Approximate solutions of time fractional Kawahara and modified Kawahara equations by Fractional complex transform

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Abstract
In this paper, fractional complex transform with new iterative method (NIM) is used to obtain approximate solutions for the nonlinear time fractional Kawahara and modified Kawahara equations based on He’s fractional derivative. Fractional complex transform is proposed to convert time fractional Kawahara and modified Kawahara equations to the nonlinear ordinary differential equations and then NIM is applied to the new obtained equations. The obtained approximate solutions are compared with the exact solutions to verify the applicability, efficiency and accuracy of the method.

Keywords: Fractional complex transforms, He’s fractional derivative, New iterative method, Time fractional Kawahara and modified Kawahara equations.

1 Introduction
In the past few decades, the topic of fractional calculus has become more popular and applied in various fields of science and engineering, such as fluid mechanics, diffusive transport, electrical networks, electromagnetic theory, different branches of physics, biological sciences and groundwater problems [1, 2, 3, 4]. Fractional calculus allows integration and differentiation of arbitrary order. Many mathematicians and researchers have tried to model several physical or biological processes using fractional differential equations. Solving these equations is turn out to be wide area of research and interest for researchers from various fields. Some of the recent analytical and numerical methods for solving linear and nonlinear fractional differential equations are the Adomian decomposition method ADM[5, 6], Variational iteration method VIM [7, 8], Homotopy-perturbation method HPM [9], Homotopy analysis method [10], Finite element method [11] and so on. In recent times, an iterative method was proposed by Daftardar-Gejji and Jafari [12, 13] which is known as new iterative method (NIM). This method is very useful and simple in fractional calculus for solving linear and nonlinear fractional partial differential equations.
In literature several useful transforms like the laplace transform, the fourier transform, the backlund transformation, the integral transform, and the local fractional integral transforms suggested to solve numerous problems. In recent times, the fractional complex transform has been proposed in [14, 15, 16] to convert fractional-order differential equations based on He’s fractional derivative [17, 18] into integer order differential equations, and further can be solved by any computational method.
Nonlinear wave phenomena arise in various parts of science and engineering such as dispersion, diffusion, reaction and convection. Starting from the classical technique of the Inverse scattering transform [19], various powerful mathematical methods such as bilinear transformation [20], the tanh–sech method [21], extended tanh method [22], exp–function method [23], sine–cosine method [24] have been proposed for obtaining exact and approximate analytic solutions. The Kawahara and modified Kawahara equations have becomes the subject of active and broad research area in recent decades [25, 26, 27]. The kawahara equation was first suggested by Kawahara in 1972 [28] for describing solitary-wave propagation in media. It arises in the theory of magneto–acoustic waves in plasmas and in theory of shallow water waves with surface tension. Likewise the modified Kawahara equation has extensive applications in physics such as plasma waves, capillary-gravity water waves, water waves with surface tension, etc. [29, 30, 31, 32]. In the present paper we have applied fractional complex transform with new iterative method to find approximate solutions of time fractional Kawahara and modified Kawahara equations which are respectively given below as:

\[ \frac{\partial^\alpha u}{\partial t^\alpha} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial x^2} = 0 \quad 0 < \alpha \leq 1 \]  

(1.1)

with initial condition

\[ u(x,0) = \frac{105}{169} sech^4 \left( \frac{x}{2\sqrt{13}} \right) \]  

(1.2)

\[ \frac{\partial^\alpha u}{\partial t^\alpha} + u^2 \frac{\partial u}{\partial x} + p \frac{\partial^3 u}{\partial x^3} + q \frac{\partial^2 u}{\partial x^2} = 0 \quad 0 < \alpha \leq 1 \]  

(1.3)

where p, q are nonzero real constants and initial condition is,

\[ u(x,0) = \frac{3p}{\sqrt{-16q}} sech^2(Kx), \quad K = \frac{1}{2} \sqrt{\frac{-p}{5q}} \]  

(1.4)

Eq.(1.1) and (1.3) becomes the original Kawahara and modified Kawahara equations for $\alpha = 1$ [28].

