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Research Article

A multiple-scale power series method for solving nonlinear ordinary differential equations

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Abstract

The power series solution is a cheap and effective method to solve nonlinear problems, like the Duffing-van der Pol oscillator, the Volterra population model and the nonlinear boundary value problems. A novel power series method by considering the multiple scales R_k in the power term $(t/R_k)^k$ is developed, which are derived explicitly to reduce the ill-conditioned behavior in the data interpolation. In the method a huge value times a tiny value is avoided, such that we can decrease the numerical instability and which is the main reason to cause the failure of the conventional power series method. The multiple scales derived from an integral can be used in the power series expansion, which provide very accurate numerical solutions of the problems considered in this paper.

Keywords and phrases: Duffing-van der Pol oscillator; Volterra population model; Power series method; Multiple scales; Recursion formula.

1 Introduction

The differential transform method [1] has been used to find the periodic solutions of strongly nonlinear oscillator. Qaisi [2] has used the power series approach to solve undamped and unforced Duffing equation, and Schovanec and White [3] used the Taylor series method. Chen [4] has used a target function method for the solution of the Duffing oscillator, while the Laplace decomposition methods were introduced by Yusufoglu [5] and Khuri [6]. Yue et al. [7] have applied the optimal scale polynomial interpolation technique to obtain the periodic solutions of the Duffing equation, and Dai et al. [8] have used the multiple scale time domain collocation method for solving nonlinear dynamical systems.

The power series method (PSM) is a classical one to solve ordinary differential equations (ODEs), which is closely related to the Taylor series method, but does not need an elaborate differential process to derive the expansion coefficients. However, the PSM is not used as a general purpose algorithm, because it is necessary to derive the algebraic equation of recursion formula case by case. In this paper, we solve nonlinear problems through an effective combination of the power series expansion method together with a sequence of multiple scales. Usually, the resulting nonlinear algebraic equations are severely ill-conditioned when higher order power terms are taken into account in the series solution. So a major issue is how to reduce the ill-condition of the resulting nonlinear algebraic equations.

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According to this principle we can derive a formula to compute the multiple scales explicitly. On the other hand, the recursion formula can be used to generate the new coefficients from the older ones through a nonlinear equation, which provides a semi-analytic solution of nonlinear ODE.
The organization of this paper is given as follows. In Section 2 we propose a new multiple-scale power series method (MSPSM) for the numerical solutions of some nonlinear problems. In Section 3 we apply the MSPSM to solve a Duffing-van der Pol equation, comparing to exact solution and displaying the high accuracy of the MSPSM. The Volterra population model of a species within a closed system is solved by the MSPSM in Section 4, while two nonlinear boundary value problems (BVPs) are solved in Section 5 by the MSPSM. Finally, the conclusions are drawn in Section 6.

2 A multiple-scale power series method

The PSM gives a solution $x(t)$ of a nonlinear ODE in the form of a power series:

$$x(t) = \sum_{k=0}^{\infty} A_k t^k. \tag{2.1}$$

It is easy to derive the formula and implement it to solve the ODE by inserting Eq. (2.1) into the desired ODE, and compute the solution $x(t)$ at any time t when A_k are computed from the derived recursion formula. However, in the numerical solution we can only take finite power terms and compute the solution to a finite time:

$$x(t) = \sum_{k=0}^{n} A_k t^k, \;\; t \le t_f, \tag{2.2}$$

which is limited by the capacity of computer, and t_f is limited by the radius of convergence.
Although the series solution in Eq. (2.2) is convergent for all t within a radius of convergence; however, the increase of n might lead to a divergent solution when $t > 1$ due to the appearance of t^k. In order to overcome this defect, Liu and Jhao [9] have proposed

$$x(t) = \sum_{k=0}^{n} a_k \left(\frac{t}{R_0}\right)^k, \;\; t \le t_f, \tag{2.3}$$

where R_0 is a given characteristic length. However, we can further modify the power series solution in Eq. (2.3) by

$$x(t) = \sum_{k=0}^{n} a_k \left(\frac{t}{R_k}\right)^k, \;\; t \le t_f, \tag{2.4}$$

