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Research Article

Reproducing kernel method with Taylor expansion for linear Volterra integro-differential equations

Azizallah Alvandi[1], Mahmoud Paripour[2*]

(1) *Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran*

(2) *Department of Computer Engineering and Information Technology, Hamedan University of Technology, Hamedan, 65155-579, Iran*

Abstract

This research aims of the present a new and single algorithm for linear integro-differential equations (LIDE). To apply the reproducing Hilbert kernel method, there is made an equivalent transformation by using Taylor series for solving LIDEs. Shown in series form is the analytical solution in the reproducing kernel space and the approximate solution u_N is constructed by truncating the series to N terms. It is easy to prove the convergence of u_N to the analytical solution. The numerical solutions from the proposed method indicate that this approach can be implemented easily which shows attractive features.

Keywords: Reproducing Hilbert kernel, Integro-differential equation, Approximate solution.

1 Introduction

In mathematical modeling of real-life problems, we need to deal with functional equations, e.g. partial differential equations, integral and integro-differential equation, stochastic equations and others. Many mathematical formulations of physical phenomena contain integro-differential equations, these equations arise in fluid dynamics, biological models and chemical kinetics. Numerical modeling of integral and integro-differential equations have been paid attention by many scholars. Several numerical methods have been developed for the solution of the integro-differential equations. The iterated Galerkin methods have been proposed in [1]. Compact finite difference method has been used for integro-differential equations by Zhao and Corless [2]. Moreover, in [3], there are found mixed interpolation collocation methods to solve first- and second-order Volterra linear integro-differential equations. For methods using a quadrature rule, degenerate kernels, interpolation or extrapolation, homotopy perturbation, Taylor expansion, Chebyshev collocation and Wavelet-Galerkin [4-11].

Recently, the applications of reproducing kernel method (RKM) have become of great interest for scholars [12-25]; In this paper, the use of RKM to solve linear volterra integro-differential equations of the form is introduced.

$$\sum_{i=0}^{p} c_i(x)u^{(i)}(x) = f(x) + \int_a^x k(x,t)u(t)\mathrm{d}t, \qquad x,t \in [a,b], \tag{1.1}$$

under the initial conditions

$$u^{(i)}(a) = d_i, \qquad i = 0,1,\cdots,p-1,$$

*Corresponding author. Email address: paripour@gmail.com

where $u(x)$ is an unknown function to be determined, the known continuous functions $c_i(x)$, $f(x)$ and $k(x,t)$ are defined on the interval [a, b].
The rest of the paper is organized as the following: the introduction is expressed in Section 1. In the next section, we define a reproducing kernel space and give its corresponding reproducing kernel. Eq. (1.1) is converted into an equivalent integro-differential equation. The equivalent integro-differential equation is solved using reproducing kernel method (RKM) in Section 3. The numerical experiments are given in Section 4. Finally, in Section 5 some concluding remarks are presented.

2 Reproducing kernel Hilbert space

To illustrate the basic ideas of the reproducing kernel method, we consider the following some useful reproducing kernel spaces ${}^oW^m[a,b]$.

Definition 2.1. ${}^oW^m[a,b] = \{u^{(m-1)}(x)$ *is an absolutely continuous real value function,* $u^{(m)}(x) \in L^2[a,b]\}, u^{(i)}(a) = 0, i = 0,1,\cdots,p-1\}$. *The inner product and norm in* ${}^oW^m[a,b]$ *are given respectively by*

$$\langle u,v\rangle = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(x)v^{(m)}(x)\,\mathrm{d}x, \tag{2.2}$$

and

$$\|u\|_m = \sqrt{\langle u,u\rangle}_m, \qquad u,v \in {}^oW^m[a,b]. \tag{2.3}$$