The rest of this paper is organized as follows. In Sections 2, basic definitions are presented. In Sections 3 and 4 we give an analysis of the fractional complex transform and new iterative method. The numerical results and graphs for the time fractional Kawahara and modified Kawahara equations are presented in Section 5. Finally, we give our conclusions in Section 6.

2 Basic Definitions

In literature there are many definitions on fractional derivatives [1–4] but the most frequently used are as below

2.1. Riemann–Liouville definition:

\[ D^\alpha_x(f(x)) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-\alpha-1} f(t) dt \]  

(2.5)

where $m - 1 < \alpha < m, \quad m \in N \cup \{0\}$

2.2. Caputo’s definition:

\[ D^\alpha_x(f(x)) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \frac{d^m}{dx^m} f(t) dt \]  

(2.6)

where $m - 1 < \alpha < m, \quad m \in N \cup \{0\}$

2.3. Jumaries definition:[33]

\[ D^\alpha_x(f(x)) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-\alpha-1} [f(t) - f(0)] dt \]  

(2.7)

2.4. Xiao-Jun Yang’s definition:[34]

\[ D^{(\alpha)}_x(f(x_0)) = f^{(\alpha)}(x_0) = \left( \frac{d^\alpha f(x)}{dx^\alpha} \right)_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha(f(x) - f(x_0))}{(x-x_0)^\alpha} \]  

(2.8)
where \( \Delta \alpha (f(x) - f(x_0)) \cong \Gamma(1 + \alpha) \Delta (f(x) - f(x_0)) \)

2.5. Ji-Huan He’s fractal derivative:[15]

\[
\frac{Du}{Dx^\alpha} = \Gamma(1 + \alpha) \lim_{\Delta x \to 0} \frac{u(x_1) - u(x_2)}{(x_1 - x_2)^\alpha}
\]

(2.9)

where \( \Delta x \) does not tend to zero. It can be the thickness (L) of a porous medium.

2.6. He’s fractional derivative:[15]

\[
\frac{\partial^\alpha u}{\partial x^\alpha} = \Gamma\left(\frac{m}{m - \alpha}\right) \int_{x_0}^{x} (t-x)^{m-\alpha-1} [u_0(t) - u(t)] dt
\]

(2.10)

where \( u_0(x,t) \) is the solution of its continuous partner of the problem with the same initial condition of the fractal partner.

3 Fractional complex transform

Recently, very efficient and useful technique for solving fractional differential equations called fractional complex transform suggested by Li and He[14] has appeared. Fractional complex transform is a simple solution procedure to convert the fractional differential equations into classical differential equations, hence all the analytical methods dedicated to the advanced calculus can be applied straightforward to the fractional calculus.

Consider the following general fractional differential equation

\[
f(u, u_\alpha, u_x, u_y, u_z,\ldots) = 0
\]

(3.11)

where \( u_\alpha = \frac{\partial^\alpha u(x,y,z,t)}{\partial t^\alpha} \) denotes He’s fractional derivation, \( u \) is continuous (but not necessarily differentiable) function.

0 < \( \alpha \leq 1 \), 0 < \( \beta \leq 1 \), 0 < \( \gamma \leq 1 \), 0 < \( \lambda \leq 1 \)

Introducing the following fractional complex transforms

\[
T = \frac{qt^\alpha}{\Gamma(1 + \alpha)}, \\
X = \frac{px^\beta}{\Gamma(1 + \beta)}, \\
Y = \frac{ky^\gamma}{\Gamma(1 + \gamma)}, \\
Z = \frac{lz^\lambda}{\Gamma(1 + \lambda)}
\]

where \( p, q, k \) and \( l \) are unknown constants. Then using chain rule of differentiation and basic properties special function like gamma function fractional derivatives are converted into partial derivatives as:

