Influence of Hall Current on MHD Flow and Heat Transfer over a slender stretching sheet in the presence of variable fluid properties

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Abstract
MHD flow and heat transfer of an electrically conducting fluid over a stretching sheet with variable thickness in the presence of variable fluid properties is analyzed. Wall temperature and the velocity of the stretching sheet are assumed to vary. Also the external magnetic field perpendicular to the sheet and the effects of Hall current are taken into account. The governing nonlinear differential equations are solved numerically by an implicit finite difference scheme. To validate the numerical method, comparisons are made with the available results in the literature for some special cases and the results are found to be in excellent agreement. The effects of physical parameters on the flow and temperature fields are analyzed graphically. The Hall current gives rise to a cross flow and the variable fluid properties have strong effects on the shear stress and the Nusselt number.

Keywords: Numerical solution; variable fluid properties; variable boundary thickness; MHD flow; Hall effects; Nusselt number.

1 Introduction

During the last several decades, the study of magnetohydrodynamic (MHD) flow has attracted several scientists and researchers due to its importance and applications to various technological and industrial problems such as petroleum industries, purification of crude oil, plasma studies, magneto-hydrodynamic electrical power generation, glass manufacturing, paper production, geothermal energy extractions, and boundary layer control in the field of aerodynamics.

In view of these applications, Chakrabarti and Gupta [1] obtained similarity solution for electrically conducting fluid analyzed the effect of hydromagnetic flow and heat transfer over a stretching sheet. Andersson et al. [2] extended the work of Chakrabarti and Gupta [1] to power law fluid and Andersson [3] presented two-dimensional Navier-Stokes equations by considering uniform magnetic field for viscous

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flow past a stretching sheet. Vajravelu and Nayfeh [4] analyzed hydromagnetic flow of a dusty fluid over a stretching sheet. Ishak et al. [5] studied the effect of a uniform transverse magnetic field on the stagnation point flow toward a vertical stretching sheet. Cortell [6] examined the flow and heat transfer of an electrically conducting second grade fluid in the presence of transverse magnetic field past a semi-infinite stretching sheet. Sweet et al. [7] obtained the analytical solution of the MHD flow of a viscous fluid between two moving parallel plates via the homotopy analysis method. Further, Robert and Vajravelu [8] obtained explicit exact solutions for fourth-order nonlinear differential equations arising in the hydromagnetic flow of a second grade fluid over a stretching or shrinking sheet. Recently, Abbasbandy et al. [9] obtained both numerical and analytical solutions for Falkner-Skan flow of MHD Maxwell fluid and emphasized the variations of viscoelastic and magnetic parameters. Many researchers have made the valuable contribution to the literature of hydromagnetic flow and heat transfer over a stretching sheet by considering different geometry and obtained exact solutions [10-15]. However, not much attention has been paid to the effects of Hall currents. When the magnetic field is strong and the density of electrons is small, the Hall effect cannot be ignored as it has a significant effect on the flow pattern of an ionized gas. However, for ionized gases, the conventional MHD is not valid. In an ionized conducting fluid where the density is low and/or magnetic field is very strong, the conductivity normal to the magnetic field is reduced due to the free spiraling of electrons and a current is induced in a direction normal to both electric and magnetic fields. This phenomenon is called Hall effect. The effect of Hall current cannot be neglected when the medium is rarefied or in a strong magnetic field. The study of MHD flows with Hall current has important industrial applications in many geophysical and astrophysical situations and in several engineering problems such as Hall accelerators, Hall effect sensors, constrictions of turbines, centrifugal machines, and flight magneto-hydrodynamics. In view of these applications, it is essential to analyze the influence of Hall current on the fluid flow. Gupta [16] considered a strong magnetic field which permeated the fluid and investigated the effect of Hall current on the fluid past an infinite porous flat plate and Jana et al. [17] continued the work of Ref. [16] for different physical situations. Hossain and Rashid [18] extended the work of Gupta [16] to unsteady free convection flow. Further, Rana et al. [19] analyzed the effect of Hall current on Hartmann flow between two parallel electrically insulated infinite planes. Chaudhary and Jha [20] studied the effect of heat and mass transfer on the flow of elastico-viscous fluid past an impulsively started infinite vertical plate with mass transfer and Hall effect taken into account. Recently, Hayat et al. [21] obtained an analytical solution for unsteady three-dimensional MHD flow due to a stretching surface in a porous medium via the homotopy method.

