Communications in Numerical Analysis 2016 No.1 (2016) 11-16

Available online at www.ispacs.com/cna

Volume 2016, Issue 1, Year 2016 Article ID cna-00250, 6 Pages

doi:10.5899/2016/cna-00250

Research Article

A combined analytic-numeric approach for some boundary-value problems

Mustafa Turkyilmazoglu*

Mathematics Department, University of Hacettepe, 06532-Beytepe, Ankara, Turkey

Abstract

A combined analytic-numeric approach is undertaken in the present work for the solution of boundary-value problems in the finite or semi-infinite domains. Equations to be treated arise specifically from the boundary layer analysis of some two and three-dimensional flows in fluid mechanics. The purpose is to find quick but accurate enough solutions. Taylor expansions at either boundary conditions are computed which are next matched to the other asymptotic or exact boundary conditions. The technique is applied to the well-known Blasius as well as Karman flows. Solutions obtained in terms of series compare favorably with the existing ones in the literature.

Keywords and phrases: Direct approach, Boundary layer, Boundary-value problems, Semi-infinite domain.

1 Introduction

Solution methods for the boundary-value problems are very significant since they appear in the formulation of many physical phenomena. The aim of the present work is to develop an accurate as well as easy to implement solution scheme.

Techniques to determine solutions to boundary-value problems have now been outnumbered. The literature on the numerical solution methods of boundary layer equations is fairly extensive and we shall therefore give a very thorough fundamental sketch by quoting only the most relevant researches to the present work. Numerical solution methods are generally based on finite-difference approach. Accuracy of this method increases whenever the approximating operators have less truncation errors. Of course this brings in an unavoidable problem of numerical stability. The merits of technique can be found in [1]. Applications to boundary layer type equations are various, see for instance [2] for the flat plate. Rotating-disk boundary layer flow was also solved with a fourth-order compact finite-difference scheme in [3]. Another common method for boundary-value equations is the Runga-Kutta scheme in conjunction with a shooting procedure. The preference here is again with its straightforward implementation, unless the differential equation is wildly stiff. [4] made use of this approach for the solution of rotating-disk flow. Last, but not the least, the third class of numerical approach is the spectral technique, see for instance [5], a particular case of which is Chebyshev collocation. This technique is based on Chebyshev polynomials, the accuracy of which becomes superior to the finite-difference or Runga-Kutta numerical integration as the number of collocation points to be employed is increased. The current trend seems to be in the use of Cheyshev collocation method for the solution of boundary-layer equations of fluid dynamics, see for instance,[6], [7], [8] and [9] amongst many others. An analytic nature method

*Corresponding author. Email address: turkyilm@hotmail.com, Tel: +90 3122977850

based on homotopy analysis was also presented for some boundary layer flows [10]. See also the articles [11] and [12, 13] for some recent analytic approximate kind methods.
The main purpose of the present study is to develop an analytic-numeric technique which gives rapid solutions, besides its reliability in terms of its accuracy and stability. In line with this, the approach adopted here is simply based upon the Taylor expansion of the physical quantities near the wall or at the far field, and at the end a direct matching of the variables. As opposed to the aforementioned methods, the present approach yields a detailed solution without a need to approximate the derivatives, by reducing the boundary-value problem to a simple root finding work. Application of the developed method to flat plate and rotating disk flow problems demonstrates the success of the method.
The following strategy is pursued in the rest of the paper. In §2 the approach is shortly described. §3 contains results obtained. An alternative efficient approach is also outlined in §3. Finally our conclusions are drawn in §4.