According to [26], the space ${}^oW^m[a,b]$ is a reproducing kernel Hilbert space. There exists $R_y(x) \in {}^oW^m[a,b]$, for any $u(y) \in {}^oW^m[a,b]$ and each fixed $x \in [a,b]$, $y \in [a,b]$, such that $\langle u(y), R_x(y)\rangle = u(x)$. The reproducing kernel $R_y(x)$ can be denoted by

$$R_y(x) = \begin{cases} R_1(x,y) = \sum_{i=1}^{2m} c_i(y)x^{i-1}, & y \le x, \\ R_2(x,y) = \sum_{i=1}^{2m} d_i(y)x^{i-1}, & y > x, \end{cases} \tag{2.4}$$

where coefficients $c_i(y), d_i(y)$, $\{i = 1,2,\cdots,2m\}$, could be obtained by solving the following equations

$$\frac{\partial^i R_y(x)}{\partial x^i}|_{x=y+0} = \frac{\partial^i R_y(x)}{\partial x^i}|_{x=y-0}, \qquad i = 0,1,2,\cdots,2m-2, \tag{2.5}$$

$$(-1)^m\left(\frac{\partial^{2m-1}R_y(x)}{\partial x^{2m-1}}|_{x=y+0} - \frac{\partial^{2m-1}R_y(x)}{\partial x^{2m-1}}|_{x=y-0}\right) = 1, \tag{2.6}$$

$$\begin{cases} \frac{\partial^i R_y(a)}{\partial x^i} - (-1)^{m-i-1}\frac{\partial^{2m-i-1}R_y(a)}{\partial x^{2m-i-1}} = 0, & i = p,p+1,\cdots,m-1, \\ \frac{\partial^{2m-i-1}R_y(b)}{\partial x^{2m-i-1}} = 0, & i = 0,1,\cdots,m-1, \\ R_y^{(i)}(a) = 0, & i = 0,1,\cdots,p-1. \end{cases} \tag{2.7}$$

3 The analytical solution

3.1 Transformation of Eq. (1.1)

In this section, we convert Eq. (1.1) into an equivalent equation, which is easily solved using RKM.
Consider the integro-differential equation with the given conditions in relation (1.1). When er write the Taylor series expansion of $u(t)$ in terms of expanding around the given point x belonging to the interval $[a,b]$, this obtained in the following form

$$u(t) = u(x) + (t-x)u'(x) + \frac{(t-x)^{2!}}{2}u''(x) + \cdots + \frac{(t-x)^n}{n!}u^{(n)}(x) + \frac{(t-x)^{n+1}}{(n+1)!}u^{(n+1)}(\zeta_{x,t}), \tag{3.8}$$

where $\zeta_{x,t}$ is between x and t. By substituting relation (3.8) into Eq. (1.1), we have

$$\sum_{i=0}^{p} c_i(x)u^{(i)}(x) - \int_0^x k(x,t)\sum_{K=0}^{n}\frac{(t-x)^K}{K!}u^{(K)}(x)\mathrm{d}t + E_n(x)$$
$$= \sum_{i=0}^{p} c_i(x)u^{(i)}(x) - \sum_{K=0}^{n}\frac{u^{(K)}(x)}{K!}\int_0^x k(x,t)(t-x)^K\mathrm{d}t + E_n(x) = f(x), \tag{3.9}$$

where $u^{(0)}(x) = u(x)$ and $E_n(x) = \frac{-1}{(n+1)!}\int_0^x k(x,t)(t-x)^{n+1}u^{(n+1)}(\zeta_{x,t})\mathrm{d}t$. Alternatively, accordingly the truncated Taylor series of $u(t)$ can be used to solve the following equation

$$\sum_{i=0}^{p} c_i(x)u^{(i)}(x) - \sum_{K=0}^{n}\frac{u^{(K)}(x)}{K!}\int_0^x k(x,t)(t-x)^K\mathrm{d}t = f(x). \tag{3.10}$$

For the two cases $k=0$ and $k>0$, $\int_0^x k(x,t)(t-x)^K\mathrm{d}t$ is computable for $K = 0,1,\cdots,n$.

