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Research Article

First general solutions for unidirectional motions of rate type fluids over an infinite plate

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Abstract
Based on a simple but important remark regarding the governing equation for the non-trivial shear stress corresponding to the motion of a fluid over an infinite plate, exact solutions are established for the motion of Oldroyd-B fluids due to the plate that applies an arbitrary time-dependent shear stress to the fluid. These solutions, that allow us to provide the first exact solutions for motions of rate type fluids produced by an infinite plate that applies constant, constantly accelerating or oscillating shears stresses to the fluid, can easily be reduced to the similar solutions for Maxwell, second grade or Newtonian fluids performing the same motion. Furthermore, the obtained solutions are used to develop general solutions for the motion induced by a moving plate and to correct or recover as special cases different known results from the existing literature. Consequently, the motion problem of such fluids over an infinite plate that is moving in its plane or applies a shear stress to the fluid is completely solved.

Keywords: Oldroyd-B fluids, Infinite plate, Shear stress on the boundary, General solutions.

1 Introduction

The motion of a fluid over an infinite plate has been extensively studied due to its theoretical and practical importance. Such a motion can be generated by the flat plate that is moving in its plane with a prescribed velocity or applies a shear stress to the fluid. In the first case, exact solutions for the fluid motion due to the impulsive motion of the plate or induced by a constantly accelerating plate are well-known [1, 2] while correct starting solutions corresponding to the motion of Oldroyd-B fluids over an oscillating infinite plate are lacking in the literature.

Furthermore, exact solutions corresponding to motions of rate type fluids generated by an infinite plate that applies constant, constantly accelerating or oscillatory shear stresses to the fluid are also absent in the existing literature. This is due to the governing equations that contain partial derivatives or differential

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expressions acting on the non-trivial shear stress. Nevertheless in some physical situations, contrary to what is usually assumed, the force with which the plate is moved can be prescribed. To reiterate, in Newtonian mechanics force is the cause and kinematics is the effect (see Rajagopal [3] for a detailed discussion on the same). Prescribing the shear stress at the plate is tantamount to prescribe the (shear) force applied to the plate and Renardy [4] showed how a well-posed shear stress boundary-value problem can be formulated in this case.

The main purpose of this work is to provide exact solutions for the motion of rate type fluids induced by an infinite plate that applies an arbitrary time-dependent shear stress to the fluid. These solutions can easy be particularized to recover known solutions for second grade and Newtonian fluids performing the same motion. Moreover, they can also be used to develop general solutions for the motion of the same fluids produced by an infinite plate that is moving in its plane with an arbitrary time-dependent velocity. In order to illustrate the theoretical and practical value of general solutions that have been obtained, as well as for a check of results, three special cases are considered and some known results from the literature are recovered or corrected. Consequently, the motion problem of fluids of type Oldroyd-B, Maxwell, second grade or Newtonian over an infinite plate can be considered as being completely solved since any solution with technical relevance regarding motions of such fluids can be obtained as a special case of the present general solutions.

2 Statement of the problem

Consider an incompressible Oldroyd-B fluid at rest over an infinite flat plate situated in the (x,z) plane of a fixed Cartesian coordinate system *x*, *y* and *z*. After the time $t=0$, the plate applies a time-dependent shear stress $\tau_0 f(t)$ to the fluid in the *x*-direction. Here, τ_0 is a constant shear stress while $f(\cdot)$ is a piecewise continuous function of exponential order for $t>0$ whose value $f(0)=0$. Owing to the shear the fluid is gradually moved and its velocity is of the form

$$\mathbf{v}=\mathbf{v}(y,t)=u(y,t)\mathbf{i}, \tag{2.1}$$

where **i** is the unit vector along the *x*-direction. For such a motion, the constraint of incompressibility is identically satisfied. In the following we shall assume that the extra-stress tensor, as well as the fluid velocity, is a function of *y* and *t* only.

By neglecting the body forces and in the absence of a pressure gradient in the flow direction, the constitutive equations of Oldroyd-B fluids and the motion equations lead to the next two relevant equations (see [2, Eqs. (3.6) and (3.7)], for instance)

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau(y,t)=\mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial y},\quad \frac{\partial \tau(y,t)}{\partial y}=\rho\frac{\partial u(y,t)}{\partial t};\ \text{for}\quad y,t>0, \tag{2.2}$$

where μ is the viscosity of the fluid, ρ is its density, λ and $\lambda_r\ (<\lambda)$ are relaxation and retardation times [5], and $\tau(y,t)$ is the non-trivial shear stress. Usually, the governing equation for velocity is obtained by eliminating the shear stress $\tau(y,t)$ between Eqs. (2.2).

