The relationship of degenerate kernel and projection methods on Fredholm integral equations of the second kind

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Abstract
In this paper, we show that the degenerate kernel method for some cases, on the condition that the source function is approximated by the same way of producing degenerate kernel, becomes as a projection method. We consider two ways, including Lagrange interpolation and best approximation methods, of producing degenerate kernel approximations of more general Fredholm integral equation of the second kind. For these two ways, we show that the degenerate kernel method becomes as a Lagrange-collocation method and Galerkin method respectively.

Keywords: Fredholm integral equations of the second kind, Degenerate kernel method, Projection methods, Collocation methods, Galerkin methods.

1 Introduction
Fredholm integral equations of the second kind are of the form [1]

\[ u(x) = f(x) + \lambda \int_{\mathcal{D}} K(x, t, u(t)) \, dt, \quad x \in \mathcal{A}, \]

(1.1)

where \( x = (x_1, x_2, \ldots, x_n) \), \( t = (t_1, t_2, \ldots, t_n) \), \( \mathcal{D} \) and \( \mathcal{A} \) are regions in \( \mathbb{R}^n \), \( \lambda \) is a parameter, \( f(x) \) is the source (or data) function, \( K(x, t, u(t)) \) is the kernel of the integral equation, and \( u(x) \) is the unknown function that will be determined. For the linear case, it is assumed that \( K(x, t, u(t)) = k(x, t)u(t) \).

In this paper, a review of degenerate kernel and collocation methods are given. As well as, we introduce a modified of degenerate kernel method by approximating source function using the same way of producing degenerate kernel. Then, we show that the modified degenerate kernel method, when we use the Lagrange interpolation and best approximation methods, becomes as a collocation method and Galerkin method respectively.

2 The degenerate kernel method
The degenerate kernel method (DKM) is a well-known classical method for solving Fredholm integral equations of the second kind, and it is one of the easiest numerical methods to define and analyze [2]. This method for a given degenerate kernel is called direct computation method (DCM) [4, 5].

DKM transforms a Fredholm integral equation of the second kind to a system of algebraic equations. To handle Eq. (1.1), by using DKM, we can express the procedure as follows.

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1. Define the approximating degenerate kernel as follows

\[ K_{K_m}(x, t, u(t)) = \sum_{i=1}^{K_m} \phi_i(x) \psi_i(t, u(t)). \]  

(2.2)

We denote \( K_m \)-order approximation of the solution given by the degenerate kernel with \( \tilde{u}_{K_m}(x) \).

2. Substituting (2.2) into (1.1) gives

\[ \tilde{u}_{K_m}(x; a) = f(x) + \lambda \sum_{i=1}^{K_m} a_i \phi_i(x), \]

(2.3)

where

\[ a_i = \int_{\mathcal{B}} \psi_i(t, u(t)) dt, \quad i = 1, \ldots, K_m, \]

(2.4)

and \( a = (a_1, \ldots, a_{K_m}) \).

3. Replacing Eq. (2.3) into (2.4) leads to the following algebraic system

\[ a_i = \int_{\mathcal{B}} \psi_i(t, f(t) + \lambda \sum_{j=1}^{K_m} a_j \phi_j(t)) dt, \quad i = 1, \ldots, K_m. \]

(2.5)

4. Solving Eq. (2.5) provides values of \( a_i, i = 1, \ldots, K_m \), for substituting them into the Eq. (2.3) to obtain solution of Eq. (1.1).

3 Projection methods

In projection methods for solving Eq. (1.1) within the framework of some complete function space \( V \), usually \( C(\mathcal{A}) \) or \( L^2(\mathcal{A}) \), we choose a sequence of finite dimensional approximating subspaces \( V_m \subset V, m \geq 1 \), with \( diam(V_m) = K_m \) [3]. Assume \( V_m, m \geq 1 \) have a basis \( B_m = \{ \phi_1, \phi_2, \ldots, \phi_{K_m} \} \), we seek a function \( u_{K_m} \in \text{span}(B_m) \) to construct the residual as follows

\[ R_m(x) = u_{K_m}(x) - f(x) - \lambda \int_{\mathcal{B}} K(x, t, u_{K_m}(t)) dt. \]

(3.6)

It is clear that \( u_{K_m} \) is of the form

\[ u_{K_m}(x) = \sum_{i=1}^{K_m} c_i \phi_i(x), \quad x \in \mathcal{A}. \]

(3.7)

The coefficients \( c_i, i = 1, 2, \ldots, K_m \), are chosen by forcing \( R_m(x) \) to be approximately zero in some sense.