where R_k is a sequence of multiple scales to be determined below, and $R_0 = 1$.
The polynomial interpolation of data is an ill-posed problem and it makes the data interpolation by using the higher-order polynomials not being easy to numerical implementation. In order to overcome that difficulty, Liu and Atluri [10] have introduced a characteristic length into the high-order polynomials expansion, which improves the numerical stability and accuracy in the data interpolation. Then, Liu [11] has further proposed a half-order multiple-scale polynomial interpolation technique, which provides a highly accurate numerical result for data interpolation. Based on these studies, we propose a multiple-scale power series expansion as a mathematical tool for the numerical solution of nonlinear problem.
Comparing Eqs. (2.2) and (2.4) we have

$$A_k = \frac{a_k}{R_k^k}, \;\; k = 1, \ldots, n. \tag{2.5}$$

Our strategy is to find suitable scales of R_k, which depend on n and the collocated nodal points t_j, such that a_k is easily and less ill-posedly to be solved than solving A_k, because $A_k t^k$ is hard to be realized numerically when k is a large integer and $t > 1$ [9]. When a_k is solved we can use Eq. (2.5) to find A_k.
In order to obtain R_k, we impose the following n interpolated conditions to Eq. (2.4):

$$x(t_i) = x_i, \; t_i = it_f/n \;\; i = 1, \ldots, n. \tag{2.6}$$

Thus, we obtain a linear equations system to determine a_0 and $a_k,\ k=1,\dots,n$:

$$\begin{bmatrix} 1 & \frac{t_1}{R_1} & \frac{t_1^2}{R_2^2} & \cdots & \left(\frac{t_1}{R_n}\right)^n \\ 1 & \frac{t_2}{R_1} & \frac{t_2^2}{R_2^2} & \cdots & \left(\frac{t_2}{R_n}\right)^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{t_n}{R_1} & \frac{t_n^2}{R_2^2} & \cdots & \left(\frac{t_n}{R_n}\right)^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \tag{2.7}$$

For the optimally scaled polynomial in Eq. (2.4) we can take

$$R_k = \left(\frac{1}{n}\sum_{j=1}^{n} t_j^{2k}\right)^{1/(2k)},\ k=1,\dots,n, \tag{2.8}$$

in which $t_j = jt_f/n$ is a uniform temporal node. Such that each column of the coefficient matrix in Eq. (2.7) has the same Euclidean norm $\sqrt{n}$. The coefficient matrix in Eq. (2.7) with the above $R_k,\ k=1,\dots,n$ is an equilibrated matrix [12, 13], which leads to a much better conditioning of Eq. (2.7) than the original Vandermonde coefficient matrix with $R_k = 1,\ k=1,\dots,n$.

When t and n are quite large the multiple scales R_k for larger k cannot be obtained from Eq. (2.8), because t_j^{2k} is overflow when k is near to n. So, replacing the summation in Eq. (2.8) by the integral we can derive

$$R_k = \left(\frac{1}{n}\int_0^{t_f} t^{2k} dt\right)^{1/(2k)} = \frac{t_f^{(2k+1)/(2k)}}{[n(2k+1)]^{1/(2k)}},\ k=1,\dots,n. \tag{2.9}$$

The above two approaches of R_k in Eqs. (2.8) and (2.9) are, respectively, called the discrete and continuous MSPSM. For a long term computation, Liu and Jhao [9] have modified the power series solution (2.3) by

$$x(t) = \sum_{k=0}^{n} a_k \left(\frac{t-t_i}{R_0}\right)^k,\ \ t_i \le t \le t_i + t_f/N, \tag{2.10}$$

where the total time interval is divided into N sub-intervals. By the same token we can modify Eq. (2.4) to

$$x(t) = \sum_{k=0}^{n} a_k \left(\frac{t-t_i}{R_k}\right)^k,\ \ t_i \le t \le t_i + t_f/N, \tag{2.11}$$

where

$$R_k = \frac{(t_i + t_f/N)^{(2k+1)/(2k)}}{[n(2k+1)]^{1/(2k)}},\ k=1,\dots,n. \tag{2.12}$$

At the first sub-interval $t_1 = 0$, $a_0 = x_0$ and $a_1 = y_0 R_1$, where x_0 and y_0 are, respectively, the initial displacement and velocity of a second-order ODE; then in the subsequent time interval $t \in [t_i, t_i + t_f/N]$, a_0 and a_1 are obtained by inserting the values at the end time of the preceding interval into $a_0 = x(t_i)$ and $a_1 = R_1 y(t_i)$. These modifications are crucial in the applications of the power series method to the solutions of nonlinear ODEs in a long time interval.