In general, studies are centered on a linear or nonlinear stretching sheet. However, not much work has been carried out for a special type of non-linear stretching, namely, \( u_n(x) = U_0(x + b)^n \) at \( y = A(x + b)^{(1-n)/2} \) for different values of \( n \) (that is, a stretching sheet with variable thickness) in a thermally stratified environment. Such a study would have practical application since deforming substances like needles and nozzles have variable sheet thickness. Also variable thickness (Lee [22]) is one of the significant properties in the analysis of vibration of orthotropic plates. Special form of non linear stretching sheet with variable thickness is often used in machine design, architecture, nuclear reactor technology, naval structures, and acoustical components. Very recently, Fang et al. [23] observed the behavior of boundary layer flow over a stretching sheet with variable thickness and explained significant effects of the non-flatness of the sheet on the velocity and shear stress.

All the above mentioned researchers restricted their analyses to the hydromagnetic flow over a horizontal or a vertical plate and assumed thermo-physical properties of the fluid as constant. However, these physical properties may change with temperature; especially the viscosity and the thermal conductivity (see for details Chiam [24], Hassani [25], Subhas Abel et al. [26], and Prasad et al. [27]). To the best knowledge of the authors, a study on the combined effects of variable fluid properties and the Hall effect for flow over a slender stretching sheet with variable thickness has not been carried out. In view of this, the
problem studied here extends the work of Prasad et al. [27] to the special form of stretching sheet with variable thickness in the presence of strong magnetic field. The coupled non-linear partial differential equations governing the problem have been transformed to a system of coupled non-linear ordinary differential equations. The transformed equations are solved numerically by a second order finite difference scheme known as the Keller-box method. Computed numerical results for the flow and heat transfer characteristics are analyzed. The analysis of the results shows that the fluid flow is appreciably influenced by the sundry parameters. It is expected that the results obtained will not only provide useful information for industrial application but also complement the previous works.

2 Mathematical formulation

Consider a steady, laminar boundary layer flow of a viscous, incompressible and electrically conducting fluid in the presence of a strong magnetic field \(B(x)\) past an impermeable stretching sheet with variable thickness. The positive \(x\) coordinate is measured along the stretching sheet in the direction of motion and the positive \(y\) coordinate is measured normal to the sheet in the upward direction (see Fig. 1 for details).

![Figure 1: Schematic diagram of the stretching sheet with variable thickness model.](image)

The continuous sheet moves in its own plane with a velocity \(u_w(x)\) and temperature distribution \(T_w(x)\). An external magnetic field is applied in the positive \(y\)-direction with a constant flux density \(B_0\). In general, for an electrically conducting fluid, Hall current affects the flow in the presence of a strong magnetic field. The effect of Hall current gives rise to a cross flow and hence the flow becomes three-dimensional. We assume that there is no variation of flow quantities in the \(z\)-direction. This assumption is valid for a surface of infinite extent. The generalized Ohm’s law including Hall currents in the usual form is given by

\[
J = \sigma \left( E + \mathbf{V} \times \mathbf{B} - \frac{1}{en_e} \mathbf{J} \times \mathbf{B} + \frac{1}{en_e} \nabla p_e \right).
\]

Here \(J = (J_x, J_y, J_z)\) is the current density vector, \(E\) is the intensity vector of the electric field, \(\mathbf{V}\) is the velocity vector, \(\mathbf{B} = (0, B_0, 0)\) is the magnetic induction vector, \(\sigma\) is the electrical conductivity, and \(p_e\) is the electronic pressure. Since there is no applied or polarization voltage is imposed on the flow we have,
\( \mathbf{E} = 0 \). For weakly ionized gases, the electron pressure gradient and the ion slip effects can be neglected. The generalized Ohm’s law under the above conditions for electrically non-conducting sheet becomes \( J_y = 0 \). Hence Eq. (2.1) reduces to

\[
J_x = \frac{\sigma B_0(x)}{(1+m^2)} (mu - w) \quad \text{and} \quad J_z = \frac{\sigma B_0(x)}{(1+m^2)} (u + mw).
\]  

(2.2)

Here \( u, v, \) and \( w \) are the \( x, y, \) and \( z \) components of the velocity vector \( \mathbf{V} \) and \( m \) is the Hall parameter.