2 Description of the method

We consider differential equations over a finite $[0,a]$ or a semi-infinite interval $[0,\infty)$ of the form

$$y^{(n)}(x) = F(x, y(x), y'(x), y''(x), \cdots, y^{(n-1)}(x)), \tag{2.1}$$

or similar system of equations with properly defined boundary conditions.
The methodology usually pursued [14] is to evaluate Taylor expansion near the wall followed by an asymptotic series expansion at infinity and eventually to patch the solutions at an intermediate region over which both series are equally valid. However, this way of finding the solution involves a laborious work. We below instead introduce a simple to adhere approach.
Let the solution to (2.1) near the wall $x = 0$ be expressed by the Taylor expansion

$$y(x) \quad = \quad \sum_{n=0}^{\infty} a_n x^n, \tag{2.2}$$

where a_n are to be determined. If the radius of convergence R of (2.2) is sufficiently large, say $|x| < R$, then we choose an artificial boundary at a position, $x = x_R = R - \delta$, δ is small. At this location we obtain the Taylor expansion

$$y(x) \quad = \quad \sum_{n=0}^{\infty} b_n (x - x_R)^n. \tag{2.3}$$

It is reminded that provided this location is close enough to the asymptotic boundary, we can enforce the far field boundary conditions there, hence b_n are known. Next, using the binomial expansion, a direct matching of the coefficients in (2.2) and (2.3) enables us to derive the relation

$$b_m \quad = \quad \sum_{n=m}^{\infty} \binom{n}{m} a_n (x_R)^{n-m}. \tag{2.4}$$

Having found a_n's in this way from (2.4), the series solution (2.2) to (2.1) up to the point x_R is determined. This solution will be sufficient since the interval of convergence is well away from the boundary layer thickness. However, if required, the solution can be carried to as far distances as desired after fixing the unknowns at the far field asymptotic solution. This is the case in what follows for the Blasius flow equation. Whenever the radius of convergence of the Taylor expansion is infinite, then a direct match to the other boundary can be achieved as for the case in the Karman equation.

3 Results and discussion

3.1 A simple example

To illustrate accuracy of the proposed technique, consider the nonlinear boundary value problem

$$y'' + yy' \quad = \quad 0, \quad y(0) = 1, \quad y(1) = \frac{2}{3}, \tag{3.5}$$

whose solution is $y = \frac{2}{2+x}$. The radius of convergence of the solution appears to be $R = 2$. Therefore, if a series solution is sought at $x = 0$ in the form (2.2), it can be extended up to the other boundary point via the expansion (2.3). Due to the first boundary condition in (3.5), $a_0 = 1$ and the second boundary condition in (3.5), $b_0 = \frac{2}{3}$ in (2.4). Substituting expansion (2.2) into the differential equation and letting $m = 0$ in (2.4) leads to the following root-finding problem for a_1

$$a_{n+2} \quad = \quad -\frac{1}{(n+1)(n+2)} \sum_{k=0}^{n+1} k a_k a_{n+1-k}, \quad \sum_{n=0}^{\infty} a_n = \frac{2}{3}. \tag{3.6}$$

Numerical solution of the above algebraic equation yields the unknown coefficient $a_1 = -0.49999999987$ (with 1000 terms evaluated in the series (3.6)), whose exact value is -1/2. This verifies the accuracy of the method.

3.2 Application to the Blasius flow

The Blasius boundary layer flow problem over a flat plate is governed by

$$f''' + \frac{x_m}{2} f f'' = 0, \tag{3.7}$$

with the initial and boundary conditions

$$f(\eta = 0) = 0, \quad f'(\eta = 0) = 0, \quad f'(\eta = 1) = x_m. \tag{3.8}$$

In equations (3.7-3.8) we introduced the substitution $\eta = \frac{x}{x_m}$, so that the computational domain $[0, x_m]$ of interest is mapped onto $[0, 1]$. The boundary layer thickness is known to lie near $x = 5$ for this flow. Consider now the Taylor expansion near the wall $\eta = 0$

$$f(\eta) = \sum_{n=0}^{\infty} (-1)^n a_n \eta^{3n+2}, \tag{3.9}$$

with $a_0 = f''(0)/2 = \alpha$ to be determined during the process. Inserting (3.9) into the differential equation (3.7) generates the following recurrence relations for a_n

$$a_{n+1} = \frac{x_m}{6(n+1)(3n+4)(3n+5)} \sum_{k=0}^{n} (3k+1)(3k+2) a_k a_{n-k}, \quad (n \geq 0). \tag{3.10}$$