3 Duffing-van der Pol oscillator

We use the above novel power series expansion method to solve the following Duffing-van der Pol oscillator [14, 15]:

$$\ddot{x}(t) + [\gamma + \beta x^2(t)]\dot{x}(t) + \alpha x(t) + x^3(t) = 0, \tag{3.13}$$

of which by inserting Eq. (2.4) we can derive the following recursion formula:

$$a_{k+2} = -\frac{R_{k+2}^{k+2}}{(k+2)(k+1)}\left(C_k + \frac{\alpha a_k}{R_k^k} + \beta S_k + \frac{\gamma(k+1)a_{k+1}}{R_{k+1}^{k+1}}\right),\ \ k=0,1,\dots, \tag{3.14}$$

where

$$B_k = \sum_{j=0}^{k} \frac{a_j}{R_j^j} \frac{a_{k-j}}{R_{k-j}^{k-j}},\ \ C_k = \sum_{j=0}^{k} \frac{a_{k-j}}{R_{k-j}^{k-j}} B_j,\ \ S_k = \sum_{j=0}^{k} \frac{(k-j+1)a_{k-j+1}}{R_{k-j+1}^{k-j+1}} B_j,\ \ k = 0,1,\ldots. \tag{3.15}$$

In the above, B_k represents the quadratic term x^2, C_k the cubic term x^3, and S_k the term of $\dot{x}x^2$.
By starting from the initial conditions with $x(0) = a_0$ and $\dot{x}(0) = a_1/R_1$, we can use Eq. (3.14) to iteratively generate the coefficients $a_k,\ k = 2,\ldots,n$, and then the solution of $x(t)$ is given by Eq. (2.4).
For a special case of Eq. (3.13) with $\alpha = 3/\beta^2$ and $\gamma = 4/\beta$, it is one of the first-kind Abel equation. Chandrasekar et al. [17] showed that Eq. (3.13) can be transformed as

$$w''(z) - \frac{\beta^2}{2} w^2(z) w'(z) = 0, \tag{3.16}$$

where

$$z := e^{-2t/\beta},\ \ w := -xe^{t/\beta}. \tag{3.17}$$

Then a particular solution is available:

$$x(t) = \frac{-\sqrt{3}}{\beta\sqrt{t_0 e^{2t/\beta} - 1}}, \tag{3.18}$$

where $t_0 > 1$ is an arbitrary constant. If t_0 is given, then the initial conditions are given by

$$x(0) = \frac{-\sqrt{3}}{\beta\sqrt{t_0 - 1}},\ \ \dot{x}(0) = \frac{\sqrt{3}t_0}{\beta^2 (t_0 - 1)^{3/2}}. \tag{3.19}$$

Conversely, if $x(0)$ is given then we have

$$t_0 = 1 + \frac{3}{\beta^2 x^2(0)},\ \ \dot{x}(0) = \frac{\sqrt{3}t_0}{\beta^2 (t_0 - 1)^{3/2}}. \tag{3.20}$$

The differential transform method (DTM) has been applied by Mukherjee et al. [14] to solve the above problem with given values of $x(0) = -0.28868$ and $\beta = 3$. By Eq. (3.20) we can obtain t_0 and $\dot{x}(0)$. We apply the multiple-scale power series method (MSPSM) to solve this problem with $n = 6$ as follows:

$$x_{\text{MSPSM}} = 0.28868 + 0.1202841368t - 0.03508337872t^2 + 0.01099856740t^3$$
$$-0.003843741333t^4 + 0.001394483198t^5 - 0.0005109682262t^6, \tag{3.21}$$
$$x_{\text{DTM}} = 0.28868 + 0.120281305t - 0.080187536t^2 - 0.019916649t^3$$
$$-0.02341443t^4 - 0.019687577t^5 - 0.010351679t^6. \tag{3.22}$$

In Table 1 we compare the DTM solution provided by Mukherjee et al. [14] with the present results computed by the power series method (PSM) and the multiple-scale power series method (MSPSM).