The following assumptions are made

- Joule heating and viscous dissipation are neglected.
- The fluid is isotropic, homogeneous, and has constant viscosity and electric conductivity.
- The wall is impermeable (\( v_w = 0 \)).
- The sheet is being stretched with a velocity \( U_\infty(x) = U_0 \left( x + b \right)^n \) where \( U_0 \) is constant, \( b \) is the physical parameter related to stretching sheet, and \( n \) is the velocity exponent parameter.
- The sheet is not flat and is defined as \( y = A \left( x + b \right)^{(1-n)/2} \), where the coefficient \( A \) is chosen as small so that the sheet is sufficiently thin, to avoid pressure gradient along the sheet (\( \partial p / \partial x = 0 \)).

Under these assumptions, along with the boundary layer approximations, the governing equations can be written as (see Chaudhary and Jha [20] for details)

\[ u_x + v_y = 0, \]  

(3.3)

\[ \rho \varepsilon \left( uu_x + vv_y \right) = \left( \mu u_y \right) - \frac{\sigma B_0^2(x)}{(1+m^2)} (u + mw), \]  

(4.4)

\[ \rho \varepsilon \left( uw_x + vw_y \right) = \left( \mu w_y \right) + \frac{\sigma B_0^2(x)}{(1+m^2)} (mu - w), \]  

(5.5)

\[ \rho \varepsilon c_p \left( uT_x + vT_y \right) = \left( k(T)T_y \right)_y. \]  

(6.6)

Here, the subscript denotes partial differentiation with respect to the independent variable, \( \rho \varepsilon \) is fluid density, \( C_p \) is the specific heat at constant pressure, and \( T \) is the temperature. Further, \( \mu \) is the coefficient of viscosity which varies as an inverse function of temperature (see for details Prasad et al. [27], and Lai and Kulacki [28]) and is as follows

\[ \mu = \frac{\mu_o}{1 + \gamma (T - T_o)} \quad \text{i.e.,} \quad \frac{1}{\mu} = a \left( T - T_r \right), \]  

(7.7)

where, \( a = \frac{\gamma}{\mu_o} \) and \( T_r = T_o - \frac{1}{\gamma} \).

Here both \( a \) and \( T_r \) are constants, and their values depend on the reference state and the thermal property of the fluid. In general, \( a > 0 \) corresponds to liquid and \( a < 0 \) to gases. \( T \) is the temperature; \( T_o \) and \( \mu_o \) are the constant values of the temperature and coefficient of viscosity respectively away from the sheet. Further, the special form of magnetic field \( B_0^2(x) = B_0^2(x + b)^{1-n} \) is considered and \( k(T) \) is the temperature-dependent thermal conductivity. We consider the temperature dependent thermal conductivity in the following form (see for details Chiam [24]):

\[ k(T) = k_o \left( 1 + \frac{\varepsilon}{\Delta T \left( T - T_o \right)} \right), \]  

(8.8)
where $\Delta T = T_w - T_\infty = \frac{C}{l}(x + b)'$, $T_w$ is the sheet temperature, $T_\infty$ is the temperature of the fluid away from the sheet, $C$ is a constant, $l$ is the characteristic length, $r$ is the wall temperature, $\varepsilon$ is a small parameter known as the thermal conductivity parameter, and $k_\infty$ is thermal conductivity of the fluid away from the sheet. Substituting (2.7) and (2.8), into (2.4) to (2.6), we obtain

$$u = \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho_\infty} \frac{\partial}{\partial y} \left( \frac{\mu_\infty}{1 + \gamma(T - T_\infty)} \frac{\partial u}{\partial y} \right) - \frac{\sigma B_0^2(x)}{\rho_\infty(1 + m^2)}(u + mw),$$  

$$w = \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} = \frac{1}{\rho_\infty} \frac{\partial}{\partial y} \left( \frac{\mu_\infty}{1 + \gamma(T - T_\infty)} \frac{\partial w}{\partial y} \right) - \frac{\sigma B_0^2(x)}{\rho_\infty(1 + m^2)}(mu - w),$$  

$$\rho_\infty c_p u \frac{\partial T}{\partial x} + \left( \rho_\infty c_p v \frac{k_\infty \varepsilon}{\Delta T} \frac{\partial T}{\partial y} \right) \frac{\partial T}{\partial y} = (k_\infty \left(1 + \frac{\varepsilon}{\Delta T}(T - T_\infty)\right)) \frac{\partial^2 T}{\partial y^2}.$$  

