

Numerical study of nonlinear singular fractional differential equations arising in biology by operational matrix of shifted Legendre polynomials

D. Jabari Sabeg¹, R. Ezzati^{1*}(1) *Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.*

Copyright 2016 © D. Jabari Sabeg and R. Ezzati. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we present a new computational method for solving nonlinear singular boundary value problems of fractional order arising in biology. To this end, we apply the operational matrices of derivatives of shifted Legendre polynomials to reduce such problems to a system of nonlinear algebraic equations. To demonstrate the validity and applicability of the presented method, we present some numerical examples.

Keywords: Fractional differential equation, Operational matrix, Collocation method, Legendre polynomials.

1 Introduction

For modeling many physiology problems such as study of steady state oxygen-diffusion in a cell with Michaelis-Menten uptake kinetics [1, 2] bending of beams [3] and spring mass system [4], nonlinear differential equations are considered as essential instruments. These equations are also useful in study of the distribution of heat sources in the human head [5, 6] and tumor growth [7, 8, 9, 10, 11].

As we know, singular boundary value problems

$$y''(x) + \left(a + \frac{m}{x}\right)y'(x) = f(x, y), \quad 0 \leq x \leq 1, \quad (1.1)$$

with the initial conditions

$$\alpha_1 y(0) + \beta_1 y'(0) = A, \quad (1.2)$$

$$\alpha_2 y(1) + \beta_2 y'(1) = B, \quad (1.3)$$

arise in physiology, where $f(x, y)$ is continuous, $\frac{\partial f}{\partial y}$ exists and is continuous and also $\frac{\partial f}{\partial y} \geq 0$, $0 \leq x \leq 1$. these problems, with $m = 0, 1, 2$ and $a = 0$, arise in the study of various tumor growth problems with linear $f(x, y)$, and also, with non-linear $f(x, y)$ of the form $f(x, y) = f(y) = \frac{ny}{y+\eta}$, $n > 0$, $\eta > 0$ and $m = 2$, $a = 0$ arise in the study of steady state oxygen diffusion in a spherical cell with Michaelis-Menten uptake kinetics. A similar equation for $m = 2$

*Corresponding author. Email address: ezati@kiau.ac.ir

and $a = 0$ arise in the study of the distribution of heat sources in the human head, in which $f(x, y) = f(y) = -\delta e^{-\varepsilon y}$, $\delta > 0$, $\varepsilon > 0$. In this paper we generalize the definition of Eqs. (1.1)-(1.3) up to fractional order as following:

$$D^\alpha y(x) + \left(a + \frac{m}{x}\right) D^\beta y(x) = f(x, y), \quad 0 \leq x \leq 1, \quad 1 < \alpha \leq 2, \quad 0 < \beta \leq 1, \quad (1.4)$$

$$\alpha_1 y(0) + \beta_1 y'(0) = A, \quad (1.5)$$

$$\alpha_2 y(1) + \beta_2 y'(1) = B, \quad (1.6)$$

where A, B are constants, $f(x, y)$ is a continuous real-valued function. The main objective of this paper is to provide an introduction of a novel-method based on operational matrices of fractional derivative of shifted Legendre polynomials that have been demonstrated in [12] in order to solve equations (1.4)-(1.6). The main characteristic behind the approach using this technique is that it reduces these problems to those of solving a system of algebraic equations. The rest of the paper is organized as follows: In section 2, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory and shifted Legendre polynomials which are required for establishing our results. In Section 3 the shifted Legendre operational matrix of fractional derivative is introduced. Section 4 is devoted to applying the shifted Legendre operational matrix for the problems given in Eqs (1.4)-(1.6). In Section 5, we solve some numerical examples by using the proposed method in previous section. Finally section 6, concludes the paper.

2 Preliminaries and notations

Here, we recall some basic definitions and properties of fractional calculus that are used in this article. There are various definitions of fractional integration and differentiation, such as Grunwald-Letnikov's definition and Riemann-Liouville's definition. The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we will introduce a modified fractional differential operator D^α proposed by Caputo in his work on the theory of viscoelasticity [13].

Definition 2.1. *The Caputo fractional derivatives of order α is defined as*

$$D^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N},$$

where $\alpha > 0$ is the order of the derivative and n is the smallest integer greater than α . For the Caputo derivative we have: [14]

$$D^\alpha x^\beta = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \alpha \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geq \lceil \alpha \rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lfloor \alpha \rfloor, \end{cases}$$

$$D^\alpha c = 0, \quad (c \text{ is a constant}).$$

We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α , and the floor function $\lfloor \alpha \rfloor$ to denote the largest integer less than or equal to α . Also $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of an integer order.

