Communications in Numerical Analysis 2016 No.1 (2016) 50-56

Available online at www.ispacs.com/cna

Volume 2016, Issue 1, Year 2016 Article ID cna-00244, 7 Pages

doi:10.5899/2016/cna-00244

Research Article

Certain new unified integrals associated with the product of generalized Bessel functions

Praveen Agarwal[1*], Daniele Ritelli[2], Adem Kilicman[3], Shilpi Jain[4]

(1) *Department of Mathematics, Anand International College of Engineering, Jaipur-303012, India.*

(2) *School of Economics, Management and Statistics, Dipartimento di Scienze Statistiche, University of Bologna, via Belle Arti 41, 40126 Bologna Italy.*

(3) *Universiti Putra Malaysia, Department of Mathematics, Serdang, Malaysia.*

(4) *Department of Mathematics, Poornima College of Engineering, Jaipur-303012, India.*

Abstract

Our focus to presenting two very general integral formulas whose integrands are the integrand given in the Oberhettingers integral formula and a finite product of the generalized Bessel function of the first kind, which are expressed in terms of the generalized Lauricella functions. Among a large number of interesting and potentially useful special cases of our main results, some integral formulas involving such elementary functions are also considered.

Keywords: Gamma function, Generalized hypergeometric function ${}_pF_q$, generalized Lauricella series in several variables, Generalized Bessel function, Oberhettinger's integral formula.

2010 Mathematics Subject Classification. Primary 33B20, 33C20; Secondary 33B15, 33C05.

1 Introduction and Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{C}$, and $\mathbb{Z}_0^-$ denote the sets of positive integers, real numbers, complex numbers, and nonpositive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Recently, Baricz introduced and studied some fundamental properties and characteristics of the generalized Bessel function of the first kind, $\mathrm{w}_\nu(z)$ which are defined by (see, for example, [3, p.10, Eqn. (1.15)]; for a very recent work, see also [4, 5, 6]):

$$w_{\nu,b,c}(z) = \sum_{n\geq 0} \frac{(-1)^n c^n \left(\frac{z}{2}\right)^{\nu+2n}}{n!\,\Gamma\left(\nu+n+\frac{b+1}{2}\right)}, \tag{1.1}$$

where $\Gamma(z)$ is the familiar Gamma function (see [14, Section 1.1], see also [1]).

Here, it is worth mentioning that, Bessel function of the first kind $J_\nu(z)$ and $I_\nu(z)$ are frequently used in studying solutions of differential equations, and they are associated with a wide range of problems in important areas of mathematical physics, like problems of acoustics, radio physics, hydrodynamics, and atomic, nuclear physics, probability

*Corresponding author. Email address: goyal.praveen2011@gmail.com; Tel:+91 8387894656.

theory and statistics. These considerations have led various workers in the field of special functions for exploring the possible extensions and applications for the Bessel functions (see, [2, 7, 9, 10, 11, 13, 17]). Here we prove two very general integral formulas whose integrands are the integrand given in the Oberhettingers integral formula and a finite product of the generalized Bessel function of the first kind, which are expressed in terms of the generalized Lauricella functions. Among a large number of interesting and potentially useful special cases of our main results, some integral formulas involving such elementary functions are also considered.
The generalized Lauricella functions (see, for example, [16, p. 36, Eq. (19)]) is (*cf.* Srivastava and Daoust [15, p. 454]; see also [16, p. 37])

$$F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\begin{pmatrix} z_1\\ \vdots \\ z_n\end{pmatrix} = F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\left(\begin{matrix}[(a):\theta^{(1)},\ldots,\theta^{(n)}]:\\ [(c):\psi^{(1)},\ldots,\psi^{(n)}]:\end{matrix}\right.$$
$$\left.\begin{matrix}[(b)^{(1)}:\phi^{(1)}];\ldots;[(b)^{(n)}:\phi^{(n)}];\\ [(d)^{(1)}:\delta^{(1)}];\ldots;[(d)^{(n)}:\delta^{(n)}];\end{matrix} z_1,\ldots,z_n\right) \tag{1.2}$$
$$= \sum_{k_1,\ldots,k_n=0}^{\infty} \Omega(k_1,\ldots,k_n)\,\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_n^{k_n}}{k_n!},$$