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = q \frac{\partial u}{\partial T}, \\
\frac{\partial^\beta u}{\partial x^\beta} = p \frac{\partial u}{\partial X}, \\
\frac{\partial^\gamma u}{\partial y^\gamma} = k \frac{\partial u}{\partial Y}, \\
\frac{\partial^\lambda u}{\partial z^\lambda} = l \frac{\partial u}{\partial Z}
\]

Therefore, the fractional partial differential equations are easily converted into classical partial differential equations, which can be solved further by new iterative method. Hence it becomes simple to deal with fractional calculus.
4 The New Iterative Method

Daftardar-Gejji and Jafari [12] have introduced a new iterative method (NIM) which is very powerful technique to solve linear and nonlinear functional equations.

In this method, consider a functional equation of the form

$$u = f + L(u) + N(u)$$  (4.12)

Where $f$ is a given function, $L$ and $N$ are given linear and non-linear operator. Let $u$ be a solution of Eq.(4.12) having the series form:

$$u = \sum_{i=0}^{\infty} u_i$$  (4.13)

Since $L$ is linear

$$L\left(\sum_{i=0}^{\infty} u_i\right) = \sum_{i=0}^{\infty} L(u_i)$$

The nonlinear operator here is decomposed as:

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty} \left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}$$  (4.14)

$$= \sum_{i=0}^{\infty} G_i$$  (4.15)

where $G_0 = N(u_0)$ and $G_i = \left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}, i \geq 1$

Hence Eq.(4.12) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + \sum_{i=0}^{\infty} G_i$$  (4.16)

Further define the recurrence relation:

$$u_0 = f$$

$$u_1 = L(u_0) + G_0$$

$$u_{m+1} = L(u_m) + G_m, \quad m = 1, 2, \ldots$$  (4.17)

Then

$$(u_1 + u_2 + \cdots + u_{m+1}) = L(u_0 + u_1 + \cdots + u_m) + N(u_0 + u_1 + \cdots + u_m), \quad m = 1, 2, \ldots$$

and $u = f + \sum_{i=1}^{\infty} u_i$

4.1 Convergence of NIM

Here we present the condition of convergence of the series $\sum u_i$ which is given in detail in [35].

Theorem 4.1. If $N$ is analytic in a neighbourhood of $u_0$ and

$$\|N^{(n)}(u_0)\| = \text{Sup} \left\{\frac{N^{(n)}(u_0)(h_1, h_2, \ldots, h_n)}{\|h_i\|} \mid 1 \leq i \leq n \leq 1 \right\} \leq L,$$

for any $n$ and for some real $L > 0$ and $\|u_i\| \leq M < \frac{1}{L}, i = 1, 2, \ldots$, then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent and moreover,

$$\|G_n\| = L M^n e^{n-1}(e-1), \quad n = 1, 2, \ldots$$

Now to show boundedness of $\|u_i\|$, for every $i$ the conditions on $N^{(n)}(u_0)$ are given which are sufficient to guarantee convergence of the series.

Theorem 4.2. If $f$ is analytic and $\|N^{(n)}(u_0)\| \leq M < \frac{1}{L}$, for every $n$ then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent.
5 Numerical Application

In this section, we test two initial value problems associated with the time fractional Kawahara and modified Kawahara equations, in order to demonstrate the efficiency of the combination of FCT and NIM. Computations are done with the help of Mathematica software.

5.1 Approximate solution for time fractional Kawahara equation

Consider the time fractional Kawahara equation (1.1) with initial condition (1.2). The exact solution of Eq. (1.1) is given in [29] as

\[ u(x,t) = \frac{105}{169} \text{sech}^{4} \left( \frac{1}{2\sqrt{13}} \left( x - \frac{36t}{169} \right) \right) \]  

(5.18)

First, we convert fractional Kawahara into ordinary Kawahara equation by using fractional complex transform as described in section 3

Let

\[ T = \frac{t^\alpha}{\Gamma(1+\alpha)} \]  

(5.19)