The Taylor expansion at the endpoint $\eta = 1$ reads

$$f(\eta) = \sum_{n=0}^{\infty} b_n (\eta - 1)^{3n+2}, \tag{3.11}$$

which can be directly matched to equation (3.9) resulting in

$$b_1 \quad = \quad x_m \; = \; \sum_{n=1}^{\infty} (-1)^n (3n+2) a_n. \tag{3.12}$$

Further substitution $a_n = \alpha^{n+1} A_{n+2}$, with $A_0 = A_1 = 0$ and $A_2 = 1$ generates similar recurrence relation to (3.10), but with the advantage of new coefficients A_n being invariant during the iterations. Thus, solution to (3.7) has a radius of convergence $R = (\frac{R_1}{\alpha})^{1/3}$, where R_1 is the radius of convergence of the associated coefficients.
From the numerical treatment it appears that $R_1 \sim 30.6$, resulting in a convergence radius $R \sim 5.7$. As a result, it is safe to insert the outer boundary x_m of system (3.7-3.8) at 5.64, which is outside the boundary layer thickness, and we obtain after a rearrangement of (3.12)

$$b_1 \; = \; x_m \; = \; \sum_{n=0}^{\infty} (-1)^{n+1} (3n+5) \alpha^{n+2} A_{n+2}. \tag{3.13}$$

As a result, the problem turns out to be a root finding problem from equations (3.13) and (3.10), which is carried out by a Newton-Raphson iteration method.
Upon evaluation of the boundary layer a straightforward match can be performed so that at large distances the following behavior holds

$$f = x + \beta + \theta (x + \beta)^{-2} e^{-\frac{1}{4}(x+\beta)^2}.$$

Table 1: Comparison of the Blasius flow quantities.

	α	β	θ
Present	0.33206	-1.72079	0.15693
Chebyshev [9]	0.33206	-1.72078	0.15691

Table 1 presents a comparison between the results from the current approach and from the Chebyshev collocation numerical procedure, see [9]. The values tabulated are also fairly consistent with the available results, see for instance [2]. Hence, accuracy of the approach is justified. The mean flow computed can be next used for further assessments as for the stability of the flow.

3.3 Application to the Karman flow

The governing equations over a rotating disk due to von Karman [15] are

$$\begin{aligned} &c^2F'' - cHF' - F^2 + (G+1)^2 = 0,\\ &c^2G'' - cHG' - 2F(G+1) = 0,\\ &cH' + 2F = 0, \end{aligned} \tag{3.14}$$

together with the imposed initial and boundary conditions

$$F(\eta=0) = G(\eta=0) = H(\eta=0) = 0, \quad F(\eta=\infty) = G(\eta=\infty)+1 = 0. \tag{3.15}$$

A scaling factor c is introduced via $\eta = cx$. Similar to Blasius flow (3.9), Taylor expansions near the wall $\eta = 0$ are

$$F = \sum_{n=1}^{\infty} a_n\eta^{n-1}, \quad G = -1 + \sum_{n=1}^{\infty} b_n\eta^{n-1}, \quad H = -\frac{2}{c}\sum_{n=1}^{\infty}\frac{a_n}{n}\eta^n, \tag{3.16}$$

and considering the boundary conditions (3.15), we set $a_1 = 0$, $b_1 = 1$, $a_2 = \alpha$ and $b_2 = \beta$, with α and β are to be found. Next, inserting these into (3.14), the subsequent relations yield

$$\begin{aligned} a_{n+1} &= -\tfrac{1}{c^2n(n-1)}\textstyle\sum_{k=1}^{n-1}[\tfrac{2(n-1)}{k} - 3]a_ka_{n-k} + b_kb_{n-k},\\ b_{n+1} &= -\tfrac{1}{c^2n(n-1)}\textstyle\sum_{k=1}^{n-1}[\tfrac{2(n-1)}{k} - 4]a_kb_{n-k}, \end{aligned} \tag{3.17}$$

valid for $n \geq 2$. Computations show that for the rotating-disk boundary layer flow the radius of convergence is nearly 2.6, which is too below the boundary layer thickness (~ 5.4), and is not good enough to reach the far field asymptotic boundary conditions in (3.15).