where, for convenience,

$$\Omega(k_1,\ldots,k_n) = \frac{\prod_{j=1}^{A}(a_j)_{k_1\theta_j^{(1)}+\cdots+k_n\theta_j^{(n)}}\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{k_1\phi_j^{(1)}}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{k_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{k_1\psi_j^{(1)}+\cdots+k_n\psi_j^{(n)}}\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{k_1\delta_j^{(1)}}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{k_n\delta_j^{(n)}}}, \tag{1.3}$$

the coefficients

$$\begin{cases}\theta_j^{(m)}\ (j=1,\ldots,A);\ \phi_j^{(m)}\ (j=1,\ldots,B^{(m)});\\ \psi_j^{(m)}\ (j=1,\ldots,C);\ \delta_j^{(m)}\ (j=1,\cdots,D^{(m)});\ \forall m\in\{1,\ldots,n\}\end{cases} \tag{1.4}$$

are real and positive, and (a) abbreviates the array of A parameters $a_1,\ldots,a_A$, $(b^{(m)})$ abbreviates the array of $B^{(m)}$ parameters

$$b_j^{(m)} \quad (j=1,\ldots,B^{(m)});\quad \forall m\in\{1,\ldots,n\},$$

with similar interpretations for (c) and $(d^{(m)})$ $(m=1,\ldots,n)$; *et cetera.*
The multiple series (1.2) converges absolutely either

(i) $\Delta_i>0\ (i=1,\ldots,n),\ \forall\, z_1,\cdots,z_n\in\mathbb{C},$

or

(ii) $\Delta_i=0\ (i=1,\ldots,n),\ \forall\, z_1,\cdots,z_n\in\mathbb{C},\ |z_i|<\rho_i\ (i=1,\ldots,n).$

The multiple series (1.2) is divergent when $\Delta_i<0\ (i=1,\cdots,n)$ except for the trivial case $z_1=0,\ldots,z_n=0$. Here

$$\Delta_i \equiv 1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}\ (i=1,\ldots,n), \tag{1.5}$$

$$\rho_i=\min_{\mu_1,\ldots,\mu_n>0}\{E_i\}\quad (i=1,\cdots,n), \tag{1.6}$$

with

$$E_i = (\mu_i)^{1+\sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)}} \cdot \frac{\left\{ \prod_{j=1}^{C} \left(\sum_{i=1}^{n} \mu_i \psi_j^{(i)} \right)^{\psi_j^{(i)}} \right\} \left\{ \prod_{j=1}^{D^{(i)}} \left(\delta_j^{(i)} \right)^{\delta_j^{(i)}} \right\}}{\left\{ \prod_{j=1}^{A} \left(\sum_{i=1}^{n} \mu_i \theta_j^{(i)} \right)^{\theta_j^{(i)}} \right\} \left\{ \prod_{j=1}^{B^{(i)}} \left(\phi_j^{(i)} \right)^{\phi_j^{(i)}} \right\}}. \tag{1.7}$$

Also, for interested researchers see [15] and [16, pp. 39-40]). We also required (see [14, Section 1.5])

$$\begin{aligned} {}_pF_q \left[\begin{matrix} \alpha_1, \ldots, \alpha_p\,; \\ \beta_1, \ldots, \beta_q\,; \end{matrix} z \right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z), \end{aligned} \tag{1.8}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [14, p. 2 and pp. 4-6]):

$$\begin{aligned} (\lambda)_n :&= \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \ldots\}) \end{cases} \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{aligned} \tag{1.9}$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers.
The Oberhettinger's integral formula [12]:

$$\int_0^\infty x^{\mu-1} \left(x+a+\sqrt{x^2+2ax} \right)^{-\lambda} dx = 2\lambda\, a^{-\lambda} \left(\frac{a}{2} \right)^{\mu} \frac{\Gamma(2\mu)\,\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)}, \tag{1.10}$$

provided $0 < \Re(\mu) < \Re(\lambda)$.