The appropriate boundary conditions for the problem are

$$u(x, y) = U_w = U_0 (x + b)' \quad \text{and} \quad v(x, y) = 0,$$

$$w(x, y) = 0, T(x, y) = T_w = \frac{C}{l} (x + b)' \quad \text{at} \quad y = A(x + b)^{1-n/2},$$

$$u(x, y) \to 0, \quad w(x, y) \to 0, \quad T(x, y) \to T_\infty \quad \text{as} \quad y \to \infty,$$

where $C$ is a constant and $r$ is the wall temperature. It should be noted that the positive and negative value of $n$ indicate cases of surface stretching and surface shrinking respectively. Now we transform the system of equations (2.3) - (2.6) into a dimensionless form. To this end, let us define the dimensionless similarity variable

$$\eta = y \sqrt{\frac{n+1}{2} \frac{U_0}{\varepsilon}} (x + b)^{\frac{n-1}{2}}.$$  

Now in terms of $\eta$, we define the dimensionless stream function $\psi(x, y)$ and the dimensionless temperature distribution $\theta(\eta)$ as

$$\psi(x, y) = f(\eta) \sqrt{\frac{2}{n+1}} \frac{U_0}{\varepsilon} (x + b)^{\frac{n+1}{2}}, \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty},$$  

where $\psi(x, y)$ identically satisfies the continuity equation (2.3). With the help of equation (2.14), the velocity components can be written as

$$u = U_w f'(\eta), \quad w = U_w h(\eta), \quad v = -\sqrt{\frac{n+1}{2}} \frac{U_0}{\varepsilon} (x + b)^{\frac{n-1}{2}} \left[ f(\eta) + \eta f'(\eta) \left( \frac{n-1}{n+1} \right) \right].$$  

Here prime denotes differentiation with respect to $\eta$. With the use of equations (2.13)-(2.15), equations (2.9) - (2.11) and condition (2.12) can be reduced to

$$\left( \frac{f''}{(1 - \theta/\theta_\infty)} \right)' + f f'' - \frac{2n}{(n+1)} f'^2 - \frac{2Mn}{(1 + m^2)(1 + n)} (f' + mh) = 0,$$

$$\left( \frac{h'}{(1 - \theta/\theta_\infty)} \right)' + f h' - \frac{2n}{n+1} f'^2 h + \frac{2Mn}{(1 + m^2)(1 + n)} (m f' - h) = 0.$$
\[ [(1 + \varepsilon \theta) \theta']' + \text{Pr} \left( f \theta' - \frac{2r}{n+1} \theta f' \right) = 0, \quad (2.18) \]

\[ f(\alpha) = \alpha^{1-n} \frac{1}{1+n}, \quad f'(\alpha) = 1, \quad h(\alpha) = 0, \quad \theta(\alpha) = 1, \quad \theta(\infty) = 0, \quad f'(\infty) = 0, \quad h(\infty) = 0. \quad (2.19) \]

The non-dimensional parameters \( Mn \) and \( \text{Pr} \) respectively denote the magnetic parameter and the Prandtl number and are defined as

\[ Mn = \frac{2\sigma B_0^2}{\rho \alpha U_0} \quad \text{and} \quad \text{Pr} = \frac{\nu_\infty}{\alpha}. \]

Here \( \alpha = A \sqrt{\frac{n+1}{2} \frac{U_0}{\nu_\infty}} \) is the wall thickness parameter and \( \eta = A \sqrt{\frac{n+1}{2} \frac{U_0}{\nu_\infty}} \) indicates the plate surface. In order to facilitate the computation, we define

\[ (f')' = f'' - \frac{2n}{(n+1)} f' + \frac{2Mn}{(1+n)^2} (f' + mh) = 0, \quad (2.20) \]

\[ (h')' = h'' - \frac{2n}{n+1} f' h + \frac{2Mn}{(1+n)^2} (mf' - h) = 0, \quad (2.21) \]

\[ [(1 + \varepsilon \theta) \theta']' + \text{Pr} \left( f \theta' - \frac{2r}{n+1} \theta f' \right) = 0, \quad (2.22) \]

and the corresponding boundary conditions are \((n \neq 1)\)

\[ f(0) = \alpha^{1-n} \frac{1}{1+n}, \quad f'(0) = 1, \quad h(0) = 0, \quad \theta(0) = 1, \quad \theta(\infty) = 0, \quad h(\infty) = 0, \quad f'(\infty) = 0 \]