Hence

\[ \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial u}{\partial T} \]

Therefore Eq.(1.1) is converted into the following form

\[ \frac{\partial u}{\partial T} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \frac{\partial^5 u}{\partial x^5} = 0 \]  

(5.20)

with the initial condition (1.2) rewritten as

\[ u(x,0) = \frac{1680}{169} \frac{e^{\frac{2x}{\sqrt{13}}}}{(e^{\frac{2x}{\sqrt{13}}} + 1)^4} \]

Applying Integral operator \( I_T \) on both side of Eq.(5.20) and using initial condition we obtain the relation

\[ u(x,T) = u_0(x,T) + L(u) + N(u) \]

where

\[ L(u) = I_T \left( \frac{\partial^3 u}{\partial x^3} - \frac{\partial^3 u}{\partial x^5} \right) \]

and \( N(u) = I_T \left( -\frac{\partial^5 u}{\partial x^5} \right) \)

Taking series solution as \( u(x,T) = \sum_{i=0}^{\infty} u_i(x,T) \) and using (4.14) and (4.17)

\[ u_0 = \frac{1680}{169} \frac{e^{\frac{2x}{\sqrt{13}}}}{(e^{\frac{2x}{\sqrt{13}}} + 1)^4} \]

Applying NIM successively we get

\[ u_1 = \frac{120960 \ e^{\frac{2x}{\sqrt{13}}} \left( -1 + e^{\frac{2x}{\sqrt{13}}} \right) T}{28561 \sqrt{13} \left( 1 + e^{\frac{2x}{\sqrt{13}}} \right)^5} \]

\[ u_2 = \frac{1}{137858491849 \left( 1 + e^{\frac{2x}{\sqrt{13}}} \right)^7} \left( 9566968320 \ (e^{\frac{2x}{\sqrt{13}}} + e^{\frac{2x}{\sqrt{13}}}) + 19133936640 \ (e^{\frac{2x}{\sqrt{13}}} + e^{\frac{2x}{\sqrt{13}}}) \right) T^2 \]
Substituting (5.19) in the equation (5.21) we get approximate solution of Eq.(1.1) as

\[
- \frac{1}{137858491849 \left(1 + e^{\frac{7x}{\sqrt{13}}}\right)^{11}} \left(38267873280 \left(e^{\frac{4x}{\sqrt{13}}} + e^{\frac{7x}{\sqrt{13}}}\right) + 143504524800 \left(e^{\frac{5x}{\sqrt{13}}} + e^{\frac{6x}{\sqrt{13}}}\right)\right) T^2 \\
+ \frac{1}{1378584918 \left(1 + e^{\frac{7x}{\sqrt{13}}}\right)^{11}} \left(9754214400 \sqrt{13} \left(e^{\frac{7x}{\sqrt{13}}} - e^{\frac{4x}{\sqrt{13}}}\right) + 39016857600 \sqrt{13} \left(e^{\frac{5x}{\sqrt{13}}} + e^{\frac{6x}{\sqrt{13}}}\right)\right) T^3
\]

Continuing in the same way, remaining terms of the iteration formula (4.17) can be calculated.

The three term approximate solution is given by

\[
u(x,T) = u_0(x,T) + u_1(x,T) + u_2(x,T)
\]

Substituting (5.19) in the equation (5.21) we get approximate solution of Eq.(1.1) as