3.2 Definition of operators

We define the operator $\mathbb{L} : {}^oW^m[a,b] \longrightarrow W^1[a,b]$ as

$$\mathbb{L}(u) = \sum_{i=0}^{p} c_i(x)u^{(i)}(x) - \sum_{K=0}^{n}\frac{u^{(K)}(x)}{K!}\int_0^x k(x,t)(t-x)^K\mathrm{d}t, \tag{3.11}$$

then equation (3.11) can be written as

$$\mathbb{L}(u) = f(x). \tag{3.12}$$

It is clear that $\mathbb{L}$ is a bounded linear operator and $\mathbb{L}^*$ is the adjoint operator of $\mathbb{L}$.

3.3 Solution of Eq. (1.1)

In order to represent the analytical solution of the model problem, we can assume that $\mathbb{L} : {}^oW^m[a,b] \longrightarrow W^1[a,b]$ is an invertible bounded linear operator, choose a countable dense subset $\{x_i\}_{i=1}^{\infty}$ in $[a,b]$ and define

$$\psi_i(x) = [\mathbb{L}_y R_y(x)](x_i) = \mathbb{L}^* R_y(x_i), \qquad i = 1,2,\cdots, \tag{3.13}$$

where

$$\psi_i(x) = \sum_{j=0}^{p} c_j(x_i)\frac{\partial^j R_x(t)}{\partial t^j}|_{t=x_i} - \sum_{K=0}^{n}\frac{\frac{\partial^K R_x(t)}{\partial t^K}|_{t=x_i}}{K!}\int_0^x k(x,t)(t-x)^K\mathrm{d}t. \tag{3.14}$$

The orthonormal system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ of ${}^oW^m[a,b]$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\bar{\psi}_i(x) = \sum_{k=1}^{i}\beta_{ik}\psi_k(x), \qquad (\beta_{ii} > 0, \quad i = 1,2,\cdots), \tag{3.15}$$

where β_{ik} are orthogonal coefficients.

Theorem 3.1. *If $\{x_i\}_{i=1}^{\infty}$ is dense in $[a,b]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is complete system in ${}^oW^m[a,b]$.*

Proof. If for any $u(x) \in {}^oW^m[a,b]$, it has $\langle u(x), \psi_i(x)\rangle = 0 \quad i = 1,2,\ldots,$
namely

$$\begin{aligned}\langle u(x), \psi_i(x)\rangle &= \langle u(x), (\mathbb{L}yR_x(y)(x_i)\rangle \\ &= \mathbb{L}_y\langle u(x), R_x(y)\rangle(x_i).\end{aligned} \tag{3.16}$$

It is worth mentioning that $\{x_i\}_{i=1}^{\infty}$ is a dense set, hence $\mathbb{L}_y u(x) \equiv 0$. It follows that $u(x) \equiv 0$. So, the proof of theorem can be completed. □

In the following, the representation of the exact solution of Eq. (1.1) is given in the reproducing kernel space ${}^oW^m[a,b]$.

Theorem 3.2. *If $u(x)$ is the solution of Eq. (1.1), then*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\bar{\psi}_i(x). \tag{3.17}$$

Proof. It suffices to expand $u(x)$ to Fourier series in terms of normal orthogonal basis $\bar{\psi}_i(x)$ in ${}^oW^m[a,b]$,

$$\begin{aligned}u(x) &= \sum_{i=1}^{\infty}\langle u(x), \bar{\psi}_i(x)\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle u(x), \psi_k(x)\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle u(x), \mathbb{L}^*\varphi_k(x)\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle \mathbb{L}u(x), \varphi_k(x)\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle f(x), \varphi_k(x)\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\bar{\psi}_i(x).\end{aligned} \tag{3.18}$$

The proof is complete. □

By truncating the series of (3.17), we obtain the approximate solution

$$u_N(x) = \sum_{i=1}^{N}\sum_{k=1}^{i} \beta_{ik} f(x_k)\bar{\psi}_i(x).$$

Lemma 3.1. *Since ${}^oW^m[a,b]$ is a Hilbert space, it becomes evident that $\sum_{i=1}^{\infty}(\sum_{k=1}^{i}\beta_{ik} f(x_k))^2 < \infty$. Therefore, the sequence u_N is convergent in the sense of norm $\|.\|_m$.*