Table 1: Comparing numerical solutions obtained by the DTM, PSM and MSPSM with exact solution

t	Exact	DTM	PSM	MSPSM
0.1	-0.2769917816	-0.27743	-0.2769917816	-0.2769917797
0.2	-0.2659442680	-0.26759	-0.2659442704	-0.2659442429
0.3	-0.2554834906	-0.259	-0.2554835310	-0.2554834071
0.4	-0.2455620422	-0.2515	-0.2455623342	-0.2455619780
0.5	-0.2361380529	-0.24495	-0.2361394003	-0.2361385827

It can be seen that the present methods can provide highly accurate numerical solutions of the Duffing-van der Pol equation. When the accuracy of the DTM is in the second order, the maximum error of PSM is 1.347×10^{-6}, and the maximum error of MSPSM is 5.297×10^{-7}. The improvement of accuracy is, respectively, in the four and five orders. Fernández [15] has pointed out that the DTM is merely the Taylor series approach. Later, Bervillier [16] has further argued and enhanced the viewpoint that "the DTM exactly coincides with the traditional Taylor method," contrary to what is currently announced that the DTM is a new method for solving ODEs. Indeed, the DTM is not at all a new method, and all the algebraic operational formulas used in the DTM can be derived from the power series method.

By fixing $\beta = 3$ and $t_0 = 5$, of which $x(0)$ and $\dot{x}(0)$ can be computed from Eq. (3.19), we also apply the MSPSM to solve this problem with $n = 6$. In Fig. 1(a) we compare the coefficients R_k, a_k of the MSPSM and A_k of the power series method, while in Fig. 1(b) we compare the numerical errors of PSM and MSPSM. It can be seen that the MSPSM is more accurate than the PSM, with the maximum error 5.297×10^{-7} of the MSPSM being much smaller than 1.347×10^{-6} of the PSM.

Fernández [15] has pointed out that the Taylor series (and also the DTM) of the exact solution (3.18) converges for all $t < |t_s|$, where

$$t_s = \frac{-\beta \ln t_0}{2}. \tag{3.23}$$

For $\beta = 3$ and $t_0 = 5$ we have $t_s = -2.414156868$. However, in Fig. 2(a) we compare the coefficients a_k of the MSPSM and A_k of the PSM with $n = 150$, while in Fig. 2(b) we compare the numerical errors of PSM and MSPSM up to $t = 2.5$. It can be seen that the error of MSPSM is smaller than that of the PSM, with the maximum error 1.09×10^{-4} of the MSPSM being much smaller than 2.012 of the PSM.

For the time range over the radius of convergence, we can divide the total time interval into N subintervals, and in each subinterval we apply the above method to find the numerical solutions. In Fig. 3(a) we show the the discrete and continuous coefficients R_k of the MSPSM with $n = 219$ (with $n = 220$ the discrete MSPSM is overflow, but it is not for the continuous MSPSM), where we take $N = 10$, while in Fig. 3(b) we compare the numerical errors up to $t = 5$, of which the maximum error of the discrete MSPSM is 1.67×10^{-16} and the maximum error of the continuous MSPSM is 1.94×10^{-16}. This shows that the both the discrete and continuous MSPSM can provide very accurate solution when more power terms are used.

We extend the time range to $t_f = 8$ and with a large $n = 200$, where $N = 20$. The discrete MSPSM is not applicable, and the PSM leads to a wrong solution with a large error over 24598.6 when $t > 6$. In Fig. 4 we show the numerical errors obtained by the continuous MSPSM, of which the maximum error is 1.94×10^{-16}. The main reason of the failure of PSM is that we need to compute the numerical result of a very tiny number of A_k, as shown in Figs. 1(a) and 2(a), times a huge number, like that 8^{200} for this case.

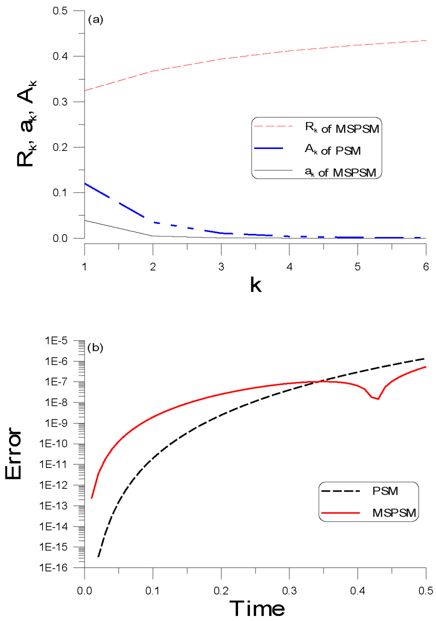

Figure 1: Comparing the power series method and the multiple-scale power series method, (a) the coefficients and (b) numerical errors.