However, in order to solve a well-posed shear stress boundary-value problem for a rate type fluid, contrary to what is usually done we follow [6] and eliminate the velocity $u(y,t)$ between Eqs. (2.2). The surprising result, namely

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial \tau(y,t)}{\partial t}=\nu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial^2 \tau(y,t)}{\partial y^2};\quad y,t>0, \tag{2.3}$$

where $\nu=\mu/\rho$ is the kinematic viscosity of the fluid, shows that for such motions of Oldroyd-B fluids the nontrivial shear stress $\tau(y,t)$ satisfies a partial differential equation of the same form as the fluid velocity $u(y,t)$ [6]. This simple but important result allows us to solve motion problems of these fluids with a given shear stress on the boundary and to develop new exact solutions for usual boundary value problems. In the following the system of partial differential equations (2.3) and $(2.2)_2$ together with the appropriate initial and boundary conditions

$$u(y,0)=0, \quad \tau(y,0)=0, \quad \left.\frac{\partial\tau(y,t)}{\partial t}\right|_{t=0}=0 \ \text{ for } \ y>0; \quad \tau(0,t)=\tau_0 f(t) \ \text{ for } \ t\geq 0, \tag{2.4}$$

will be solved using the Laplace and Fourier sine transforms. The natural conditions at infinity, namely

$$u(y,t), \quad \tau(y,t)\to 0, \quad \text{as} \quad y\to\infty, \tag{2.5}$$

have to be also satisfied.

3 Dimensionless solution of the problem

Introducing the following non-dimensional variables and functions

$$y^*=\frac{y}{\sqrt{\nu t_0}},\ t^*=\frac{t}{t_0},\ \lambda^*=\frac{\lambda}{t_0},\ \lambda_r^*=\frac{\lambda_r}{t_0},\ \tau^*=\frac{\tau}{\tau_0},\ u^*=\frac{u}{\sqrt{\nu/t_0}},\ f^*(t^*)=f(t_0t^*),\ \rho^*=\frac{\mu}{t_0\tau_0}, \tag{3.6}$$

where t_0 is a characteristic time and dropping out the star notation, we attain to the next dimensionless initial and boundary-value problem

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial\tau(y,t)}{\partial t}=\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial^2\tau(y,t)}{\partial y^2}; \quad y,t>0, \tag{3.7}$$

$$\tau(y,0)=0, \quad \left.\frac{\partial\tau(y,t)}{\partial t}\right|_{t=0}=0 \ \text{ for } \ y>0; \quad \tau(0,t)=f(t) \ \text{ for } \ t\geq 0 \ \text{ and } \ \tau(y,t)\to 0 \ \text{ as } \ y\to\infty, \tag{3.8}$$

for the shear stress $\tau(y,t)$. The non-dimensional form of Eq. $(2.2)_2$ is the same.

Applying the Laplace transform to Eq. (3.7) and bearing in mind the initial and boundary conditions (3.8), we find that

$$(1+\lambda q)qT(y,q)=(1+\lambda_r q)\frac{\partial^2 T(y,q)}{\partial y^2} \ \text{ for } y>0;\ T(0,q)=F(q) \text{ and } T(y,q)\to 0 \text{ as } y\to\infty, \tag{3.9}$$

where $T(y,q)$ and $F(q)$ are the Laplace transforms of the functions $\tau(y,t)$, respectively $f(t)$ and q is the transform parameter. Now, we apply the Fourier sine transform [7] to Eq. $(3.9)_1$, use the boundary conditions (3.9), and obtain

$$T_s(\xi,q)=\xi\sqrt{\frac{2}{\pi}}\frac{\lambda_r q+1}{\lambda q^2+(1+\lambda_r\xi^2)q+\xi^2}F(q), \tag{3.10}$$

where $T_s(\xi,q)$ is the Fourier sine transform of $T(y,q)$.

In order to come back and determine the Fourier sine transform $\tau_s(\xi,t)$ of $\tau(y,t)$, we firstly write $T_s(\xi,q)$ in the suitable form

$$T_s(\xi,q)=\xi\sqrt{\frac{2}{\pi}}\frac{\lambda_r}{\lambda}\frac{q+a(\xi)}{[q+a(\xi)]^2-[b(\xi)]^2}F(q)+\xi\sqrt{\frac{2}{\pi}}\frac{1-\lambda_r a(\xi)}{\lambda b(\xi)}\frac{b(\xi)}{[q+a(\xi)]^2-[b(\xi)]^2}F(q), \tag{3.11}$$

where $a(\xi)=\dfrac{1+\lambda_r\xi^2}{2\lambda}$ and $b(\xi)=\dfrac{\sqrt{(1+\lambda_r\xi^2)^2-4\lambda\xi^2}}{2\lambda}$.