\[
u(x,T) = \frac{1}{137858491849 \left(1 + e^{\frac{7x}{\sqrt{13}}}\right)^{11}} \left(1370427611280 \left(e^{\frac{8x}{\sqrt{13}}} + e^{\frac{9x}{\sqrt{13}}}\right) + 9592993278960 \left(e^{\frac{8x}{\sqrt{13}}} + e^{\frac{9x}{\sqrt{13}}}\right)\right) \\
+ \frac{1}{1378584918 \left(1 + e^{\frac{7x}{\sqrt{13}}}\right)^{11}} \left(28778979836880 \left(e^{\frac{2x}{\sqrt{13}}} + e^{\frac{7x}{\sqrt{13}}}\right) + 47964966394800 \left(e^{\frac{5x}{\sqrt{13}}} + e^{\frac{6x}{\sqrt{13}}}\right)\right) \\
+ \frac{1}{1378584918 \left(1 + e^{\frac{7x}{\sqrt{13}}}\right)^{11}} \left(224558006400 \sqrt{13} \left(e^{\frac{3x}{\sqrt{13}}} - e^{\frac{5x}{\sqrt{13}}}\right) + 404204411520 \sqrt{13} \left(e^{\frac{4x}{\sqrt{13}}} - e^{\frac{3x}{\sqrt{13}}}\right)\right) T^0 \\
+ \frac{1}{1378584918 \left(1 + e^{\frac{7x}{\sqrt{13}}}\right)^{11}} \left(44911601280 \sqrt{13} \left(e^{\frac{3x}{\sqrt{13}}} - e^{\frac{5x}{\sqrt{13}}}\right) + 39016857600 \sqrt{13} \left(e^{\frac{5x}{\sqrt{13}}} + e^{\frac{6x}{\sqrt{13}}}\right)\right) T^1 \\
+ \frac{1}{1378584918 \left(1 + e^{\frac{7x}{\sqrt{13}}}\right)^{11}} \left(9566968320 \left(e^{\frac{4x}{\sqrt{13}}} + e^{\frac{5x}{\sqrt{13}}}\right) + 19133936640 \left(e^{\frac{5x}{\sqrt{13}}} + e^{\frac{6x}{\sqrt{13}}}\right)\right) T^2 \\
+ \frac{1}{1378584918 \left(1 + e^{\frac{7x}{\sqrt{13}}}\right)^{11}} \left(38267873280 \left(e^{\frac{4x}{\sqrt{13}}} + e^{\frac{5x}{\sqrt{13}}}\right) + 143504524800 \left(e^{\frac{5x}{\sqrt{13}}} + e^{\frac{6x}{\sqrt{13}}}\right)\right) T^3 \\
+ \frac{1}{1378584918 \left(1 + e^{\frac{7x}{\sqrt{13}}}\right)^{11}} \left(9754214400 \sqrt{13} \left(e^{\frac{7x}{\sqrt{13}}} - e^{\frac{4x}{\sqrt{13}}}\right) + 39016857600 \sqrt{13} \left(e^{\frac{5x}{\sqrt{13}}} + e^{\frac{6x}{\sqrt{13}}}\right)\right) T^4 \\
+ \frac{1}{1378584918 \left(1 + e^{\frac{7x}{\sqrt{13}}}\right)^{11}} \left(47964966394800 \left(e^{\frac{2x}{\sqrt{13}}} + e^{\frac{7x}{\sqrt{13}}}\right) + 404204411520 \sqrt{13} \left(e^{\frac{4x}{\sqrt{13}}} - e^{\frac{3x}{\sqrt{13}}}\right)\right) T^5
\]
Table 1: The numerical results for comparison of absolute error between the exact solution with three term approximations obtained by NIM of Eq.(1.1) for $\alpha = 1$