3.1 Collocation methods

Let \( x_r \in \mathcal{A} \). In collocation methods, by selecting distinct collocation nodes \( x_r \), require

\[ R_m(x_r) = 0, \]

(3.8)

where \( r = 1, 2, \ldots, K_m \). We rewrite Eq. (3.8) as follows

\[ \sum_{i=1}^{K_m} c_i \phi_i(x_r) = f(x_r) + \lambda \int_{\mathcal{B}} K(x_r, t, u_{K_m}(t)) dt. \]

(3.9)
3.2 Galerkin methods

Let \( V = L^2(\Omega') \) or some other Hilbert function space, and let \( \langle \cdot, \cdot \rangle \) denote the inner product for \( V \). In Galerkin method, require \( R_m \) to satisfy

\[
\langle R_m(x), \phi_i(x) \rangle = 0, i = 1, 2, \ldots, \kappa_m.
\] (3.10)

We rewrite Eq. (3.10) as follows

\[
\langle u \phi_i(x) \rangle = \langle f(x), \phi_i(x) \rangle + \lambda \int_{\Omega} K(x, t, u \phi_i(t)) dt, i = 1, 2, \ldots, \kappa_m.
\] (3.11)

4 The modified degenerate kernel method (MDKM)

We propose a modification of the degenerate kernel method (MDKM) by approximating source function using the same way of producing degenerate kernel denoted by Eq. (2.2) [4]. Then we write

\[
f_{\kappa_m}(x) = \sum_{i=1}^{\kappa_m} \beta_i \phi_i(x),
\] (4.12)

where \( \beta_i, i = 1, 2, \ldots, \kappa_m, \) are known. Therefore Eqs. (2.3) and (2.5) become as follows

\[
\tilde{u}_{\kappa_m}(x; \alpha) = \sum_{i=1}^{\kappa_m} (\lambda \alpha_i + \beta_i) \phi_i(x),
\] (4.13)

and

\[
\alpha_i = \int_{\Omega} \psi_i \left( t, \sum_{j=1}^{\kappa_m} (\lambda \alpha_j + \beta_j) \phi_j(t) \right) dt, i = 1, \ldots, \kappa_m,
\] (4.14)

respectively.

**Remark 4.1.** The nonlinear algebraic systems denoted by Eqs. (2.5), (3.9), (3.11) and (4.14) are nontrivial systems to solve, and usually some variant of Newton’s method is used to find an approximating of solution. A major difficulty is that the integrals in them will need to be numerically evaluated [3]. Also, the role of initial guesses in Newton’s method is very important, for more details refer to [4].

5 The relationship of MDKM and projection methods

The claim is that if \( \phi_i(x), i = 1, 2, \ldots, \kappa_m, \) are Lagrange polynomials or orthogonal functions then MDKM becomes as a collocation method or Galerkin method respectively. To establish the first part of this claim, we show that the algebraic systems denoted by Eqs. (3.9) and (4.14) are equivalent and then the Eqs. (3.7) and (4.13) give same solutions. For the second part of claim, similar equivalences should be shown. We consider two cases for the bases functions.

Case I. **Lagrange interpolation**

In this case, we assume that \( \phi_i(x_r) = \delta_{ir} \). Therefore, Eq. (3.9) becomes as follows

\[
c_r = f(x_r) + \lambda \int_{\Omega} K(x_r, t, \sum_{j=1}^{\kappa_m} c_j \phi_j(t)) dt.
\] (5.15)

Now, we set \( c_r - f_r = \lambda \alpha_r \), where \( f_r = f(x_r) \), we find

\[
\alpha_r = \int_{\Omega} K(x_r, t, \sum_{j=1}^{\kappa_m} (f_j + \lambda \alpha_j) \phi_j(t)) dt.
\] (5.16)
On the other hand, from Eqs. (2.2) and (4.12) we find \( \psi_i(t, u(t)) = K(x_i, t, u(t)) \) and \( \beta_i = f_i \) respectively. Thus, the algebraic systems (4.14) and (5.16) are equivalent. In this case, by assuming \( \kappa_m = m^6 \), the bases is as follows

\[
B_k = \left\{ \prod_{i=1}^{m} l_{i}(x_i)|r_i = 1, 2, \ldots, m \right\},
\]