Lemma 3.2. *If $u(x) \in {}^oW^m[a,b]$, there exists a constant c such that $|u(x)| \leq c\|u\|_m$.*

Proof.

$$|u(x)| = |\langle u(y), R_x(y)|\rangle \leq \|u(y)\|\|R_x(y)\|_m,$$

with a constant c such that

$$|u(x) \leq c\|u\|_m.$$

The proof of the lemma can be completed. □

According to [26], the following theorem can be obtained

Theorem 3.3. *Assume that $\|u_N(x)\|$ is bounded and Eq. (3.12) has a unique solution. If $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[0,1]$, then N-term approximate solution $u_N(x)$ converges to the exact solution $u(x)$ of Eq. (3.12) and the exact solution is expressed as*

$$u(x) = \sum_{i=1}^{\infty} B_i \bar{\psi}_i(x), \tag{3.19}$$

where $B_i = \sum_{k=1}^{i} \beta_{ik} f(x_k)$.

4 Numerical experiments

There are given 4 examples in this section with exact solutions. In these examples we take $N = 10$ and $m > n$ where n is the number of terms of the Taylor series and N is that of the Fourier series of the unknown function $u(x)$.

Example 4.1. *Considering the linear Volterra integro-differential equation [9], we have*

$$\begin{cases} u'(x) + u(x) + \int_0^x tu(t)\mathrm{d}t = (x^2+2x+1)e^{-x} + 5x^2 + 8, & 0 \le x \le 1, \\ u(0) = 10, \end{cases}$$

with the exact solution $u(x) = 10 - xe^{-x}$.

We tried to find the approximate solution by the proposed method for $n = 2$. The comparison between the exact solution and the approximate solution and the absolute errors in spaces $W^5[0,1], W^6[0,1]$ are graphically shown in figure 1, respectively. The absolute errors between $u(x)$ and $u_{10}(x)$ in spaces $W^5[0,1], W^6[0,1]$ are shown in Table 1. However, by increasing m, the behavior improves.

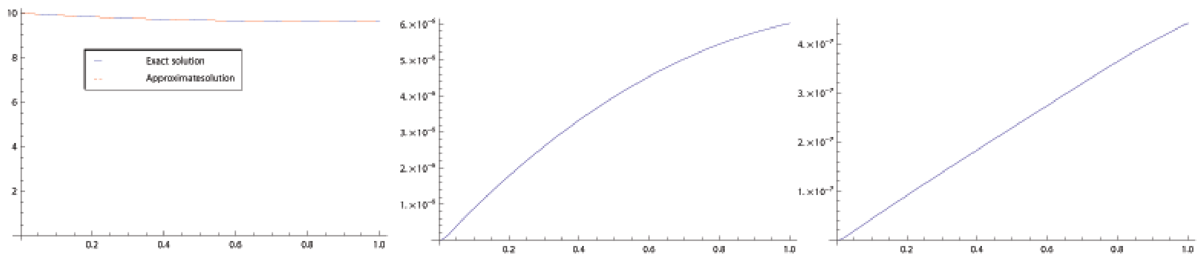

Figure 1: The figures of the approximate solution, the absolute errors in W^5 and W^6, respectively left to right.

Table 1: Numerical results of Ex. 4.1.

Node	$\|u_N(x)-u(x)\|_{W^5}, N=10$	$\|u_N(x)-u(x)\|_{W^6}, N=10$
0	0	0
0.1	8.88713E-7	4.38892E-7
0.2	1.79025E-6	9.15135E-7
0.3	2.60324E-6	1.38127E-7
0.4	3.33138E-6	1.84034E-7
0.5	3.97741E-6	2.29456E-7
0.6	4.54291E-6	2.74487E-7
0.7	5.02855E-6	3.19634E-7
0.8	5.43442E-6	3.64149E-7
0. 9	5.76043E-6	4.05433E-7
1	6.00774E-6	4.42064E-7