By applying the inverse Laplace transform to Eq. (3.11) and using the convolution theorem, we find that

$$\tau_s(\xi,t)=\xi\sqrt{\frac{2}{\pi}}\frac{\lambda_r}{\lambda}\int_0^t f(t-s)\mathrm{ch}[b(\xi)s]\mathrm{e}^{-a(\xi)s}ds+\xi\sqrt{\frac{2}{\pi}}\frac{1-\lambda_r a(\xi)}{\lambda b(\xi)}\int_0^t f(t-s)\mathrm{sh}[b(\xi)s]\mathrm{e}^{-a(\xi)s}\mathrm{d}s. \quad (3.12)$$

Finally, applying the inverse Fourier sine transform to Eq. (3.12), it results that

$$\tau(y,t)=\frac{2}{\pi}\frac{\lambda_r}{\lambda}\int_0^\infty \xi\sin(y\xi)\int_0^t f(t-s)\mathrm{ch}[b(\xi)s]\mathrm{e}^{-a(\xi)s}\mathrm{d}s\mathrm{d}\xi$$
$$+\frac{2}{\lambda\pi}\int_0^\infty \frac{1-\lambda_r a(\xi)}{b(\xi)}\xi\sin(y\xi)\int_0^t f(t-s)\mathrm{sh}[b(\xi)s]\mathrm{e}^{-a(\xi)s}\mathrm{d}s\mathrm{d}\xi. \quad (3.13)$$

The shear stress $\tau(y,t)$, given by Eq. (3.13), clearly satisfies the initial conditions (3.8). However, in this form, the boundary condition (3.8) seems to be dissatisfied. In order to remove this drawback, we integrate by parts the two integrals from Eq. (3.13) with respect to *s*. Lengthy, but straightforward computations show that shear stress $\tau(y,t)$ can be presented in the simpler but equivalent form

$$\tau(y,t)=f(t)-\frac{2}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi}\int_0^t f'(t-s)\left\{\mathrm{ch}[b(\xi)s]+\frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)s]\right\}\mathrm{e}^{-a(\xi)s}\mathrm{d}s\mathrm{d}\xi, \quad (3.14)$$

that clearly satisfies the boundary condition $(3.8)_3$.
The corresponding expression for the non-dimensional velocity, namely

$$u(y,t)=-\frac{2}{\rho\pi}\int_0^\infty \cos(y\xi)\int_0^t\int_0^s f'(s-\sigma)\left\{\mathrm{ch}[b(\xi)\sigma]+\frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)\sigma]\right\}\mathrm{e}^{-a(\xi)\sigma}\mathrm{d}\sigma\mathrm{d}s\mathrm{d}\xi, \quad (3.15)$$

is obtained introducing Eq. (3.14) in the dimensionless form of Eq. $(2.2)_2$, integrating the result with respect to time from 0 to *t* and bearing in mind both the corresponding initial condition from Eqs. (2.4) and the natural condition (2.5) at infinity. An equivalent form for velocity, namely

$$u(y,t)=\frac{2}{\rho\pi\lambda}\int_0^\infty \xi^2\cos(y\xi)\int_0^t\int_0^s f(s-\sigma)\left\{\lambda_r\mathrm{ch}[b(\xi)\sigma]+\frac{1-\lambda_r a(\xi)}{b(\xi)}\mathrm{sh}[b(\xi)\sigma]\right\}\mathrm{e}^{-a(\xi)\sigma}\mathrm{d}\sigma\mathrm{d}s\mathrm{d}\xi, \quad (3.16)$$

can be also obtained applying the Laplace transform to Eq. (3.14), introducing the result into the equality (obtained from Eq. $(2.2)_2$ by applying the Laplace transform)

$$\rho\bar{u}(y,q)=\frac{1}{q}\frac{\partial T(y,q)}{\partial y};\quad y>0, \quad (3.17)$$

where $\bar{u}(y,q)$ is the Laplace transform of $u(y,t)$ and then applying the inverse Laplace transform. Lengthy but direct computations clearly show that the two expressions of velocity, given by Eqs. (3.15) and (3.16), are equivalent.

The solutions (3.13), (3.14), (3.15) and (3.16) give the dimensionless shear stress and velocity distributions corresponding to the motion of an Oldroyd-B fluid due to an infinite plate that applies an arbitrary time-dependent shear stress to the fluid. They allow us to provide exact solutions for any motion with technical relevance of these fluids. Such exact solutions are lack in the literature for motions of rate type fluids over an infinite plate and their value for theory and practice can be significant. The solutions corresponding to the motion induced by an infinite plate that applies a constant or an oscillating shear stress to the fluid, for instance, are obtained taking $f(t)=H(t)$, respectively $f(t)=H(t)\sin(\omega t)$ or $f(t)=H(t)\cos(\omega t)$ into anyone of the above relations. The corresponding velocity fields, as they result from Eq. (3.15), are