2 Unified Integrals Involving Generalized Bessel Functions

Theorem 2.1. *If* $x > 0$ *;* $\lambda, \mu, \nu_j, b, c \in \mathbb{C}$ with $\Re(\nu_j) > -1$, $0 < \Re(\mu) < \Re(\lambda + \nu_j)$ $(j = 1, \ldots, n)$. *Then there holds the following integral formula:*

$$\begin{aligned} &\int_0^\infty x^{\mu-1} \left(x+a+\sqrt{x^2+2ax} \right)^{-\lambda} \prod_{j=1}^{n} w_{\nu_j} \left(\frac{y_j}{x+a+\sqrt{x^2+2ax}} \right) dx \\ &= 2^{1-\mu} a^{\mu-\lambda} \left(\prod_{j=1}^{n} \frac{(\frac{y_j}{2a})^{\nu_j}}{\Gamma(\nu_j + \frac{1+b}{2})} \right) \frac{\Gamma(2\mu)\Gamma(1+\lambda+\sum_{j=1}^{n}\nu_j)\Gamma(\lambda-\mu+\sum_{j=1}^{n}\nu_j)}{\Gamma(\lambda+\sum_{j=1}^{n}\nu_j)\Gamma(1+\lambda+\mu+\sum_{j=1}^{n}\nu_j)} \\ &\cdot F_{2:1;\ldots;1}^{2:0;\ldots;0} \left[\begin{matrix} \left[1+\lambda+\sum_{j=1}^{n}\nu_j : 2,\ldots,2\right], \left[\lambda-\mu+\sum_{j=1}^{n}\nu_j : 2,\ldots,2\right] : \\ \left[1+\lambda+\mu+\sum_{j=1}^{n}\nu_j : 2,\ldots,2\right], \left[\lambda+\sum_{j=1}^{n}\nu_j : 2,\ldots,2\right] : \end{matrix} \right. \\ &\qquad \left. \begin{matrix} \text{———};\ldots;\text{———}; \\ [\nu_1 + \frac{1+b}{2} : 1];\ldots;[\nu_n + \frac{1+b}{2} : 1]; \end{matrix} -c\frac{y_1^2}{4a^2},\ldots,-c\frac{y_n^2}{4a^2} \right]. \end{aligned} \tag{2.1}$$

Theorem 2.2. *For* $\lambda, \mu, \nu_j, b, c \in \mathbb{C}$ with $\Re(\nu_j) > -1,\ 0 < \Re(\mu) < \Re(\lambda+\nu_j)$ $(j = 1,\ldots,n)$ and $x > 0$. Then there holds the following integral formula*:*

$$\begin{aligned}
&\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}\prod_{j=1}^n w_{\nu_j}\left(\frac{xy_j}{x+a+\sqrt{x^2+2ax}}\right)dx\\
&=2^{1-\mu}a^{\mu-\lambda}\left(\prod_{j=1}^n\frac{(\frac{y_j}{4})^{\nu_j}}{\Gamma(\nu_j+\frac{1+b}{2})}\right)\frac{\Gamma(\lambda-\mu)\Gamma(1+\lambda+\sum_{j=1}^n\nu_j)\Gamma(2\mu+2\sum_{j=1}^n\nu_j)}{\Gamma(\lambda+\sum_{j=1}^n\nu_j)\Gamma(1+\lambda+\mu+2\sum_{j=1}^n\nu_j)}\\
&\cdot F_{2:1;\ldots;1}^{2:0;\ldots;0}\left[\begin{matrix}\left[1+\lambda+\sum_{j=1}^n\nu_j:2,\ldots,2\right], & \left[2\mu+2\sum_{j=1}^n\nu_j:4,\ldots,4\right]:\\ \left[1+\lambda+\mu+2\sum_{j=1}^n\nu_j:4,\ldots,4\right], & \left[\lambda+\sum_{j=1}^n\nu_j:2,\ldots,2\right]:\end{matrix}\right.\\
&\qquad\left.\begin{matrix}\text{———};\ldots;\text{———};\\ [\nu_1+\frac{1+b}{2}:1];\ldots;[\nu_n+\frac{1+b}{2}:1];\end{matrix}\ -c\frac{y_1^2}{16},\ldots,-c\frac{y_n^2}{16}\right].
\end{aligned} \tag{2.2}$$