2.1 Shifted Legendre polynomials

The well-known shifted Legendre polynomials of order i are defined on the interval $[0, 1]$ and satisfy in the following recursive formula:

$$p_{i+1}(x) = \frac{(2i+1)(2x-1)}{(i+1)} p_i(x) - \frac{i}{i+1} p_{i-1}(x), \quad i = 1, 2, \dots,$$

where $p_0(x) = 1$ and $p_1(x) = 2x - 1$. The analytic form of the shifted Legendre polynomial $p_i(x)$ of degree i is given by

$$p_i(x) = \sum_{k=0}^i (-1)^{i+k} \frac{(i+k)!x^k}{(i-k)!(k!)^2}. \tag{2.7}$$

For these polynomials, by using orthogonality properties, we have:

$$\int_0^1 p_i(x)p_j(x) dt = \begin{cases} \frac{1}{2i+1}, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases}$$

A function $y(x) \in L^2[0, 1]$ may be expressed in terms of shifted Legendre polynomials as

$$y(x) = \sum_{j=0}^{\infty} c_j p_j(x),$$

where the coefficients c_j are given by $c_j = (2j + 1) \int_0^1 p_j(x)y(x) dx$.

In practice, only the first $(m + 1)$ terms shifted Legendre polynomials are considered. Therefore, $y(x)$ can be written in the form

$$y(x) \simeq \sum_{j=0}^m c_j p_j(x) = C^T B(x), \tag{2.8}$$

where

$$C^T = [c_0, c_1, \dots, c_m], \tag{2.9}$$

$$B(x) = [p_0(x), p_1(x), \dots, p_m(x)]^T. \tag{2.10}$$

3 Operational matrix of derivative and fractional calculus

The derivative of the vector $B(x)$ can be expressed by

$$\frac{dB(x)}{dx} = D^{(1)}B(x), \tag{3.11}$$

where $D^{(1)}$ is the $(m + 1) \times (m + 1)$ operational matrix of derivative given by

$$D^{(1)} = (d_{ij}) = \begin{cases} 2(2j + 1), & \text{for } j = i - k, \\ 0, & \text{otherwise.} \end{cases} \begin{cases} k = 1, 3, \dots, m, & \text{if } m \text{ odd,} \\ k = 1, 3, \dots, m - 1, & \text{if } m \text{ even,} \end{cases}$$

By using Eq. (3.11), it is clear that

$$\frac{d^n B(x)}{dx^n} = (D^{(1)})^n B(x), \tag{3.12}$$

where $n \in N$ and the superscript, in $D^{(1)}$, denote matrix powers. Thus

$$(D^{(1)})^n = D^{(n)}, \quad n = 1, 2, \dots \tag{3.13}$$

Let $B(x)$ be shifted Legendre vector defined in (2.10), so suppose $\alpha > 0$ so,

$$D^\alpha B(x) = D^{(\alpha)}B(x), \tag{3.14}$$

where $D^{(\alpha)}$ is the $(m + 1) \times (m + 1)$ operational matrix of fractional derivative of order α in the Caputo sense and is defined as follows [12]:

$$D^{(\alpha)} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,1,k} & \dots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,m,k} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^i \theta_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^i \theta_{i,1,k} & \dots & \sum_{k=\lceil\alpha\rceil}^i \theta_{i,m,k} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^m \theta_{m,0,k} & \sum_{k=\lceil\alpha\rceil}^m \theta_{m,1,k} & \dots & \sum_{k=\lceil\alpha\rceil}^m \theta_{m,m,k} \end{pmatrix}, \quad (3.15)$$

where $\theta_{i,j,k}$ is given by

$$\theta_{i,j,k} = (2j + 1) \sum_{l=0}^j \frac{(-1)^{(i+j+k+l)} (i+k)! (l+j)!}{(i-k)! k! \Gamma(k-\alpha+1) (j-l)! (l!)^2 (k+l-\alpha+1)}. \quad (3.16)$$