$$u(y,t)=-\frac{2}{\rho\pi}\int_0^\infty \cos(y\xi)\int_0^t\left\{\mathrm{ch}[b(\xi)s]+\frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)s]\right\}\mathrm{e}^{-a(\xi)s}\mathrm{d}s\mathrm{d}\xi, \tag{3.18}$$

$$u_s(y,t)=-\frac{2\omega}{\rho\pi}\int_0^\infty \cos(y\xi)\int_0^t\int_0^s \cos[\omega(s-\sigma)]\left\{\mathrm{ch}[b(\xi)\sigma]+\frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)\sigma]\right\}\mathrm{e}^{-a(\xi)\sigma}\mathrm{d}\sigma\mathrm{d}s\mathrm{d}\xi, \tag{3.19}$$

respectively,

$$\begin{aligned}u_c(y,t)=&-\frac{2}{\rho\pi}\int_0^\infty \cos(y\xi)\int_0^t\left\{\mathrm{ch}[b(\xi)s]+\frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)s]\right\}\mathrm{e}^{-a(\xi)s}\mathrm{d}s\mathrm{d}\xi\\ &+\frac{2\omega}{\rho\pi}\int_0^\infty \cos(y\xi)\int_0^t\int_0^s \sin[\omega(s-\sigma)]\left\{\mathrm{ch}[b(\xi)\sigma]+\frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)\sigma]\right\}\mathrm{e}^{-a(\xi)\sigma}\mathrm{d}\sigma\mathrm{d}s\mathrm{d}\xi.\end{aligned} \tag{3.20}$$

In order to obtain Eqs. (3.18)-(3.20) we have used the known result $H'(t)=\delta(t)$, where $\delta(\cdot)$ is the Dirac delta function, and the identity

$$(\delta*h)(t)=\int_0^t \delta(t-s)\,h(s)\,\mathrm{d}s=\int_0^t \delta(s)\,h(t-s)\mathrm{d}s=h(t). \tag{3.21}$$

Of course, Eqs. (3.18)-(3.20) can be further processed but we leave this problem to the reader. Furthermore, as expected, Eq. (3.18) can be obtained as a limiting case of Eq. (3.20) for $\omega=0$.

It is worth pointing out that the present solutions (3.18)-(3.20) represent the first exact solutions corresponding to motions of rate type fluids induced by an infinite plate that applies constant or oscillating shear stresses to the fluid. Indeed, this is true since the solutions of Tong *et al*. [8] and Fetecau and Kannan [9] as well as those of Nazish *et al*. [10] and Sohail *et al*. [11] do not correspond to a constant, respectively oscillating shear stress on the boundary.

Now, it is important to notice that the solutions for Maxwell, second grade and Newtonian fluids performing the same motion are immediately obtained as limiting cases of Eqs. (3.14) and (3.15). Indeed, by making $\lambda_r=0$ into these equalities, the solutions corresponding to Maxwell fluids are obtained. The equality (3.14), for instance, becomes

$$\tau_M(y,t)=f(t)-\frac{2}{\pi}\int_0^\infty\frac{\sin(y\xi)}{\xi}\int_0^t f'(t-s)\left\{\mathrm{ch}[c(\xi)s]+\frac{1}{2\lambda c(\xi)}\mathrm{sh}[c(\xi)s]\right\}\mathrm{e}^{-s/2\lambda}\mathrm{d}s\mathrm{d}\xi, \tag{3.22}$$

where $c(\xi)=\sqrt{1-4\lambda\xi^2}/(2\lambda)$.

By now letting $\lambda\to 0$ into Eqs. (3.14) and (3.15), the solutions

$$\tau_{SG}(y,t)=f(t)-\frac{2}{\pi}\int_0^\infty\frac{\sin(y\xi)}{\xi(1+\lambda_r\xi^2)}\int_0^t f'(t-s)\exp\left(-\frac{\xi^2 s}{1+\lambda_r\xi^2}\right)\mathrm{d}s\mathrm{d}\xi, \tag{3.23}$$

$$u_{SG}(y,t)=-\frac{2}{\rho\pi}\int_0^\infty\frac{\cos(y\xi)}{1+\lambda_r\xi^2}\int_0^t\int_0^s f'(s-\sigma)\exp\left(-\frac{\xi^2\sigma}{1+\lambda_r\xi^2}\right)\mathrm{d}\sigma\mathrm{d}s\mathrm{d}\xi, \tag{3.24}$$

corresponding to second grade fluids are obtained. Simple computations show that Eq. (3.23) is the non-dimensional form of Eq. (3.20) obtained in [12] by a different technique. The solutions corresponding to Newtonian fluids are obtained by making $\lambda_r=0$ into Eqs. (3.23) and (3.24).