(2.23)

where the prime denotes differentiation with respect to \( \xi \). For all practical purposes, the important physical quantities of interest are the horizontal skin friction \( C_{f_h} \), transverse skin friction \( C_{f_z} \) and the Nusselt number \( Nu_x \) defined by

\[ C_{f_h} = \frac{2 \nu_\infty}{U_w^2} \int_{y=A(x+b)}^{y=A(x+b)+l_y} f''(0), \quad C_{f_z} = \frac{2 \nu_\infty}{U_w^2} \int_{y=A(x+b)}^{y=A(x+b)+l_y} w''(0), \]

\[ Nu_x = \frac{U_w}{\nu_\infty} \frac{(x+b)(T_y - T_x)(y=A(x+b))}{(T_s - T_x)} = \frac{n+1}{2 \text{Re}_x} h'(0), \quad (2.24) \]

where \( \text{Re}_x = \frac{U_w(x+b)}{\nu_\infty} \) is the local Reynolds number.

3 Exact solutions for some special cases

Here we present exact solutions for certain special cases and these solutions serve as a baseline for computing general solutions through numerical schemes. We notice that in the absence of variable fluid properties, magnetic field, and Hall current, heat transfer Eq. (2.20) reduces to those of Fang et al. [23].
Further, in the absence of variable fluid properties, Hall current, heat transfer and variable thickness of the wall, Eq. (2.20) reduces to those of Andersson [3], Van Gorder and Vajravelu [8], Vajravelu and Rollins [11], Pop and Na [12], Fang et al. [13], and Fang and Zhang [14].

3.1. Absence of variable fluid properties and Hall current for the case of a flat plate (i.e., $\theta_c \to \infty, n = 1$, $m = 0$ and $b = 0$)

In the limiting case of $\theta_c \to \infty$, $n = 1$ and $m = 0$, since the boundary layer flow and heat transfer problem degenerates. The solution for the velocity in the presence of a magnetic field turns out to be $f(\eta) = 1 - e^{-\beta \eta}$ where $\beta = \sqrt{1 + Mn}$. The solution for the temperature field can be written as a two parameter solution in terms of the confluent hypergeometric function, namely, the Kummer’s function, $\phi$, to wit:

$$\theta(\eta) = e^{-\frac{\beta \eta}{2}} \phi(a_1, b_1, -\alpha),$$

where $a_0 = \Pr \beta^2$, $a_1 = a_0 - r$, $b_1 = 1 + a_0$, $z = -a_0 e^{-\beta \eta}$.

3.2. No variable fluid properties, no magnetic field, no Hall currents, no heat transfer but in the presence of variable boundary thickness (i.e., $\theta_c \to \infty, Mn = m = 0$, $n \neq 1, r = 0$)

Case (i): When $n = -1/3$, Eq. (2.20) becomes

$$f'' + ff' + f'^2 = 0$$

with the boundary conditions

$$f(0) = 2\alpha, f'(0) = 1, f'(\infty) = 0.$$  

On integrating (3.25) twice yields to

$$f' + \frac{f'^2}{2} = (\beta + 2\alpha)\eta + (2\alpha^2 + 1),$$  

where $\beta = f''(0)$. To obtain finite solution it is essential to consider $\beta = -2\alpha$.

Thus (3.27) reduces to

$$f' + \frac{f'^2}{2} = (2\alpha^2 + 1).$$  

The solution is

$$f(\xi) = \sqrt{2 + 4\alpha^2} \tanh\left[\frac{\sqrt{2 + 4\alpha^2}}{2} \xi + \tanh^{-1}\left(\frac{2\alpha}{\sqrt{2 + 4\alpha^2}}\right)\right].$$  

and

$$f'(\xi) = 1 + 2\alpha^2 \text{Sech}^2\left[\frac{\sqrt{2 + 4\alpha^2}}{2} \xi + \tanh^{-1}\left(\frac{2\alpha}{\sqrt{2 + 4\alpha^2}}\right)\right].$$  

Case (ii): When $n = -1/2$, Eq. (2.20) becomes

$$f'' + ff' + 2f'^2 = 0$$

with the boundary conditions

$$f(0) = 3\alpha, f'(0) = 1, f'(\infty) = 0.$$
Eq. (3.31) is equivalent to
\[ \frac{1}{f} \frac{d}{d\xi} \left[ f^{3/2} \frac{d}{d\xi} \left( f^{-1/2} f' + \frac{2}{3} f^{3/2} \right) \right] = 0, \]  
(3.33)