Proof. By applying (1.1) to the integrand of (2.1) and then interchanging the order of integral sign and summation, we get

$$\begin{aligned}
\mathscr{I}=&\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}\\
&\cdot\sum_{k_1=0}^\infty(-c)^{k_1}\frac{\left(\frac{y_1}{2(x+a+\sqrt{x^2+2ax})}\right)^{\nu_1+2k_1}}{k_1!\,\Gamma(\nu_1+k_1+\frac{1+b}{2})}\cdots\sum_{k_n=0}^\infty(-c)^{k_n}\frac{\left(\frac{y_n}{2(x+a+\sqrt{x^2+2ax})}\right)^{\nu_n+2k_n}}{k_n!\,\Gamma(\nu_n+k_n+\frac{1+b}{2})}dx
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{I}=&\sum_{k_1,\cdots,k_n=0}^\infty\frac{(-c)^{k_1}(y_1/2)^{\nu_1+2k_1}}{k_1!\Gamma(\nu_1+\frac{1+b}{2})(\nu_1+\frac{1+b}{2})_{k_1}}\cdots\frac{(-c)^{k_n}(y_n/2)^{\nu_n+2k_n}}{k_n!\Gamma(\nu_n+\frac{1+b}{2})(\nu_n+\frac{1+b}{2})_{k_n}}\\
&\cdot\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-\nu_1-\cdots-\nu_n-2k_1-\cdots-2k_n}dx.
\end{aligned} \tag{2.3}$$

In view of the conditions given in Theorem 2.1, since

$$\begin{gathered}
\Re(\nu_j)>-1,\ 0<\Re(\mu)<\Re(\lambda+\nu_j)\le\Re(\lambda+\nu_j+2k_j)\\
(k\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\quad\text{and}\quad j=1,\ldots,n),
\end{gathered}$$

we can apply the integral formula (1.10) to the integral in (2.3) and obtain the following expression:

$$\begin{aligned}
\mathscr{I}=&2^{1-\mu}a^{\mu-\lambda}\sum_{k_1,\ldots,k_n=0}^\infty\frac{(-c)^{k_1}(y_1/2)^{\nu_1+2k_1}}{k_1!\Gamma(\nu_1+\frac{1+b}{2})(\nu_1+\frac{1+b}{2})_{k_1}}\cdots\frac{(-c)^{k_n}(y_n/2)^{\nu_n+2k_n}}{k_n!\Gamma(\nu_n+\frac{1+b}{2})(\nu_n+\frac{1+b}{2})_{k_n}}\\
&\cdot\frac{\Gamma(2\mu)\Gamma(\lambda-\mu+\nu_1+\cdots+\nu_n+2k_1+\cdots+2k_n)}{\Gamma(1+\lambda+\mu+\nu_1+\cdots+\nu_n+2k_1+\cdots+2k_n)}\\
&\cdot(\lambda+\nu_1+\cdots+\nu_n+2k_1+\cdots+2k_n)a^{-(\nu_1+\cdots+\nu_n+2k_1+\cdots+2k_n)}.
\end{aligned}$$