As a check of results, let us take $f(t)=H(t)$ into Eqs. (3.23) and (3.24). Direct computations clearly show that these solutions reduce to

$$\tau_{SG}(y,t)=H(t)\left[1-\frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda_r\xi^2}\exp\left(-\frac{\xi^2 t}{1+\lambda_r\xi^2}\right)\frac{\sin(y\xi)}{\xi}\mathrm{d}\xi\right], \tag{3.25}$$

$$u_{SG}(y,t)=\frac{y}{\rho}-\frac{2}{\rho\pi}\int_0^\infty \frac{1}{\xi^2}\left\{1-\exp\left(-\frac{\xi^2 t}{1+\lambda_r\xi^2}\right)\cos(y\xi)\right\}\mathrm{d}\xi, \tag{3.26}$$

which are the dimensionless forms of Eqs. (4.1) that have been obtained by Fetecau and Kannan [9] using a different technique.

4 Applications

The general solutions (3.13), (3.14), (3.15) and (3.16) for Oldroyd-B fluids, as well as those of Maxwell fluids obtained for $\lambda_r=0$, are new in the literature. In the following we shall use Eq. (3.14) in order to develop general solutions for the motion of the same fluids due to an infinite plate that is moving in its plane with an arbitrary time-dependent velocity. In order to do that, let us consider again an Oldroyd-B fluid at rest over an infinite flat plate. After time $t=0$ the plate is moving in its plane along the x - axis with the velocity $U_0 g(t)$ where U_0 is a constant velocity and the function $g(\cdot)$ satisfies the same properties as the previous function $f(\cdot)$. Due to shear the fluid above the plate begins to move. Its velocity is given by Eq. (2.1) and the constitutive and motion equations again reduce to Eqs. (2.2). Eliminating $\tau(y,t)$ instead of $u(y,t)$ between these equations, as usually, we recover the governing equation for velocity, namely

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t}=\nu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial^2 u(y,t)}{\partial y^2};\quad y,t>0. \tag{4.27}$$

This equation is identical as form to Eq. (2.3) that is satisfied by the shear stress. Its dimensionless form and the corresponding initial and boundary conditions are also identical to those from Eqs. (3.7) and (3.8) with $u(y,t)$ and $g(t)$ instead of $\tau(y,t)$ and $f(t)$.

Consequently, the dimensionless velocity field $u(y,t)$ corresponding to such a motion, as it results from Eq. (3.14), is

$$u(y,t)=g(t)-\frac{2}{\pi}\int_0^\infty\frac{\sin(y\xi)}{\xi}\int_0^t g'(t-s)\left\{\mathrm{ch}[b(\xi)s]+\frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)s]\right\}\mathrm{e}^{-a(\xi)s}\mathrm{d}s\mathrm{d}\xi. \tag{4.28}$$

The solution (4.28) is also new in the literature and the corresponding shear stress can easy be obtained using the dimensionless form of Eq. $(2.2)_1$. The solutions corresponding to Maxwell, second grade and Newtonian fluids can be immediately obtained as limiting cases of these solutions. In order to bring to light the theoretical importance of general solution (4.28), let us consider some motions with technical relevance of Oldroyd-B fluids.

4.1. Flow due to the impulsive motion of the plate (Stokes' first problem)

In this case taking $g(t)=H(t)$ into Eq. (4.28) and again using the property (3.21), we obtain the dimensionless form of a known result, namely (see [13], Eq. (3.6) with $\beta=0$)

$$u(y,t)=H(t)\left\{1-\frac{2}{\pi}\int_0^\infty\frac{\sin(y\xi)}{\xi}\left[\mathrm{ch}[b(\xi)t]+\frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)t]\right]\mathrm{e}^{-a(\xi)t}\mathrm{d}\xi\right\}. \tag{4.29}$$

By making λ_r or $\lambda \to 0$ into Eq. (4.29), the solutions corresponding to Maxwell, respectively second grade fluids are again obtained. The solution for second grade fluids, for example, is (see [1, Eq. (3.14)] for its dimensional form)

$$u_{SG}(y,t) = H(t)\left[1 - \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda_r\xi^2}\exp\left(-\frac{\xi^2 t}{1+\lambda_r\xi^2}\right)\frac{\sin(y\xi)}{\xi}\mathrm{d}\xi\right]. \tag{4.30}$$