Integrating (3.33) once reduces to following form
\[-\frac{1}{2} f'^2 + ff'' + f^2 f' = -\frac{1}{2} + 3\alpha\beta + 9\alpha^2 \]
(3.34)

Applying for free boundary condition we obtain
\[ \beta = -3\alpha + \frac{1}{6\alpha}. \]
(3.35)

An integration of (3.34) leads to
\[ f^{-\frac{1}{2}} f' + \frac{2}{3} f^\frac{3}{2} = \frac{2}{3} (3\alpha)^\frac{3}{2} + \frac{1}{\sqrt{3}\alpha}. \]
(3.36)

The final solution is
\[ \eta + D = \frac{1}{2d^2} \ln \left[ \frac{f + d \sqrt{f + d^2}}{(d - \sqrt{f})^2} \right] + \sqrt{3} \frac{d}{d^2} \tan^{-1} \left( \frac{2\sqrt{f + d}}{d\sqrt{3}} \right), \]
(3.37)

where \( d = \left[ (3\alpha)^{3/2} + \frac{3}{2\sqrt{3}\alpha} \right]^{1/3} \) and \( D = \frac{1}{2d^2} \ln \left( \frac{3\alpha + d\sqrt{3\alpha} + d^2}{(d - \sqrt{3\alpha})^2} \right) + \sqrt{3} \frac{d}{d^2} \tan^{-1} \left( \frac{2\sqrt{3\alpha} + d}{d\sqrt{3}} \right). \)

4 Numerical procedure

The system of Eqs. (2.20)-(2.22) with conditions (2.23) has no exact analytical solutions, they are solved numerically via a second order finite difference scheme. These equations are highly non-linear and coupled ordinary differential equations with variable coefficients. Exact analytical solutions are not possible for the complete set of equations. Hence, we use an efficient numerical method with second order finite difference scheme known as the Keller-box method (For details see Cebeci and Bradshaw [29], Keller [30], and Vajravelu and Prasad [31]). For the sake of brevity, the details of the numerical solution procedure are not presented here. For numerical calculations, a uniform step size of \( \Delta\eta = 0.01 \) is found to be satisfactory and the shooting error was controlled with an error tolerance of \( 10^{-6} \) in all the cases. In order to validate the method used in this study and to judge the accuracy of the present analysis, the horizontal skin friction and the wall temperature gradient are compared with the previously published results of Andersson et al. [2], Grubka and Bobba [32], Chen [33], Ali [34], and Prasad et al. [27] for several special cases, and the results are found to be in good agreement: The results are shown in Tables I and II.
Table I: Comparison of skin friction \( f''(0) \) for different values of the magnetic Parameter when 
\[ \theta_r \rightarrow \infty, \varepsilon = 0.0, m = 0, n = 1. \]

<table>
<thead>
<tr>
<th>( Mn )</th>
<th>( Mn = 0.0 )</th>
<th>( Mn = 0.5 )</th>
<th>( Mn = 1.0 )</th>
<th>( Mn = 1.5 )</th>
<th>( Mn = 2.0 )</th>
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<tbody>
<tr>
<td>Exact solution</td>
<td>-1.000</td>
<td>-1.2247</td>
<td>-1.414235</td>
<td>-1.5811388</td>
<td>-1.732050</td>
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<tr>
<td>Andersson et al. [2] for ( n = 1 )</td>
<td>-1.00000</td>
<td>-1.2249</td>
<td>-1.4140</td>
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<tr>
<td>Present results</td>
<td>-1.000174</td>
<td>-1.224753</td>
<td>-1.4144499</td>
<td>-1.581139</td>
<td>-1.732203</td>
</tr>
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</table>

Table II: Comparison of wall temperature gradient \( \theta'(0) \) for different values of Prandtl number and magnetic parameter when \( \theta_r \rightarrow \infty, \varepsilon = 0.0, r = 0.0, n = 1. \)

<table>
<thead>
<tr>
<th>Pr</th>
<th>Present results</th>
<th>Grubka &amp; Bobba [32]</th>
<th>Chen [33]</th>
<th>Ali [34]</th>
<th>Prasad et al. [27] for linear stretching</th>
<th>Present results</th>
<th>Prasad et al. [27] for linear stretching</th>
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<td>-</td>
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5 Results and discussion