Example 4.2. *Considering r the following third order linear Volterra integro-differential equation [8], we have*

$$\begin{cases} u^{(3)} - xu^{(2)} - 4\int_0^x x^2t^3u(t)\mathrm{d}t = \frac{4}{7}x^9 - \frac{8}{5}x^7 - x^6 + 6x^2 - 6, \qquad 0 \le x,t \le 1, \\ u(0) = 1, u'(0) = 2, u^{(2)}(0) = 0, \end{cases}$$

with the exact solution $u(x) = -x^3 + 2x + 1$. *In this stage, we can find the approximate solution by the proposed method for* $n = 3$. *The comparison between the exact solution and the approximate solution and the absolute errors in spaces* $W^4[0,1], W^5[0,1]$ *are graphically shown in figure 2, respectively. The absolute errors between* $u(x)$ *and* $u_{10}(x)$ *in spaces* $W^4[0,1], W^5[0,1]$ *are shown in Table 2. However, by increasing m, the behavior improves.*

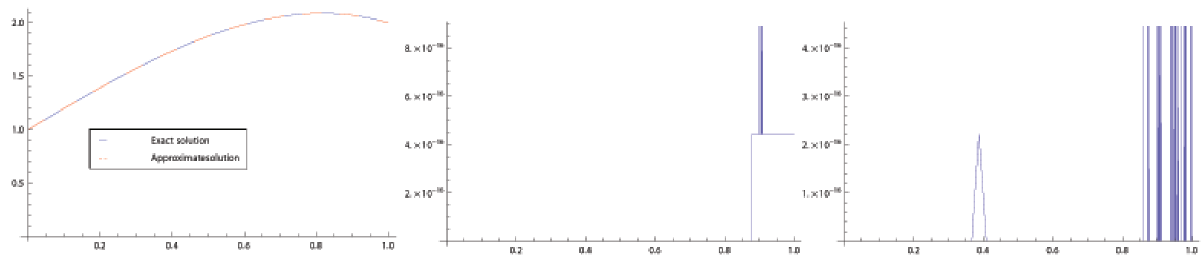

Figure 2: The figures of the approximate solution, the absolute errors in W^4 and W^5, respectively left to right.

Table 2: Numerical results of Ex. 4.2.

Node	$\lvert u_N(x) - u(x)\rvert_{W^4}, N = 10$	$\lvert u_N(x) - u(x)\rvert_{W^5}, N = 10$
0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0. 9	4.44089E-16	0
1	4.44089E-16	2.22045E-16

Example 4.3. *Considering the first order linear Volterra integro-differential equation [9], we have*

$$\begin{cases} u'(x) + u(x) - \int_0^x e^{(t-x)}u(t)\mathrm{d}t = 0, \qquad 0 \le x \le 1, \\ u(0) = 1, \end{cases}$$

with the exact solution $u(x) = e^{-x}\cosh(x)$. *There was an effort made to approximate solution by the proposed method for* $n = 5$. *The comparison between the exact solution and the approximate solution and the absolute errors in spaces* $W^6[0,1], W^7[0,1]$ *is graphically shown in figure 3, respectively. The absolute errors between* $u(x)$ *and* $u_{10}(x)$ *in spaces* $W^6[0,1], W^7[0,1]$ *are shown in Table 3. However, by increasing m, the behavior improves.*

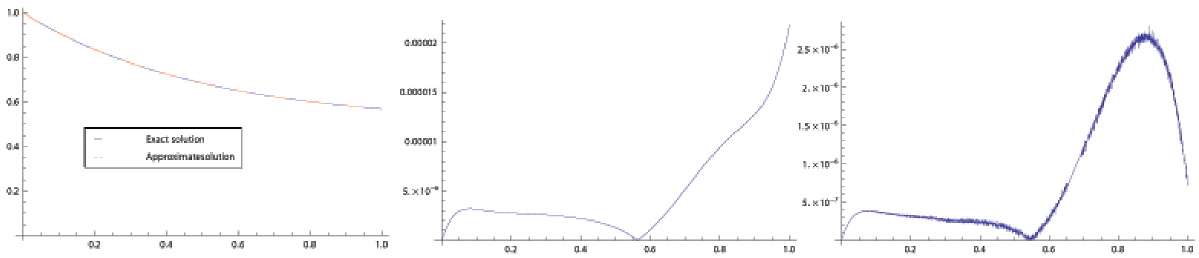

Figure 3: The figures of the approximate solution, the absolute errors in W^6 and W^7, respectively, left to right.