| x  | t    | Exact Solution | NIM solution for $\alpha = 1$ | NIM solution for $\alpha = \frac{2}{3}$ | Absolute error $|u - u_3|$ |
|----|------|----------------|---------------------------------|----------------------------------------|------------------|
| -5 | 0.02 | 0.2539853437   | 0.2539853435                    | 0.2528761694                         | 2.27500 × 10^{-10} |
|    | 0.04 | 0.2536250379   | 0.2536250361                    | 0.2520150099                         | 1.81933 × 10^{-9}  |
|    | 0.06 | 0.2532650172   | 0.2532650111                    | 0.2512939452                         | 6.13795 × 10^{-9}  |
|    | 0.08 | 0.2529052822   | 0.2529052677                    | 0.2506512815                         | 1.45438 × 10^{-8}  |
|    | 0.1  | 0.2525458332   | 0.2525458048                    | 0.2500613981                         | 2.83953 × 10^{-8}  |
| 0  | 0.02 | 0.6213013414   | 0.62130134141                  | 0.6212945518                         | 1.76636 × 10^{-13} |
|    | 0.04 | 0.6213000402   | 0.6213000402                   | 0.6212835735                         | 2.82618 × 10^{-12} |
|    | 0.06 | 0.6212978715   | 0.6212978715                   | 0.6212705216                         | 1.43068 × 10^{-11} |
|    | 0.08 | 0.6212948354   | 0.6212948354                   | 0.6212559099                         | 4.52161 × 10^{-11} |
|    | 0.1  | 0.62129093197  | 0.6212909318                   | 0.6212400166                         | 1.10391 × 10^{-10} |
| 5  | 0.02 | 0.2547068095   | 0.2547068097                    | 0.2558204358                         | 2.27670 × 10^{-10} |
|    | 0.04 | 0.2550679685   | 0.2550679703                   | 0.2566887941                         | 1.82204 × 10^{-9}  |
|    | 0.06 | 0.2554294112   | 0.2554294173                   | 0.2574184171                         | 6.15166 × 10^{-9}  |
|    | 0.08 | 0.2557911370   | 0.2557911516                   | 0.2580706620                         | 1.45871 × 10^{-8}  |
|    | 0.1  | 0.2561531457   | 0.2561531742                   | 0.2586709670                         | 2.85011 × 10^{-8}  |

Figure 1: Approx. soln of Eq.(1.1), for $\alpha = 1$

Figure 2: Exact. soln of Eq.(1.1) for $\alpha = 1$

5.2 Approximate solution for modified time fractional Kawahara equation

Next we consider the modified time fractional Kawahara equation (1.3) with initial condition (1.4). The exact solution for the classical modified Kawahara equation is given by [29]

$$ u(x,t) = \frac{3p}{\sqrt{10q}} \text{sech}^2 \left[ K(x - ct) \right], \quad c = \frac{25q - 4p^2}{25q} \tag{5.22} $$

As described in Section 3, we make the transformation

$$ T = \frac{t^\alpha}{1(1 + \alpha)} \tag{5.23} $$
then Eq.(1.3) converts to the following form

\[ \frac{\partial u}{\partial T} + u^2 \frac{\partial u}{\partial x} + p \frac{\partial^3 u}{\partial x^3} + q \frac{\partial^5 u}{\partial x^5} = 0 \] (5.24)

with the initial condition (1.4) rewritten as

\[ u(x, 0) = \frac{6 \sqrt{\frac{1}{2}} e^{\frac{\sqrt{\frac{1}{2}}}{\sqrt{-q}}} p}{(1 + e^{\frac{\sqrt{\frac{1}{2}}}{\sqrt{-q}}})^3 \sqrt{-q}} \] (5.25)

Applying Integral operator \( I_T \) on both side of (5.23) and using initial condition we obtain the relation

\[ u(x, T) = u_0(x, T) + L(u) + N(u) \]

where

\[ L(u) = -I_T \left( p \frac{\partial^3 u}{\partial x^3} + q \frac{\partial^5 u}{\partial x^5} \right) \quad \text{and} \quad N(u) = -I_T \left( u^2 \frac{\partial u}{\partial x} \right) \]

Taking series solution as \( u(x, T) = \sum_{i=0}^{\infty} u_i(x, T) \) and using (4.14) and (4.17)

\[ u_0 = \frac{6 \sqrt{\frac{1}{2}} e^{\frac{\sqrt{\frac{1}{2}}}{\sqrt{-q}}} p}{(1 + e^{\frac{\sqrt{\frac{1}{2}}}{\sqrt{-q}}})^3 \sqrt{-q}} \]

\[ u_1 = \frac{24 \sqrt{\frac{1}{2}} e^{\frac{1}{\sqrt{-q}}} \left( -1 + e^{\frac{1}{\sqrt{-q}}} \right) p^3 \sqrt{-q}}{125 \left( 1 + e^{\frac{1}{\sqrt{-q}}} \right)^3 \left( -q \right)^{\frac{1}{2}}} \]