4.2. Flow caused by a variably accelerating plate

By now letting $g(t) = t^\alpha$ $(\alpha > 0)$ into Eq. (4.28), we find the velocity field

$$u(y,t) = t^\alpha - \frac{2\alpha}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi}\int_0^t (t-s)^{\alpha-1}\left\{\mathrm{ch}[b(\xi)s] + \frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)s]\right\}\mathrm{e}^{-a(\xi)s}\mathrm{d}s\mathrm{d}\xi\ , \tag{4.31}$$

corresponding to the motion induced by a variable accelerating plate. Of special interest is the case $\alpha = 1$ corresponding to the motion induced by a constantly accelerating plate. In this case we recover the dimensionless form

$$u(y,t) = t - \frac{2}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi^3}d\xi + \frac{2}{\pi}\int_0^\infty \left\{\mathrm{ch}[b(\xi)t] + \frac{1+(\lambda_r - 2\lambda)\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)t]\right\}\frac{\sin(y\xi)}{\xi^3}\mathrm{e}^{-a(\xi)t}\mathrm{d}\xi, \tag{4.32}$$

of the equality (18) from [2]. A simple analysis clearly shows that $u(y,t)$ given by Eqs. (4.31) and (4.32) satisfies all imposed initial and boundary conditions.

4.3. Motion produced by an oscillating plate (Stokes' second problem)

Let us now assume that the infinite plate oscillates according to

$$\mathbf{v}(0,t) = H(t)\sin(\omega t)\mathbf{i} \quad \text{or} \quad \mathbf{v}(0,t) = H(t)\cos(\omega t)\mathbf{i}. \tag{4.33}$$

By considering $g(t) = H(t)\sin(\omega t)$ into Eq. (4.28) we obtain the velocity field

$$\begin{aligned} u_s(y,t) &= H(t)\sin(\omega t) \\ &- \frac{2}{\pi}\omega H(t)\int_0^\infty \frac{\sin(y\xi)}{\xi}\int_0^t \cos[\omega(t-s)]\left[\mathrm{ch}[b(\xi)s] + \frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)s]\right]\mathrm{e}^{-a(\xi)s}\mathrm{d}s\mathrm{d}\xi, \end{aligned} \tag{4.34}$$

corresponding to the second problem of Stokes for Oldroyd-B fluids. If $\lambda \to 0$, Eq. (4.34) takes the simplified form

$$u_{sSG}(y,t) = H(t)\left\{\sin(\omega t) - \frac{2}{\pi}\omega\int_0^\infty \frac{\sin(y\xi)}{\xi(1+\lambda_r\xi^2)}\int_0^t \cos[\omega(t-s)]\exp\left(-\frac{\xi^2 s}{1+\lambda_r\xi^2}\right)\mathrm{d}s\mathrm{d}\xi\right\}, \tag{4.35}$$

corresponding to second grade fluids. By evaluating the second integral, we recover the non-dimensional form

$$\begin{aligned} u_{sSG}(y,t) = H(t)\Bigg\{&\sin(\omega t) - \frac{2}{\pi}\omega\cos(\omega t)\int_0^\infty \frac{\xi\sin(y\xi)}{\xi^4 + \omega^2(1+\lambda_r\xi^2)^2}d\xi \\ &- \frac{2}{\pi}\omega^2\sin(\omega t)\int_0^\infty \frac{(1+\lambda_r\xi^2)\sin(y\xi)}{\xi[\xi^4+\omega^2(1+\lambda_r\xi^2)^2]}d\xi + \frac{2}{\pi}\omega\int_0^\infty \frac{\xi\sin(y\xi)}{\xi^4+\omega^2(1+\lambda_r\xi^2)^2}\exp\left(-\frac{\xi^2 t}{1+\lambda_r\xi^2}\right)\mathrm{d}\xi\Bigg\}, \end{aligned} \tag{4.36}$$

of Eq. (3.9) from [14]. Of course, in view of the entry 1 of Table 5 from [7], this result is equivalent to that obtained by Christov and Jordan [15, Eq. (3.8)].

As regards the solution corresponding to the cosine oscillations of the plate, namely

$$u_c(y,t)=H(t)\left\{\cos(\omega t)-\frac{2}{\pi}\int_0^\infty\frac{\sin(y\xi)}{\xi}\left[\mathrm{ch}[b(\xi)t]+\frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)t]\right]\mathrm{e}^{-a(\xi)t}\mathrm{d}\xi\right.$$
$$\left.+\frac{2}{\pi}\omega\int_0^\infty\frac{\sin(y\xi)}{\xi}\int_0^t\sin[\omega(t-s)]\left[\mathrm{ch}[b(\xi)s]+\frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)s]\right]\mathrm{e}^{-a(\xi)s}\mathrm{d}s\mathrm{d}\xi\right\}, \tag{4.37}$$

it corrects the result obtained in [16, Eq. (3.10)]. By making $\lambda\to 0$ into Eq. (4.37), we recover the dimensionless form of Eq. (2.5) from [15]. Furthermore, as it was to be expected, Eq. (4.37) reduces to the solution corresponding to the first problem of Stokes given by Eq. (4.29) if $\omega=0$.
The starting solutions (4.34) and (4.37) corresponding to Oldroyd-B fluids, as well as the solution (4.35) for second grade fluids, can be written as a sum of permanent and transient solutions. The permanent solutions corresponding to Oldroyd-B fluids, namely