In order to get a clear insight into the physical problem, numerical computations have been carried out for different values of the non-dimensional parameters such as the Hall parameter \( m \), the fluid viscosity parameter \( \theta_r \), the power index parameter \( n \), the variable thickness \( \alpha \). Throughout the computation, the values of the wall temperature parameter \( r \), the Prandtl number \( Pr \), and the magnetic parameter \( Mn \) are fixed. Analytical solutions are obtained for the special cases of \( \theta_r \rightarrow \infty \), \( n = 1 \), \( m = 0 \), and \( Mn = 0 \). We presented the numerical results graphically for the horizontal velocity profile \( f''(\eta) \), the transverse velocity profile \( h(\eta) \), and the temperature field \( \theta(\eta) \) for different values of pertinent parameters in Figs. 2-6. It can be perceived from these graphs that all three profiles \( f''(\eta), \theta(\eta) \) and \( h(\eta) \) decrease monotonically and tend asymptotically to zero as the distance increases from the boundary. The computed numerical values for the horizontal skin friction \( f''(0) \), the transverse skin friction \( h'(0) \), and the rate of heat transfer \( \theta'(0) \) are tabulated in Table III.

Figures 2(a) through 2(c) depict the effect of increasing values of \( \theta_r \) on \( f''(\eta), \theta(\eta), \) and \( h(\eta) \) respectively. It is observed that \( f''(\eta) \) decreases with increasing values of \( \theta_r \). That is, as \( \theta_r \rightarrow 0 \), the momentum boundary layer thickness decreases and the velocity distribution becomes linear for higher values of \( \alpha \). This is even true in the case of \( h(\eta) \). This is due to the fact that for a given fluid, when \( \theta_r \) is smaller, then the temperature difference between the wall and the ambient fluid is larger. The results explicitly manifest that \( \theta_r \) is the indicator of the variable fluid viscosity with temperature which has a substantial effect on \( f''(\eta) \) and \( h(\eta) \) and hence on the skin friction. In the case of \( \theta(\eta) \), the impact is quite the opposite. Figures 3(a) to 3(c) exhibit the effects of \( n \), respectively on \( f'(\eta), \theta(\eta), \) and \( h(\eta) \).
It can be seen that for increasing values of $n$ both $f'(\eta)$ and $h(\eta)$ increase. This implies that the momentum boundary layer thickness increases but exactly the opposite is true in the case of the transverse velocity distribution. That is, there is a significant effect of $n$ (for $n < 0$, $n = 0$ and $n > 0$) on the flow pattern: Here, it may be observed that for negative value of $n$, the sheet is shrinking along the axis, and is stretching for positive $n$. Figures 4(a) to 4(c) are the graphical representations of $f'(\eta)$, $\theta(\eta)$, and $h(\eta)$ for increasing values of $\alpha$. It is noticed that all the profiles increase as $\alpha$ increases. Figures 5(a) to 5(c) demonstrate the effect of increasing values of $m$ on $f'(\eta)$, $\theta(\eta)$, and $h(\eta)$. It is very clear from these figures that $f'(\eta)$ increases and $\theta(\eta)$ decreases. This is due to the fact that the effective conductivity $\left[\sigma / (1 + m^2)\right]$ decreases with increasing $m$. This in turn reduces the magnetic damping force on $f'(\eta)$ and hence increases propelling effect on $f'(\eta)$. Further, the transverse flow in the z-direction initially increases with $m$, reaching a maximum for $m = 1.5$ and 1.0 respectively when $n=1.0$ and $n = 5.0$ and then decreases. For large values of $m$, the term $\left[1/(1 + m^2)\right]$ becomes very small and hence the resistive effect of the magnetic field is diminished. The effects of $\epsilon$ and $\alpha$ on $\theta$ can be seen from Figure 6. Temperature of the fluid is found to increase with increasing values of $\epsilon$. That is, the assumption of temperature dependent thermal conductivity suggests a reduction in the magnitude of the velocity by a quantity $\partial k(T)/\partial y$ which can be seen in Eq. (2.6). Therefore, the rate of cooling is much faster for the coolant material having small thermal conductivity parameter. By a careful scrutiny of table III, one can observe that $f''(0)$ decrease as $\alpha$ increases for $n < 1$. The opposite is true for $n \geq 1$. This is because of the induced mass transfer. This momentum transfer accelerates fluid particles downstream. Similarly we notice a significant change in the velocity field for positive values of $n$, the stretching sheet case.
Fig. 2: (a) Horizontal velocity profiles (b) Temperature profiles for different values of $\Theta_r$ and $\alpha$ with $n = 2.0, \varepsilon = 0.1, m = 1.0, Mn = 1.0, r = 1.0, Pr = 2.0$.

