[\square\]

3 Special Cases

In this section, we present the generalized form of classical and modified Bessel functions which are the special cases of \( k \)-Bessel function defined (1.7). Also, we prove two corollaries which are the special cases of obtained theorems in Section 2.
Case 1. If we set \( c = 1 \) in (1.7), then we get another definition of \( k\)-Bessel function. We call it the classical \( k\)-Bessel function

\[
J_{v}^{1}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{k})^n}{\Gamma(k+in+k)n!}
\]  

(3.16)

Case 2. If we set \( c = -1 \) in (1.7), then we get another definition of \( k\)-Bessel function. We call it the modified \( k\)-Bessel function

\[
I_{v}^{1}(z) = \sum_{n=0}^{\infty} \frac{(\frac{z}{k})^n}{\Gamma(k+in+k)n!}
\]  

(3.17)

Corollary 3.1. Assume that the conditions of Theorem 2.1 are satisfied. Then the following integral formula holds:

\[
\int_{0}^{\infty} z^{\alpha-1} \left( z + a + \sqrt{z^2 + 2az} \right)^{-\beta} J_{v}^{0} \left( \frac{y}{z + a + \sqrt{z^2 + 2az}} \right) dz = \frac{(y)^{-\beta} \Gamma(2\alpha) a^{\alpha-\beta} z^{\alpha-1}}{2^{\alpha+1+k} z^{\alpha+\beta}} \times \Psi_{3} \left[ \left( \frac{\beta+\frac{\alpha}{2}}{2}, \frac{\beta+\frac{\alpha+1}{2}}{2} \right) ; \frac{|y|^2}{4ka^2} \right].
\]

Corollary 3.2. Assume that the conditions of Theorem 2.1 are satisfied. Then the following integral formula holds:

\[
\int_{0}^{\infty} z^{\alpha-1} \left( z + a + \sqrt{z^2 + 2az} \right)^{-\beta} I_{v}^{0} \left( \frac{y}{z + a + \sqrt{z^2 + 2az}} \right) dz = \frac{(y)^{-\beta} \Gamma(2\alpha) a^{\alpha-\beta} z^{\alpha-1}}{2^{\alpha+1+k} z^{\alpha+\beta}} \times \Psi_{3} \left[ \left( \frac{\beta+\frac{\alpha}{2}}{2}, \frac{\beta+\frac{\alpha+1}{2}}{2} \right) ; \frac{|y|^2}{4ka^2} \right].
\]

Corollary 3.3. Assume that the conditions of Theorem 2.13 are satisfied. Then the following integral formula holds:

\[
\int_{0}^{\infty} z^{\alpha-1} \left( z + b + \sqrt{z^2 + 2bz} \right)^{-\beta} J_{v}^{0} \left( \frac{yz}{z + b + \sqrt{z^2 + 2bz}} \right) dz = \frac{(y)^{-\beta} \Gamma(2\alpha) a^{\alpha-\beta} z^{\alpha-1}}{2^{\alpha+1+k} z^{\alpha+\beta}} \times \Psi_{3} \left[ \left( \frac{\beta+\frac{\alpha}{2}}{2}, \frac{\beta+\frac{\alpha+1}{2}}{2} \right) ; \frac{|y|^2}{4ka^2} \right].
\]

Corollary 3.4. Assume that the conditions of Theorem 2.13 are satisfied. Then the following integral formula holds:

\[
\int_{0}^{\infty} z^{\alpha-1} \left( z + b + \sqrt{z^2 + 2bz} \right)^{-\beta} I_{v}^{0} \left( \frac{yz}{z + b + \sqrt{z^2 + 2bz}} \right) dz = \frac{(y)^{-\beta} \Gamma(2\alpha) a^{\alpha-\beta} z^{\alpha-1}}{2^{\alpha+1+k} z^{\alpha+\beta}} \times \Psi_{3} \left[ \left( \frac{\beta+\frac{\alpha}{2}}{2}, \frac{\beta+\frac{\alpha+1}{2}}{2} \right) ; \frac{|y|^2}{4ka^2} \right].
\]
Remark 3.1. In this paper, we introduced two integral representation for $k$-Bessel function. If letting $k = 1$ and $c = \pm 1$ respectively, then we obtained the results of classical Bessel function $J_v(z)$ and modified Bessel function $I_v(z)$.

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