$$u_{sp}(y,t)=\sin(\omega t)$$
$$-\frac{2}{\pi}\omega\int_0^\infty\frac{\sin(y\,\xi)}{\xi}\int_0^\infty\cos[\omega(t-s)]\left[\mathrm{ch}[b(\xi)s]+\frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)s]\right]\mathrm{e}^{-a(\xi)s}\mathrm{d}s\mathrm{d}\xi, \tag{4.38}$$

$$u_{cp}(y,t)=\cos(\omega t)$$
$$+\frac{2}{\pi}\omega\int_0^\infty\frac{\sin(y\,\xi)}{\xi}\int_0^\infty\sin[\omega(t-s)]\left[\mathrm{ch}[b(\xi)s]+\frac{1-\lambda_r\xi^2}{2\lambda b(\xi)}\mathrm{sh}[b(\xi)s]\right]\mathrm{e}^{-a(\xi)s}\mathrm{d}s\mathrm{d}\xi, \tag{4.39}$$

can be further processed to give the simple forms

$$u_{sp}(y,t)=\mathrm{e}^{-my}\sin(\omega t-ny),\quad u_{cp}(y,t)=\mathrm{e}^{-my}\cos(\omega t-ny), \tag{4.40}$$

where $2m^2=\sqrt{\alpha^4+\beta^2}-\alpha^2$, $2n^2=\sqrt{\alpha^4+\beta^2}+\alpha^2$, $\alpha^2=\dfrac{\omega^2(\lambda-\lambda_r)}{1+\lambda_r^2\omega^2}$ and $\beta=\dfrac{\omega(1+\lambda\lambda_r\omega^2)}{1+\lambda_r^2\omega^2}$.

By now letting $\lambda\to 0$ into Eqs. (4.40), the dimensionless forms of the solutions obtained by Rajagopal [17] are recovered. In the case of Newtonian fluids, when λ and $\lambda_r\to 0$, Eqs. (4.40) reduce to the dimensionless forms

$$u_{Nsp}(y,t)=\mathrm{e}^{-y\sqrt{\frac{\omega}{2}}}\sin\left(\omega t-y\sqrt{\frac{\omega}{2}}\right),\quad u_{Ncp}(y,t)=\mathrm{e}^{-y\sqrt{\frac{\omega}{2}}}\cos\left(\omega t-y\sqrt{\frac{\omega}{2}}\right), \tag{4.41}$$

of classical solutions (see for instance [18, Eqs. (3.17) and (3.12)]). Finally, in the view of governing equations (3.7) and (4.27) for velocity and shear stress, it results that the expressions of dimensionless permanent shear stresses corresponding to the motion of an Oldroyd-B fluid due to an infinite plate that applies oscillating shear stresses $\tau_0\sin(\omega t)$ or $\tau_0\cos(\omega t)$ to the fluid are given by the same expressions appearing in the right parts of Eqs. (4.40). The corresponding velocity fields, namely

$$u_{sp}(y,t)=\frac{\sqrt{m^2+n^2}}{\rho\omega}\mathrm{e}^{-my}\sin(\omega t-ny-\varphi),\quad u_{cp}(y,t)=\frac{\sqrt{m^2+n^2}}{\rho\omega}\mathrm{e}^{-my}\cos(\omega t-ny-\varphi), \tag{4.42}$$

where $tg\varphi=m/n$ are obtained using again the dimensionless form of Eq. $(2.2)_2$. For Newtonian fluids, when λ and $\lambda_r\to 0$, Eqs. (4.42) reduce to

$$u_{Nsp}(y,t)=\frac{1}{\rho\sqrt{\omega}}\mathrm{e}^{-y\sqrt{\frac{\omega}{2}}}\sin\left(\omega t-y\sqrt{\frac{\omega}{2}}-\frac{\pi}{4}\right),\quad u_{Ncp}(y,t)=\frac{1}{\rho\sqrt{\omega}}\mathrm{e}^{-y\sqrt{\frac{\omega}{2}}}\cos\left(\omega t-y\sqrt{\frac{\omega}{2}}-\frac{\pi}{4}\right), \tag{4.43}$$

which are just the dimensionless forms of the permanent solutions (3.20) and (3.22) from [19].

5 Conclusions

In this work, contrary to what is usually solved in the existing literature, a shear stress boundary value problem is analytically studied. It corresponds to unsteady unidirectional motion of an Oldroyd-B fluid over an infinite plate that applies an arbitrary time-dependent shear stress to the fluid. The dimensionless solutions for velocity and shear stress that have been obtained by means of integral transforms, allow us to provide exact solutions for motions of rate type fluids due to an infinite plate that applies constant, constantly accelerating or oscillating shear stresses to the fluid. Such solutions, which are lack in the literature for rate type fluids, can be easy reduced to the known solutions for second grade or Newtonian fluids performing the same motion.