And we have

$$\begin{aligned}
\mathscr{I}=&2^{1-\mu}a^{\mu-\lambda}\sum_{k_1,\cdots,k_n=0}^\infty\frac{(-c)^{k_1}(y_1/2)^{\nu_1+2k_1}}{k_1!\Gamma(\nu_1+\frac{1+b}{2})(\nu_1+\frac{1+b}{2})_{k_1}}\cdots\frac{(-c)^{k_n}(y_n/2)^{\nu_n+2k_n}}{k_n!\Gamma(\nu_n+\frac{1+b}{2})(\nu_n+\frac{1+b}{2})_{k_n}}\\
&\cdot\frac{\Gamma 2\mu)\Gamma(\lambda-\mu+\nu_1+\cdots+\nu_n+2k_1+\cdots+2k_n)}{\Gamma(1+\lambda+\mu+\nu_1+\cdots+\nu_n+2k_1+\cdots+2k_n)}\\
&\cdot(\lambda+\nu_1+\cdots+\nu_n+2k_1+\cdots+2k_n)\cdot a^{-(\nu_1+\cdots+\nu_n+2k_1+\cdots+2k_n)}.
\end{aligned}$$

Therefore we find that

$$\begin{aligned}\mathscr{I} =&2^{1-\mu}\, a^{\mu-\lambda}\Big(\prod_{j=1}^{n}\frac{(\frac{y_j}{2a})^{\nu_j}}{\Gamma(\nu_j+\frac{1+b}{2})}\Big)\frac{\Gamma(2\mu)\Gamma(\lambda-\mu+\sum_{j=1}^{n}\nu_j)\Gamma(1+\lambda+\sum_{j=1}^{n}\nu_j)}{\Gamma(1+\lambda+\mu+\sum_{j=1}^{n}\nu_j)\Gamma(\lambda+\sum_{j=1}^{n}\nu_j)}\\ &\cdot\sum_{k_1,\cdots,k_n=0}^{\infty}\frac{(\lambda-\mu+\sum_{j=1}^{n}\nu_j)_{2k_1+\cdots+2k_n}(1+\lambda+\sum_{j=1}^{n}\nu_j)_{2k_1+\cdots+2k_n}}{(\lambda+\sum_{j=1}^{n}\nu_j)_{2k_1+\cdots+2k_n}(1+\lambda+\mu+\sum_{j=1}^{n}\nu_j)_{2k_1+\cdots+2k_n}}\\ &\cdot\frac{1}{(\nu_1+\frac{1+b}{2})_{k_1}\cdots(\nu_n+\frac{1+b}{2})_{k_n}}\frac{(-cy_1^2/4a^2)^{k_1}}{k_1!}\cdots\frac{(-cy_n^2/4a^2)^{k_n}}{k_n!}.\end{aligned}\tag{2.4}$$

Finally, we interpret the multiple series in (2.4) as a special case of the general hypergeometric series in several variables defined by (1.2). We are thus conclude to the (2.1). The assertion (2.2) of the Theorem 2.2 can be proved by a similar argument. □

Remark 2.1. *If we set $n=1$ in* (2.1) *and* (2.2) *we can arrive at the (2.1) and (2.2)*et al. *[8] and for $c=b=1$ in* (2.1) *and* (2.2) *we can arrive at (2.1) and (2.2) in Choi and Agarwal [8].*

3 Special Cases

Here we give only two example. For our purpose the particular cases of the function $w_{\nu,b,c}(z)$ given by (1.1), is worthy to mention here.