4 Description of the method

A computational approach based on the operational matrix of derivative of shifted Legendre polynomials is offered in this section in order to solve nonlinear singular boundary value problem (1.4) with the mixed conditions (1.5) and (1.6). Clearly, from Eq. (2.8), we can approximate the solution of (1.4), $y(x)$, as follows:

$$y(x) = C^T B(x), \quad (4.17)$$

where C and $B(x)$ are defined in Eqs. (2.9) and (2.10). Also, by using Eq. (3.15), we conclude that

$$D^\alpha y(t) = C^T D^{(\alpha)} B(x), \quad (4.18)$$

$$D^\beta y(t) = C^T D^{(\beta)} B(x). \quad (4.19)$$

Now, by substituting Eqs. (4.17), (4.18) and (4.19) in Eq. (1.4) we have:

$$C^T D^{(\alpha)} B(x) + (a + \frac{m}{x}) C^T D^{(\beta)} B(x) = f(x, C^T B(x)). \quad (4.20)$$

Also, by using Eqs. (1.5), (1.6), (3.13) and (4.17), we conclude that

$$\alpha_1 C^T B(0) + \beta_1 C^T D^{(1)} B(0) = A, \quad (4.21)$$

$$\alpha_2 C^T B(1) + \beta_2 C^T D^{(1)} B(1) = B. \quad (4.22)$$

To find the solution $y(x)$, we first collocate Eq. (4.20) at $(m - 1)$ points. To this end, we choose suitable collocation points as the first $(m - 1)$ shifted Legendre roots of $p_{m+1}(x)$. Eq. (4.20) together with Eqs. (4.21) and (4.22) generate $(m + 1)$ nonlinear equations which can be solved using Newton's iterative method. Consequently, we can obtain $y(x)$ given in Eq. (4.17).

5 Numerical examples

Here, we apply the proposed method in previous section to solve some numerical examples. All computations are done by using Mathematica 9 software.

Example 5.1. As the first example, we Consider the following singular two point boundary value problem:

$$y^{(\alpha)}(x) + \frac{1}{x}y^{(\beta)}(x) = -e^y, \quad 1 < \alpha \leq 2, \quad 0 < \beta \leq 1, \quad (5.23)$$

$$y'(0) = 0, \quad y(1) = 0,$$

for $\alpha = 2, \beta = 1$ the exact solution is $y(x) = 2\ln\left(\frac{4-2\sqrt{2}}{(3-2\sqrt{2})x^2+1}\right)$. The approximate solution for $m = 15$ is graphically shown in Figure 1.

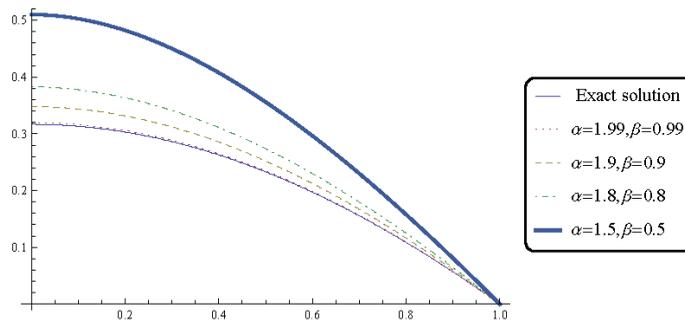


Figure 1: Comparison of the behavior of $y(x)$ for $m = 15$, with exact solution, for Example 5.1.

Example 5.2. Consider the following singular two point boundary value problem:

$$y^\alpha(x) + \left(1 + \frac{1}{x}\right)y^\beta(x) = \frac{5x^3(5x^3e^y - 5 - x)}{4 + x^5}, \quad (5.24)$$

$$y(0) = \ln\left(\frac{1}{4}\right), \quad y(1) + 5y'(1) = \ln\left(\frac{1}{5}\right) - 5.$$

for $\alpha = 2, \beta = 1$, the exact solution of this example is $y(x) = \ln\left(\frac{1}{4+x^5}\right)$, The approximate solution for $m = 15$ is graphically shown in Figure 2.

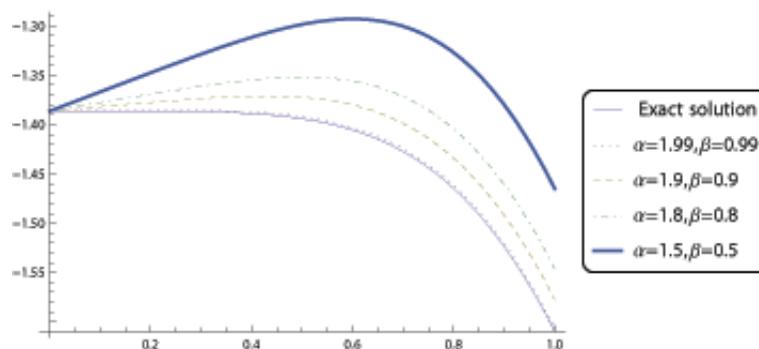


Figure 2: Comparison of the behavior of $y(x)$ for $m = 15$, with exact solution, for Example 5.2.