Instead, let us concentrate at the far field and assume series like solutions

$$F = \sum_{n=1}^{\infty} A_ne^{-n\eta}, \quad G = -1 + \sum_{n=1}^{\infty} A_ne^{-n\eta}, \quad H = H_\infty + \frac{2}{c}\sum_{n=1}^{\infty}\frac{A_n}{n}e^{-n\eta}, \tag{3.18}$$

which satisfy the infinity boundary conditions in (3.15). We should bear in mind that this form of solutions is also a result of Taylor expansion. Upon substitution of (3.18) into (3.14), we see that the stretching factor c satisfies $c^2 + cH_\infty = 0$ and the coefficients are found from

$$\begin{aligned} A_n &= -\tfrac{1}{c^2n(n-1)}\textstyle\sum_{k=1}^{n-1}[\tfrac{2n}{k} - 3]A_kA_{n-k} + B_kB_{n-k},\\ B_n &= -\tfrac{1}{c^2n(n-1)}\textstyle\sum_{k=1}^{n-1}[\tfrac{2n}{k} - 4]A_kB_{n-k}, \end{aligned} \tag{3.19}$$

valid for $(n \geq 2)$. Because $\lim|\frac{A_n}{A_{n+1}}|$ and $\lim|\frac{B_n}{B_{n+1}}|$ are less than unity, the interval of convergence of (3.19) extends to infinity, making it possible a direct match to the wall (3.16), giving

$$a_{m+1} = \sum_{n=1}^{\infty} A_n\frac{(-n)^m}{m},\ b_{m+1} = \sum_{n=1}^{\infty} B_n\frac{(-n)^m}{m},\ -\frac{2}{c}\frac{a_{m+1}}{m+1} = H_\infty + \frac{2}{c}\sum_{n=1}^{\infty}\frac{A_n}{n}\frac{(-n)^m}{m}, \tag{3.20}$$

which holds for all positive integer m, and specifically when $m = 0$ (3.20) results in

$$\sum_{n=1}^{\infty} A_n = 0, \quad \sum_{n=1}^{\infty} B_n = 1, \quad \sum_{n=1}^{\infty} \frac{A_n}{n} = \frac{c^2}{2}, \tag{3.21}$$

that collapses onto the method introduced in [16]. Equations (3.19) and (3.21) show that we have at hand a zero finding problem involving three parameters c^2, A_1 and B_1. Having found these, we know in turn the wall expansion coefficients a_m and b_m from (3.20).
As an alternative, a further replacement of the coefficients by $\bar{A}_n = c^2 A_n$ and $\bar{B}_n = c^2 B_n$ peels of the parameter c from (3.19) and converts the boundary conditions (3.21) to

$$\sum_{n=1}^{\infty} \bar{A}_n = 0, \quad \sum_{n=1}^{\infty} \bar{B}_n = 1/c^2, \quad \sum_{n=1}^{\infty} \frac{\bar{A}_n}{n} = \frac{1}{2}. \tag{3.22}$$

Thus, the second of equations (3.22) becomes redundant, and the problem simplifies to finding out only the parameters $\bar{A}_n$ and $\bar{B}_n$ from the given initial guesses $\bar{A}_1$ and $\bar{B}_1$. The second of equations (3.22) eventually fixes the unknown c^2.

Table 2: Comparison of the Von Karman flow quantities.

	$F'(0)$	$G'(0)$	$H(\infty)$	A_1	B_1
Present	0.51023	-0.61592	-0.88446	0.92487	1.20221
Chebyshev [9]	0.51023	-0.61592	-0.88447	0.92486	1.20221

Table 2 presents a comparison between the approximate results from the current approach and from the Chebyshev collocation numerical procedure, see [9]. The values tabulated are also fairly consistent with the available results, see for instance [16].

4 Conclusion

A self-consistent and rigorous expansion technique has been pursued in this study to compute solutions for the steady two-three dimensional boundary layer flows. In place of approximating the solution of a differential equation, the reduction of it to a root finding problem constitutes the simplicity of the present method. Particular attention is paid to the Blasius and Karman boundary layers. The presented method is no doubt applicable to other boundary layer type equations, such as Falkner-Skan-Cooke, Bodewat and Ekman equations. It is also a simple matter to cover the cases of impermeable wall conditions and also normal uniform magnetic field.