Applying NIM successively we get rest of the terms of series solution. The three term approximate solution of Eq. (5.23) is given as

\[ u(x, T) = u_0(x, T) + u_1(x, T) + u_2(x, T) \] (5.26)

replacing \( T \) by \( \frac{\mu}{\Gamma(\alpha+\mu)} \) in the equation (5.25), we get solution of (1.3) as

\[ u(x, t) = \frac{6 \sqrt{\frac{1}{2}} e^{\frac{1}{\sqrt{-q}}} p}{1953125 \sqrt{3} \left( 1 + e^{\frac{1}{\sqrt{-q}}} \right)^8 \sqrt{-q} \sqrt{-q + 312500} \left( 1 + e^{\frac{1}{\sqrt{-q}}} \right)^7 \sqrt{-q} \frac{\mu^3}{\Gamma(\alpha+\mu)}} \]

\[ 9765625 \left( 1 + e^{\frac{1}{\sqrt{-q}}} \right)^{10} \sqrt{-q} \left( -q \right)^{\frac{13}{2}} \]

\[ 6 \sqrt{\frac{1}{2}} e^{\frac{1}{\sqrt{-q}}} p \left[ 5000 \sqrt{3} \left( 1 + e^{\frac{1}{\sqrt{-q}}} \right)^6 \sqrt{-q} \sqrt{-q + 312500} \left( 1 - 4e^{\frac{1}{\sqrt{-q}}} + e^{\frac{2}{\sqrt{-q}}} \right) p^4 \left( -q \right)^{\frac{1}{2}} q^4 \frac{\mu^2}{\Gamma(\alpha+\mu)^2} \]

\[ + \frac{9765625 \left( 1 + e^{\frac{1}{\sqrt{-q}}} \right)^{10} \sqrt{-q} \left( -q \right)^{\frac{13}{2}}}{1 + e^{\frac{1}{\sqrt{-q}}} \sqrt{-q}} \]
the equations are very near to each other. From these surfaces it can be observed that approximate solution and exact solution are of both equations respectively with their exact solutions. Also fig. 1 and 2 shows surfaces for approximate solution of Eq. (1.1) and exact solution of classical Kawahara and modified Kawahara equation is demonstrated for the absolute errors of Eq. (1.1) and Eq.(1.3) respectively with their exact solutions.

In tables 1 and 2 the computational results are obtained to get approximate solution of equations (1.1) and (1.3) respectively with $p = 0.001$ and $q = -1$. In Eq.(1.3) by using fractional complex transform with NIM. This technique provides accurate numerical solutions even if lower order approximations are used. The accuracy for the time fractional Kawahara and modified Kawahara equation is demonstrated for the absolute errors of Eq. (1.1) and Eq.(1.3) respectively with their exact solutions.

Also fig. 1 and 2 shows surfaces for approximate solution of Eq. (1.1) and exact solution of classical Kawahara equation and fig. 3 and 4 shows surfaces for approximate solution of Eq. (1.3) and exact solution of classical modified Kawahara equation. From these surfaces it can be observed that approximate solution and exact solution are of both the equations are very near to each other.
6 Conclusions

In this study, we have obtained the approximate solutions of time fractional Kawahara and modified Kawahara equations based on He’s fractional derivation by using fractional complex transform with new iterative method. It is seen that the solutions obtained converges very rapidly to the exact solutions in only second order approximations i.e. approximate solutions are very near to the exact solutions. We can conclude from the numerical results that the present technique is straightforward, efficient and provides very high accuracy. This combination of FCT with NIM is very simple, reliable and powerful technique for finding approximate solutions of many fractional physical models arising in science and engineering.

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