6 Conclusion

In this paper, first, we introduced nonlinear singular boundary value problems of fractional order arising in biology. Then, we applied the shifted Legendre polynomials as the basis to solve these problems. By use of the operational matrices of derivatives of these basic functions, we converted such problems to an algebraic system. Finally, by solving this algebraic system, we presented the approximate solution of the problem.

Acknowledgements

The authors thank referee and editor for their useful technical comments and valuable suggestions to improve the readability of the paper, which led to a significant improvement of the paper.

References

- [1] H. S. Lin, Oxygen diffusion in a spherical cell with nonlinear oxygen uptake Kinetics, *J. Theor. Biol*, 60 (1976) 449-457.
[http://dx.doi.org/10.1016/0022-5193\(76\)90071-0](http://dx.doi.org/10.1016/0022-5193(76)90071-0)
- [2] D. L. S. McElwain, A re-examination of oxygen diffusion in a spherical cell with Michaelis-Menten oxygen uptake Kinetics, *J. Theor. Biol*, 71 (1978) 255-263.
[http://dx.doi.org/10.1016/0022-5193\(78\)90270-9](http://dx.doi.org/10.1016/0022-5193(78)90270-9)
- [3] S. S. Ganji, A. Barari, D. D. Ganji, Approximate analysis of two-mass-spring systems and buckling of a column, *Computers and Mathematics with Applications*, 61 (2011) 1088-1095.
<http://dx.doi.org/10.1016/j.camwa.2010.12.059>
- [4] J. B. Paiva, A. V. Mendona, A coupled boundary element differential equation method formulation for platebeam interaction analysis, *Engineering Analysis with Boundary Elements*, 34 (2010) 456-462.
<http://dx.doi.org/10.1016/j.enganabound.2009.10.017>
- [5] U. Flesch, The Distribution of heat sources in the human head: A theoretical consideration, *J. Theor. Biol*, 54 (1975) 285-287.
[http://dx.doi.org/10.1016/S0022-5193\(75\)80131-7](http://dx.doi.org/10.1016/S0022-5193(75)80131-7)
- [6] B. F. Gray, The Distribution of heat sources in the human head: A theoretical consideration, *J. Theor. Biol*, 82 (1980) 473-476.
[http://dx.doi.org/10.1016/0022-5193\(80\)90250-7](http://dx.doi.org/10.1016/0022-5193(80)90250-7)
- [7] A. D. Conger, M. C. Ziskin, Growth of mammalian multicellular tumor spheroids, *Cancer Res*, 43 (1983) 556-60.
- [8] J. A. Adam, A simplified mathematical model of tumor growth, *Math. Biosci*, 81 (1986) 224-229.
[http://dx.doi.org/10.1016/0025-5564\(86\)90119-7](http://dx.doi.org/10.1016/0025-5564(86)90119-7)
- [9] J. A. Adam, A mathematical model of tumor growth II: effect of geometry and spatial non-uniformity on stability, *Math. Biosci*, 86 (1987) 183-211.
[http://dx.doi.org/10.1016/0025-5564\(87\)90010-1](http://dx.doi.org/10.1016/0025-5564(87)90010-1)
- [10] J. A. Adam, S. A. Maggelakis, Mathematical model of tumor growth IV: effect of necrotic core, *Math. Biosci*, 97 (1989) 121-136.
[http://dx.doi.org/10.1016/0025-5564\(89\)90045-X](http://dx.doi.org/10.1016/0025-5564(89)90045-X)
- [11] A. C. Burton, Rate of growth of solid tumor as a problem of diffusion, *Growth*, 30 (1966) 157-176.
- [12] A. Saadatmandi, M. Dehghan, A new operational matrix for solving fractional-order differential equations, *J. Comput. Math. Appl*, 59 (2010) 1326-1336.
<http://dx.doi.org/10.1016/j.camwa.2009.07.006>

- [13] M. Caputo, Linear models of dissipation whose Q is almost frequency independent. Part II, J. Roy Austral. Soc, 13 (1967) 529-539.
<http://dx.doi.org/10.1111/j.1365-246X.1967.tb02303.x>
- [14] K. Diethelm, N. J. Ford, A. D. Freed, Yu. Luchko, Algorithms for the fractional calculus: A selection of numerical methods, Comput. Methods Appl. Mech. Eng, 194 (2005) 743-73.
<http://dx.doi.org/10.1016/j.cma.2004.06.006>