Acknowledgements

The author thanks referee and editor for their useful technical comments and valuable suggestions to improve the readability of the paper, which led to a significant improvement of the paper.

References

[1] J. D. Anderson, Computational fluid dynamics, McGraw-Hill, Inc, (1995).

[2] L. Rosenhead, Laminar boundary layers, Oxford University Press, (1963).

[3] M. R. Malik, The neutral curve for stationary disturbances in rotating-disk flow, J. Fluid Mech, 164 (1986) 275-287.
http://dx.doi.org/10.1017/S0022112086002550

[4] M. Turkyilmazoglu, J. W. Cole, J. S. B. Gajjar, Absolute and convective instabilities in the compressible boundary layer on a rotating disk, Theoret. Comput. Fluid Dyn, 14 (2000) 21-37.
http://dx.doi.org/10.1007/s001620050123

[5] C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang, Spectral methods in fluid dynamics, Springer-Verlag, (1988).
http://dx.doi.org/10.1007/978-3-642-84108-8

[6] O. R. Burggraf, P. W. Duck, Spectral computation of triple-deck flows, Numerical and Physical Aspects of Aerodynamic Flows, 399 (1982) 145-158.
http://dx.doi.org/10.1007/978-3-662-12610-3_9

[7] G. Danabasoglu, S. Biringen, A Chebyshev matrix method for spatial modes of the Orr-Sommerfeld equation, NASA Contr. Rep. No. 4247 (1989).

[8] M. Turkyilmazoglu, J. S. B. Gajjar, On the absolute instability of the attachment-line and swept-hiemenz boundary layers, Theoret. Comput. Fluid Dyn, 13 (1999) 57-75.
http://dx.doi.org/10.1007/s001620050004

[9] M. Turkyilmazoglu, Unsteady mhd flow with variable viscosity: Applications of spectral scheme, International Journal of Thermal Sciences, 49 (2010) 563-570.
http://dx.doi.org/10.1016/j.ijthermalsci.2009.10.007

[10] M. Turkyilmazoglu, A homotopy treatment of analytic solution for some boundary layer flows, Int. J. Nonlinear Sci. Numer. Simulation, 10 (2009) 885-890.
http://dx.doi.org/10.1515/IJNSNS.2009.10.7.885

[11] M. Turkyilmazoglu, Effective computation of exact and analytic approximate solutions to singular nonlinear equations of LaneEmdenFowler type, Applied Mathematical Modelling, 37 (2013) 7539-7548.
http://dx.doi.org/10.1016/j.apm.2013.02.014

[12] S. A. Shehzad, T. Hussain, T. Hayat, M. Ramzan, A. Alsaedi, Boundary layer flow of third grade nanofluid with newtonian heating and viscous dissipation, Journal of Central South University, 22 (2015) 360-367.
http://dx.doi.org/10.1007/s11771-015-2530-x

[13] M. B. Ashraf, T. Hayat, A. Alsaedi, S. A. Shehzad, Convective heat and mass transfer in mhd mixed convection flow of Jeffrey nanofluid over a radially stretching surface with thermal radiation, Journal of Central South University, 22 (2015) 1114-1123.
http://dx.doi.org/10.1007/s11771-015-2623-6

[14] W. G. Cochran, The flow due to a rotating disk, Proc. Camb. Phil. Soc, 30 (1934) 365-375.
http://dx.doi.org/10.1017/S0305004100012561

[15] T. V. Kármán, Über laminare und turbulente reibung, Zeitschnnift fur angewantee Mathematik und Mechanik, 1 (2010) 233-252.
http://dx.doi.org/10.1002/zamm.19210010401

[16] J. A. D. Ackroyd, On the steady flow produced by a rotating disc with either surface suction or injection, J. Eng. Phys, 12 (1978) 207-220.
http://dx.doi.org/10.1007/bf00036459