(5.17)

where \( l_i \) are Lagrange polynomials. From Eq. (3.7), we find \( c_r = u_{\kappa m}(x_r) \), therefore, Eq. (3.7) and Eq. (4.13) provide the \( \kappa_m \)-order Lagrange interpolation of solution as follows

\[
u_{\kappa m}(x) = \sum_{r_1, r_2, \ldots, r_m = 1} u_{\kappa m, r} \prod_{i=1}^{m} l_{i}(x_i), x \in \mathcal{A},
\]

(5.18)

where \( u_{\kappa m, r} = u_{\kappa m}(x_r) \).

Case II. The best approximation

Similar to the Galerkin methods, let \( V = L^2(\mathcal{A}) \) or some other Hilbert function space and \( (\cdot, \cdot) \) denote the inner product for \( V \), and let \( \phi_i(x), i = 1, 2, \ldots, \kappa_m \), are orthogonal. Then, from Eq. (3.11) by setting \( \langle u_{\kappa m}(x), \phi_i(x) \rangle = \langle f(x), \phi_i(x) \rangle = \lambda_i \alpha_i \), we have

\[
\alpha_i = \left( \int_{\mathcal{A}} K(x, t, u_{\kappa m}(t))dt, \phi_i(x) \right), i = 1, 2, \ldots, \kappa_m.
\]

(5.19)

On the other hand, from Eqs. (2.2) and (4.12) we find \( \psi_i(t, u(t)) = \langle K(x, t, u(t)), \phi_i(x) \rangle \) and \( \beta_i = \langle f(x), \phi_i(x) \rangle \) respectively. Thus, the algebraic systems (4.14) and (5.19) are equivalent. From Eq. (3.7), we find \( c_i = \langle u_{\kappa m}(x), \phi_i(x) \rangle \), therefore, Eq. (3.7) and Eq. (4.13) provide the best approximation of solution in \( V_m \) as follows

\[
u_{\kappa m}(x) = \sum_{i=1}^{\kappa_m} (u_{\kappa m}(x), \phi_i(x)) \phi_i(x), x \in \mathcal{A}.
\]

(5.20)

We collect the above analysis and present the following summary.

**Theorem 5.1.** Assume that the base functions \( \phi_i(x), i = 1, 2, \ldots, \kappa_m \), are Lagrange polynomials or orthogonal functions then MDKM becomes as a collocation method or Galerkin method respectively.

**Proof.** See the analysis preceding the theorem statement.

\[\square\]

6 Test example

Consider the following linear two-dimensional Fredholm integral equation of the second kind with a non-degenerate kernel [2, 6]

\[
u(x, y) = 1 - \frac{\sin(\sqrt{\pi}x) \sin(\sqrt{\pi}y)}{5xy} + \frac{1}{2} \int_{0}^{\sqrt{\pi}} \int_{0}^{\sqrt{\pi}} \cos(xs) \cos(ys)u(s, t)dsdt.
\]

(6.21)

We evaluate Eq. (6.21) by MDKM using the Lagrange interpolation and best approximation methods for comparing with collocation and Galerkin methods respectively. For this purpose, we consider two cases as follows.

Case 1. Comparison between MDKM and Lagrange-collocation method

By choosing four Chebychev or equally spaced collocation nodes, to make a degenerate approximation of the kernel as well as an approximation of same order to the source function, the MDKM gives the exact solution of Eq. (6.21). For equally spaced collocation nodes, we find

\[
\begin{align*}
a_1 &= \frac{\pi}{2}, \\
a_2 &= a_3 = a_4 = 0,
\end{align*}
\]
and 
\[
\hat{a}_4(x, y; \alpha) = \frac{1}{3} (5\alpha_1 - \pi + 5) + \frac{1}{5\sqrt{\pi}} (-5\alpha_1 + 5\alpha_3 + \pi) y \\
+ \frac{1}{5\sqrt{\pi}} (\sqrt{\pi}(-5\alpha_1 + 5\alpha_2 + \pi) - (-5\alpha_1 + 5\alpha_2 + 5\alpha_3 - 5\alpha_4 + \pi)) x.
\]

Therefore, MDKM gives exact solution of Eq. (6.21) as follows
\[
u(x, y) = \hat{u}_4(x, y) = 1.
\]