Table 3: Numerical results of Ex. 4.3.

Node	$\lvert u_N(x)-u(x)\rvert_{W^6}, N=10$	$\lvert u_N(x)-u(x)\rvert_{W^7}, N=10$
0	0	0
0.1	3.12933E-6	3.72109E-7
0.2	2.72793E-6	3.16225E-7
0.3	2.55976E-6	2.67057E-7
0.4	2.17746E-6	2.34321E-7
0.5	1.27074E-6	1.17419E-7
0.6	1.07808E-6	2.88075E-7
0.7	5.13952E-6	1.20459E-6
0.8	9.43276E-6	2.31163E-6
0. 9	1.27501E-5	2.62441E-6
1	2.18428E-5	7.50394E-7

Example 4.4. *Considering the first order linear Volterra integro-differential equation [27], we have*

$$\begin{cases} u'(x)+\int_0^x tu(t)\mathrm{d}t = -1+\frac{1}{2}x^2 - xe^x, \qquad 0\le x\le 1, \\ u(0)=0, \end{cases}$$

the corresponding exact solution is given by $u(x)=1-e^x$.

Let $n=5$ and applying the reproducing kernel method. The comparison between the exact solution and the approximate solution and the absolute errors in spaces $W^6[0,1], W^7[0,1]$ is graphically shown in figure 4, respectively. The absolute errors between $u(x)$ and $u_{10}(x)$ in spaces $W^6[0,1], W^7[0,1]$ are shown in Table 4. However, by increasing m, the behavior improves.

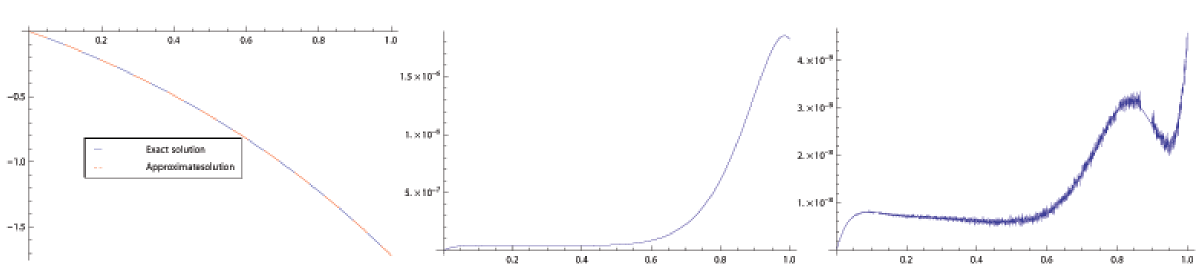

Figure 4: The figures of the approximate solution, the absolute errors in W^6 and W^7, respectively left to right.

Table 4: Numerical results of Ex. 4.4.

Node	$\lvert u_N(x)-u(x)\rvert_{W^6}, N=10$	$\lvert u_N(x)-u(x)\rvert_{W^7}, N=10$
0	0	0
0.1	4.34853E-8	8.16755E-9
0.2	4.16036E-8	7.21391E-9
0.3	4.17865E-8	7.11712E-9
0.4	4.24348E-8	5.67206E-9
0.5	4.88258E-8	5.98150E-9
0.6	8.58927E-8	7.72599E-9
0.7	2.24325E-7	1.65949E-8
0.8	6.06379E-7	2.96067E-8
0. 9	1.34911E-6	2.57549E-8
1	1.82209E-6	4.59614E-8

5 Concluding remarks

In this study, we developed an efficient and computationally attractive method to solve the linear Volterra integro-differential equations. We used the Taylor series and solved examples with our proposed method. The method can be easily implemented and its algorithm is simple and efficient to approximate the unknown function. However, to obtain better results, this recommended to use larger parameter m. The convergence accuracy of this method was examined in several numerical examples.

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