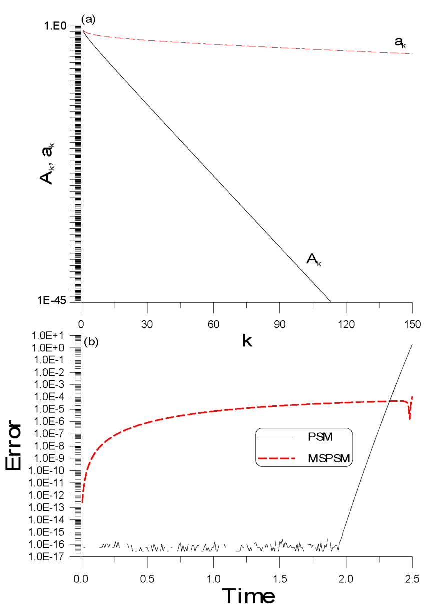

Figure 2: Comparing the PSM and the MSPSM to a larger time range, (a) the coefficients, and (b) the numerical errors.

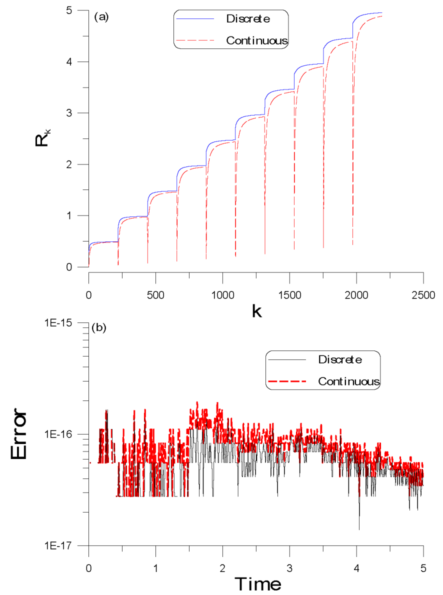

Figure 3: Beyond the convergence region of the power series method, (a) showing the discrete and continuous multiple scales of the MSPSM, and (b) comparing the numerical errors.

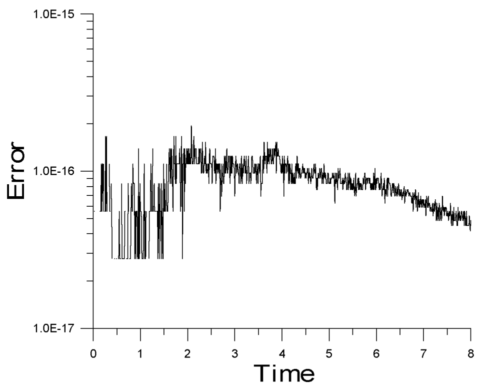

Figure 4: Far beyond the convergence region, showing the numerical errors obtained by the continuous MSPSM.

4 The model of Volterra population

For the model of Volterra population [18, 19] in a closed system:

$$\frac{dp}{d\bar{t}} = ap - bp^2 - cp\int_0^{\bar{t}} p(s)ds, \quad p(0) = p_0, \tag{4.24}$$

describing the growth of a species, $a > 0$ is the coefficient of birth rate, and $b > 0$ is the crowding coefficient while $c > 0$ is the toxicity coefficient, which indicates that the population evolution tends to zero in a long term.
Upon letting

$$\eta = \frac{c}{ab}, \quad t = \frac{c\bar{t}}{b}, \quad u = \frac{bp}{a}, \tag{4.25}$$