Similarly, by choosing four Chebychev or equally spaced collocation nodes, to make Lagrange base functions, Lagrange-collocation method gives the exact solution of Eq. (6.21). For equally spaced collocation nodes, we find
\[
c_i = 1, i = 1, 2, 3, 4,
\]
and
\[
u(x, y) = \frac{1}{3} (c_1(\sqrt{\pi} - x)(\sqrt{\pi} - y) + c_2 x(\sqrt{\pi} - y) + c_3(\sqrt{\pi} - x)y + c_4 xy).
\]

Therefore, Lagrange-collocation method gives exact solution of Eq. (6.21) as follows
\[
u(x, y) = u_4(x, y) = 1.
\]

Case 2. **Comparison between MDKM and Galerkin method**

For this case, Starting with the monomials \(\{1, x, y, xy\}\), we apply the Gram- Schmidt procedure to construct a system of orthogonal polynomials \(B_4 = \{\varphi_i(x, y)\}_{i=1}^4\) as an orthogonal base with the following inner product
\[
\langle \varphi_i, \varphi_j \rangle = \int_0^{\sqrt{\pi}} \int_0^{\sqrt{\pi}} \varphi_i(x, y)\varphi_j(x, y)dx dy.
\]

In this case, using the MDKM, we find
\[
\left\{
\begin{array}{l}
\alpha_1 = \frac{S_1^2(\pi)}{5\pi}, \\
\alpha_2 = -\frac{\sqrt{3}Si(\pi)(\pi Si(\pi) - 4)}{5\pi^{3/2}}, \\
\alpha_3 = \frac{-\sqrt{3}Si(\pi)(\pi Si(\pi) - 4)}{5\pi^{3/2}}, \\
\alpha_4 = \frac{1}{5}\frac{(\pi Si(\pi) - 4)^2}{\pi^{3/2}},
\end{array}
\right.
\]
and
\[
\hat{u}_4(x, y; \alpha) = \frac{1}{5\pi}\left(5\alpha_1 - \pi + 5\right) - 5\sqrt{3}\pi^{5/2}\alpha_2 - 5\sqrt{3}\pi^{5/2}\alpha_3 + 15\pi^{5/2}\alpha_4
\]
\[
- 16\pi^2Si^2(\pi) + 96\pi Si(\pi) + 5\pi^3 - 144 + 2x(\sqrt{\pi}(5\sqrt{3}\pi^{5/2}\alpha_2 + 3(-5\pi^{5/2}\alpha_4
\]
\[
+ 4\pi^2Si^2(\pi) - 28\pi Si(\pi) + 48) + 6y(5\pi^{5/2}\alpha_4 - 3(\pi Si(\pi) - 4)^2))
\]
\[
+ 2\sqrt{\pi}y(5\sqrt{3}\pi^{5/2}\alpha_3 - 15\pi^{5/2}\alpha_4 + 12\pi^2Si^2(\pi) - 84\pi Si(\pi) + 144),
\]
where \(Si(z) = \int_0^z \frac{\sin(t)}{t}dt\). Therefore, MDKM gives exact solution of Eq. (6.21) as follows
\[
u(x, y) = \hat{u}_4(x, y) = 1.
\]

Similarly, by applying the Galerkin method for solving Eq. (6.21), we find
\[
c_1 = \sqrt{\pi}, c_2 = c_3 = c_4 = 0,
\]
and
\[
u(x, y) = \frac{1}{\pi^{3/2}} \left[\pi c_1 - \sqrt{3}c_2(\pi - 2\sqrt{\pi}x) - (\sqrt{\pi} - 2y)(\sqrt{\pi}c_3 - 3c_4(\sqrt{\pi} - 2x))\right).
\]

Therefore, Galerkin method gives exact solution of Eq. (6.21) as follows
\[
u(x, y) = u_4(x, y) = 1.
\]

For the results of applying the degenerate kernel method to Eq. (6.21) refer to [2].
7 Conclusion

In this paper, the relationship of degenerate kernel and projection methods was studied. It was shown that the modified degenerate kernel method (MDKM) becomes as a projection method. In fact, the MDKM for the bases of Lagrange polynomials and orthogonal functions became equivalent to the Lagrange-collocation and Galerkin methods respectively. Also, extension of the discussion to a system of equations can be accommodated. We pointed out that the corresponding analytical and numerical results are obtained using Mathematica.

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References