For $b=c=1$ in (1.1), we obtain the familiar Bessel function $J_\nu(z)$ defined by (see [3])

$$J_\nu(z)=\sum_{n\geq 0}\frac{(-1)^n\left(\frac{z}{2}\right)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)},\quad (z\in\mathbb{C}).\tag{3.1}$$

For $b=-c=1$ in (1.1), we obtain the modified Bessel function $I_\nu(z)$ defined by (see [3])

$$I_\nu(z)=\sum_{n\geq 0}\frac{1\left(\frac{z}{2}\right)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)},\quad (z\in\mathbb{C}).\tag{3.2}$$

For $b-1=c=1$ in (1.1), we obtain the spherical Bessel function $K_\nu(z)$ defined by

$$K_\nu(z)=\sum_{n\geq 0}\frac{(-1)^n\left(\frac{z}{2}\right)^{\nu+2n}}{n!\,\Gamma\left(\nu+n+\frac{3}{2}\right)},\quad (z\in\mathbb{C}).\tag{3.3}$$

We now present the new unified integrals in terms of the spherical Bessel function $K_\nu(z)$ by setting $b=2$ and $c=1$ in (2.1) and (2.2), we obtain two further pairs of integral formulae, reads as follows.

Corollary 3.1. *Let the condition of Theorem 2.2 be satisfied. Then the following integral formula holds true.*

$$\begin{aligned}&\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}\prod_{j=1}^{n}K_{\nu_j}\left(\frac{y_j}{x+a+\sqrt{x^2+2ax}}\right)dx\\ &=2^{1-\mu}\,a^{\mu-\lambda}\left(\prod_{j=1}^{n}\frac{(\frac{y_j}{2a})^{\nu_j}}{\Gamma(\nu_j+\frac{1+b}{2})}\right)\frac{\Gamma(2\mu)\Gamma(1+\lambda+\sum_{j=1}^{n}\nu_j)\Gamma(\lambda-\mu+\sum_{j=1}^{n}\nu_j)}{\Gamma(\lambda+\sum_{j=1}^{n}\nu_j)\Gamma(1+\lambda+\mu+\sum_{j=1}^{n}\nu_j)}\\ &\cdot F^{2:0;\ldots;0}_{2:1;\ldots;1}\left[\begin{array}{l}\left[1+\lambda+\sum_{j=1}^{n}\nu_j:2,\ldots,2\right],\left[\lambda-\mu+\sum_{j=1}^{n}\nu_j:2,\ldots,2\right]:\\ \left[1+\lambda+\mu+\sum_{j=1}^{n}\nu_j:2,\ldots,2\right],\left[\lambda+\sum_{j=1}^{n}\nu_j:2,\ldots,2\right]:\end{array}\right.\\ &\qquad\qquad\left.\begin{array}{l}\overline{\qquad\qquad};\ldots;\overline{\qquad\qquad};\\ [\nu_1+\frac{3}{2}:1];\ldots;[\nu_n+\frac{3}{2}:1];\end{array}-\frac{y_1^2}{4a^2},\ldots,-\frac{y_n^2}{4a^2}\right].\end{aligned}\tag{3.4}$$

Corollary 3.2. *Let the condition of Theorem 2.2 be satisfied. Then the following integral formula holds true.*

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}\prod_{j=1}^{n}K_{\nu_j}\left(\frac{xy_j}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$=2^{1-\mu}a^{\mu-\lambda}\left(\prod_{j=1}^{n}\frac{(\frac{y_j}{4})^{\nu_j}}{\Gamma(\nu_j+\frac{1+b}{2})}\right)\frac{\Gamma(\lambda-\mu)\Gamma(1+\lambda+\sum_{j=1}^{n}\nu_j)\Gamma(2\mu+2\sum_{j=1}^{n}\nu_j)}{\Gamma(\lambda+\sum_{j=1}^{n}\nu_j)\Gamma(1+\lambda+\mu+2\sum_{j=1}^{n}\nu_j)}$$
$$\cdot F^{2:0;\ldots;0}_{2:1;\ldots;1}\left[\begin{array}{l}\left[1+\lambda+\sum_{j=1}^{n}\nu_j:2,\ldots,2\right],\quad\left[2\mu+2\sum_{j=1}^{n}\nu_j:4,\ldots,4\right]:\\ \left[1+\lambda+\mu+2\sum_{j=1}^{n}\nu_j:4,\ldots,4\right],\quad\left[\lambda+\sum_{j=1}^{n}\nu_j:2,\ldots,2\right]:\end{array}\right.$$
$$\left.\begin{array}{l}\text{———};\ldots;\text{———};\\ [\nu_1+\frac{3}{2}:1];\ldots;[\nu_n+\frac{3}{2}:1];\end{array}\;-\frac{y_1^2}{16},\ldots,-\frac{y_n^2}{16}\right] \tag{3.5}$$