Fig. 2: (c) Transvers velocity profiles for different values of $\Theta_r$ and $n$ with $\alpha = 0.2, \varepsilon = 0.1, m = 1.0, Mn = 1.0, r = 1.0, Pr = 2.0$. 

\[ \alpha = 0.2, \cdots \cdots \alpha = 0.0 \] 

\[ n = 1.0, 5.0 \] 

\[ \Theta_r = -1.5, -2.0, -3.0, -5.0, -10.0 \] 

\[ \Theta_r = -10.0, -5.0, -3.0, -2.0, -1.5 \] 

\[ \varepsilon = 0.2, \varepsilon = 0.1, m = 1.0, Mn = 1.0, r = 1.0, Pr = 2.0 \]
Fig. 3: (a) Horizontal velocity profiles (b) Temperature profiles (c) Transverse velocity profiles for different values of $n$ and $\alpha$ with $\tilde{\theta}_{0} = -0.5$, $\varepsilon = 0.1$, $m = 1.0$, $Mn = 1.0$, $r = 1.0$, $Pr = 2.0$. 
Fig. 4: (a) Horizontal velocity profiles (b) Temperature profiles (c) Transverse velocity profiles for different values of $\alpha$ and $n$ with $\beta = -5.0, \, \nu = 0.1, \, m = 1.0, \, M_n = 1.0, \, r = 1.0, \, Pr = 2.0$. 

\[ f'(\eta) = -0.5, -0.2, 0.0, 0.5, 1.0 \quad n = 2.0, \quad n = 5.0 \]

\[ \theta(\eta) = -0.5, -0.2, 0.0, 0.5 \quad n = 2.0, \quad n = 5.0 \]

\[ h(\eta) = -0.5, -0.2, 0.0, 0.5 \quad n = 2.0, \quad n = 5.0 \]
Fig. 5: (a) Horizontal velocity profiles (b) Temperature profiles (c) Transverse velocity profiles for different values of $m$ and $n$ with $\theta_i = -5.0$, $\alpha = 0.1$, $\gamma = 0.2$, $Mn=1.0$, $r = 1.0$, $Pr = 2.0$. 

![](image1.png)

![](image2.png)

![](image3.png)
Fig. 6: Temperature profiles for different values of $\varepsilon$ and $\alpha$ with $\theta_1=-5.0$, $m=1.0$, $n=2.0$, $Mn=1.0$, $r=1.0$, $Pr=2.0$. 

$\alpha=0.2$, $\alpha=0.0$ 

$\varepsilon = -0.0, 0.1, 0.2, 0.4$
Table III: values of Skin friction, transverse skin friction and wall temperature gradient for different values of the pertinent parameters with \( pr = 2.0 \), \( r = 1.0 \), \( Mn =1.0 \)

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<th>( \alpha )</th>
<th>( \theta_r )</th>
<th>( m )</th>
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<td>( h^*(0) )</td>
<td>( \theta'(0) )</td>
<td>( f^*(0) )</td>
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<th>( \alpha )</th>
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<td>( \theta'(0) )</td>
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<table>
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<th>( m )</th>
<th>( n )</th>
<th>( \varepsilon )</th>
<th>( \theta_r )</th>
<th>( \alpha=0.0 )</th>
<th>( \alpha=0.2 )</th>
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<td>( f^*(0) )</td>
<td>( h^*(0) )</td>
<td>( \theta'(0) )</td>
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| 10.0 | -1.421613 | 0.050647 | -0.904341 | -1.330092 | 0.049788 | -0.727622 |
6 Conclusion

Some of the interesting findings of the study are summarized below.

- An increase in the wall thickness parameter leads to reduction in the skin-friction coefficient in the $x$-direction for $n < 1$ while quite the opposite is true for $n \geq 1$.
- Hall current has a strong effect on the shear stress and the Nusselt number.
- For large values of the Hall parameter $m$, the resistive effect of the magnetic field is reduced.
- The effect of the variable thermal conductivity parameter is to enhance the temperature field.
- The variable thermophysical property parameters have strong effects on the skin friction and the heat transfer characteristics.

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