Eq. (4.24) is non-dimensionalized to a nonlinear Volterra differential-integral equation:

$$\eta \frac{du}{dt} = u - u^2 - u \int_0^t u(s)ds, \ \ u(0) = u_0. \tag{4.26}$$

There are many approximate and numerical solutions for Volterra's population model, we name a few [18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. We can solve Eq. (4.26) by applying the MSPSM with the following recursion formula:

$$a_{k+1} = \frac{R_{k+1}^{k+1}}{\eta(k+1)} \left(\frac{a_k}{R_k^k} - B_k - D_k \right), \ \ k = 0, 1, \ldots, \tag{4.27}$$

where

$$B_k = \sum_{j=0}^{k} \frac{a_j}{R_j^j} \frac{a_{k-j}}{R_{k-j}^{k-j}}, \ \ C_k = \frac{a_{k-1}}{k R_{k-1}^{k-1}}, \ \ D_k = \sum_{j=0}^{k} \frac{a_j}{R_j^j} C_{k-j}, \ \ k = 1, 2, \ldots, \tag{4.28}$$

in which B_k represents the quadratic term u^2, C_k the integral term $\int_0^t u(s)ds$, and D_k the product term $u\int_0^t u(s)ds$. For the purpose of comparison we let

$$v(t) = \int_0^t u(s)ds, \tag{4.29}$$

and then we have a system of two first-order ODEs:

$$\dot{u}(t) = \frac{1}{\eta}[u(t) - u^2(t) - u(t)v(t)], \ \ u(0) = u_0,$$
$$\dot{v}(t) = u(t), \ \ v(0) = 0. \tag{4.30}$$

We apply the fourth-order Runge-Kutta method (RK4) to integrate the above ODEs with a time increment Δt. Under the values of $n = 300$, $\eta = 0.5$, $u_0 = 0.1$ and $\Delta t = 0.001$, in Fig. 5(a) we compare u obtained by the RK4 and MSPSM to $t = 1$, of which we can observe that these two curves are almost coincident, with the absolute difference being shown in Fig. 5(b), whose maximum value of difference is 8.66×10^{-15}.

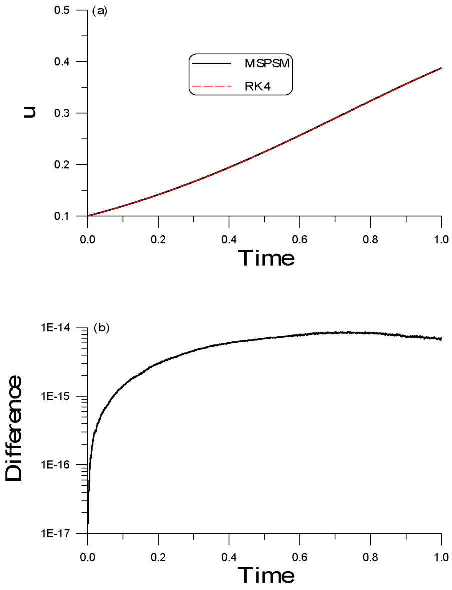

Figure 5: For the Volterra population model, (a) comparing the solutions obtained by the RK4 and the MSPSM, and (b) comparing the numerical differences.

5 Nonlinear boundary value problems

We can also apply the MSPSM to solve nonlinear boundary value problems, which needs the help from the fictitious time integration method (FTIM) developed by Liu and Atluri [33] to solve nonlinear equation $\mathbf{F}(\mathbf{x}) = 0$:

$$\frac{d\mathbf{x}}{dt} = -\frac{\nu}{1+t}\mathbf{F}(\mathbf{x}). \tag{5.31}$$

Example 5.1. *First we consider*

$$\begin{aligned} u''(x) &= \frac{3}{2}u^2(x), \\ u(0) &= 4, \;\; u(1) = 1, \\ u(x) &= \frac{4}{1+x^2}. \end{aligned} \tag{5.32}$$

Let $y = u'(0)$ be an unknown and we apply the MSPSM to solve Eq. (5.32). For an initial guess of $y_0 = u'(0)$ we can obtain the corresponding end value u_i^f by using the MSPSM which is compared with the exact value. Then we apply the FTIM to solve y by the iteration:

$$y_{i+1} = y_i - \frac{\nu\Delta t}{1+t_i}(u_i^f - 1), \tag{5.33}$$

until the convergence criterion $|u_i^f - 1| < \varepsilon$ is satisfied. Under $\Delta t = 0.01$, $\nu = 10$, $n = 20$, and $\varepsilon = 10^{-10}$ we apply the MSPSM and FTIM to solve this problem. In Fig. 6(a) we show the residual $|u_i^f - 1|$, which is convergence with 154 steps. In Fig. 6(b) we compare u obtained by the exact solution and the MSPSM, of which we can observe that these two curves are almost coincident, with the absolute error being shown in Fig. 6(c), whose maximum value is 9.28×10^{-11}.