4 Concluding Remarks

The results provided in this paper are easily converted in terms of the various Bessel functions after suitable parametric replacements, and further, trigonometric functions, hyperbolic functions and exponential function. The explicit details of those special cases are left to the interested reader. We would like to emphasize that the results derived in this paper may be potentially useful due mainly to the demonstrated production of a variety of specialized integral formulas, in particular, associated with such elementary functions as trigonometric functions, hyperbolic functions and exponential function as well as various Bessel functions.

Acknowledgements

The authors thank referee and editor for their useful technical comments and valuable suggestions to improve the readability of the paper, which led to a significant improvement of the paper.

References

[1] P. Agarwal, A Study of New Trends and Analysis of Special Function, LAP LAMBERT Academic Publishing, (2013).

[2] W. N. Bailey, Some infinite integrals involving Bessel functions, Proc. London Math. Soc, 40 (2) (1936) 37-48. *http://dx.doi.org/10.1112/plms/s2-40.1.37*

[3] Á. Baricz, Generalized Bessel Functions of the First Kind, Springer-Verlag Berlin, Heidelberg, (2010). *http://dx.doi.org/10.1007/978-3-642-12230-9*

[4] Á. Baricz, Geometric properties of generalized Bessel functions of complex order, Mathematica, 48 (71) (1) (2006) 13-18.

[5] Á. Baricz, Geometric properties of generalized Bessel functions, Publ. Math. Debrecen, 73 (1-2) (2008) 155-178.

[6] Á. Baricz, Jorden-Type Inequalities for Generalized Bessel Functions, J. Inequal. Pure and Appl. Math, Article 39, 9 (2) (2008).

[7] Y. A. Brychkov, Handbook of Special Functions, Derivatives, Integrals, Series and Other Formulas, CRC Press, Taylor & Francis Group, Boca Raton, London, and New York, (2008).

[8] J. Choi, P. Agarwal, Certain unified integrals associated with Bessel functions, appear in Boundary Value Problems, (2013).
http://dx.doi.org/10.1186/1687-2770-2013-95

[9] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher Transcendental Functions, McGraw-Hill Book Company, New York, Toronto, and London, II (1953).

[10] A. Gray, G. B. Mathews, A treatise on Bessel functions and their applications to physics. 2nd ed, prepared by A. Gray and T.M. MacRobert. London: Macmillan; (1922).

[11] Y. L. Luke, Integrals of Bessel functions, NewYork: McGraw-Hill; (1962).

[12] F. Oberhettinger, Tables of Mellin Transforms, Springer-Verlag, New York, (1974).
http://dx.doi.org/10.1007/978-3-642-65975-1

[13] S. O. Rice, On contour integrals for the product of two Bessel functions, Quart. J. Math. Oxford Ser. 6 (1935) 52-64.
http://dx.doi.org/10.1093/qmath/os-6.1.52

[14] H. M. Srivastava, J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York, (2012).

[15] H. M. Srivastava, M. C. Daoust, A note on the convergence of Kampé de Fériet's double hypergeometric series, Math. Nachr, 53 (1985) 151-159.
http://dx.doi.org/10.1002/mana.19720530114

[16] H. M. Srivastava, P. W. Karlsson, Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, (1985).

[17] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge Mathematical Library edition, Camdridge University Press, 1995, Reprinted 1996.