Example 5.2. *The Bratu problem is a nonlinear BVP that is used as a benchmark problem to test the accuracy of many numerical methods for solving the BVPs. The Bratu problem is given by [34]*

$$u''(x) = -\lambda \exp(u(x)),$$
$$u(0) = u(1) = 0, \tag{5.34}$$

which has a closed-form solution:

$$u(x) = -2\ln \frac{\cosh\left[\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right]}{\cosh\frac{\theta}{4}}, \tag{5.35}$$

where θ *is solved from the following equation:*

$$\theta = \sqrt{2\lambda}\cosh\frac{\theta}{4}. \tag{5.36}$$

We test the case of $\lambda = 2.3$. *From Eq. (5.36) we have* $\theta > 0$, *and there exist two intersection points of the two functions* θ *and* $\sqrt{2\lambda}\cosh(\theta/4)$, *and thus there are two roots* $\theta = 3.73351$ *rad and* $\theta = 6.5765693$ *rad of Eq. (5.36).*
Under $\Delta t = 0.02$, $\nu = 20$, $n = 10$, *and* $\varepsilon = 10^{-10}$ *we apply the MSPSM and FTIM to solve this problem. In Fig. 7(a) we show the residual* $|u_i^f - 1|$, *which is convergence with 304 steps. In Fig. 7(b) we compare u obtained by the exact solution with* $\theta = 3.73351$ *and the MSPSM, of which we can observe that these two curves are almost coincident, with the absolute error being shown in Fig. 7(c), whose maximum value is* 1.13×10^{-4}.

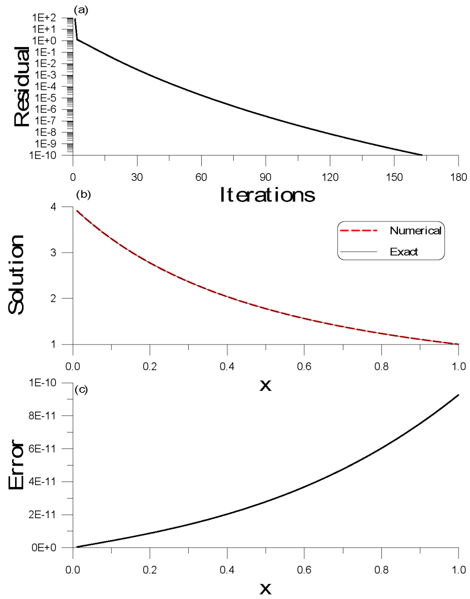

Figure 6: For example 5.1 of the boundary value problems, (a) convergence speed, (b) comparing the solutions obtained by exact solution and the MSPSM, and (c) the numerical error.

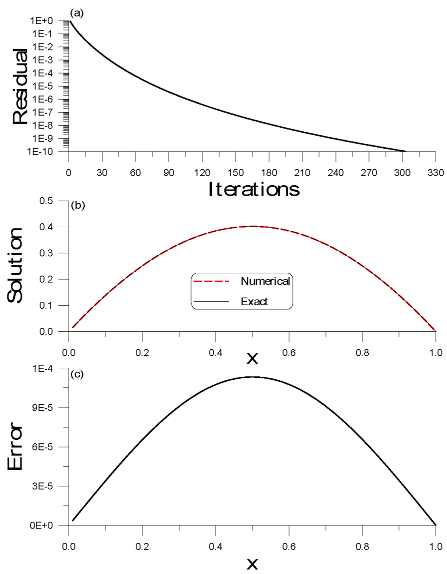

Figure 7: For the Bratu problem, (a) convergence speed, (b) comparing the solutions obtained by exact solution and the MSPSM, and (c) the numerical error.

6 Conclusions

We have developed a quite powerful multi-scale power series method (MSPSM) to solve nonlinear problems, including the Duffing-van der Pol oscillator, the Volterra population model and two nonlinear boundary value problems being investigated in this paper. The multi-scales can be derived explicitly through an integral operation, which is used to reduce the ill-condition in the data interpolation. The present MSPSM is very accurate because we can employ many power terms up to 300 in the series solution. For a long-time solution we suggested to use the multi-interval MSPSM to find the solution, whose accuracy and stability are much better than that using a single-interval MSPSM.

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