As an application, a general solution is developed for the velocity field corresponding to the motion of the same fluids induced by a moving plate. For illustration, as well as for a check of results, three special cases are considered and different known solutions are recovered as limiting cases. They correspond to motions with uniform, accelerated or oscillating velocity of the plate. The velocity field (4.34) corresponding to the sine oscillations of the plate is new in the literature while the solution (4.37) for cosine oscillations of the plate corrects a result obtained in [16, Eq. (3.10)]. In all cases, the solutions corresponding to Maxwell, second grade or Newtonian fluids performing the same motions can be obtained as special cases of general solutions and the motion problem of these fluids over an infinite plate can be considered as being completely solved.

Now, in order to get some physical insight of results that have been obtained, some numerical calculations have been carried out for different values of physical parameters and time t. The dimensionless velocity profiles corresponding to two motions with technical relevance are presented in Figs. 1 and 2 for three different values of the time *t*. They correspond to the motion induced by an infinite plate that applies a constant shear to the fluid or due to the plate that is moving with a constant velocity in its plane. The fluid velocity, as expected, is an increasing function of time in both cases. It smoothly decreases from maximum values at the wall to the asymptotic value for large values of *y*. Furthermore, the values of $u(y,t)$ at any distance y are always higher for $t=t_1$ than for $t=t_2$ if $t_1>t_2$. The velocity boundary condition, in the second case, is clearly satisfied.

As regards the oscillating motions of fluids, their starting solutions are written as a sum of steady-state (permanent) and transient solutions. Such solutions are important for those who want to eliminate the transients from their rheological measurements or experiments. Consequently, an important problem regarding the technical relevance of these solutions is to find the approximate time after which the fluid is moving according to the steady-state solutions. More exactly, we have to determine the required time to reach the steady-state by comparing the corresponding steady-state and starting solutions at different values of the time *t*. This time, with an error of order 10^{-3}, is determined for comparison in Figs. 3, 5, 7 and 4, 6, 8 for motions due to sine and cosine oscillations of the plate. In both cases, as expected, it is a decreasing function with respect to ω and λ_r and increases for increasing values of λ.

The main findings which directly result from our graphical representations are:

- The dimensionless velocity of the fluid corresponding to the motion due to the plate that applies a constant shear to the fluid (when $\tau(0,t)=H(t)$) is greater than that corresponding to the motion induced by the plate that is moving with a constant velocity (when $u(0,t)=H(t)$).
- The required time to reach the steady-state for oscillatory motions of Maxwell or second grade fluids is greater, respectively lower in comparison with Newtonian fluids.
- The required time to reach the steady-state for oscillating motions of Oldroyd-B fluids is greater or lower in comparison with Maxwell, respectively second grade fluids.
- The required time to reach the steady state for motions due to sine oscillations of the plate is higher in comparison to that corresponding to motions induced by cosine oscillations of the plate. This is obvious because at $t=0$ the velocity of the plate is zero.

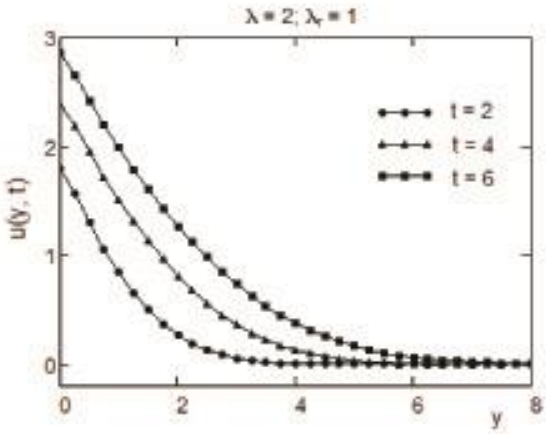


Figure 1: Profiles of the velocity u(y,t) given by Eq. (3.18) for different values of t.

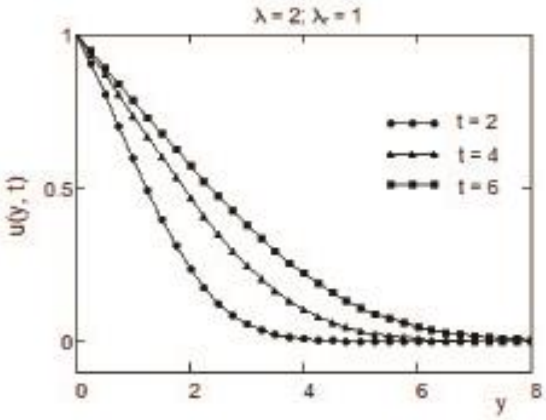
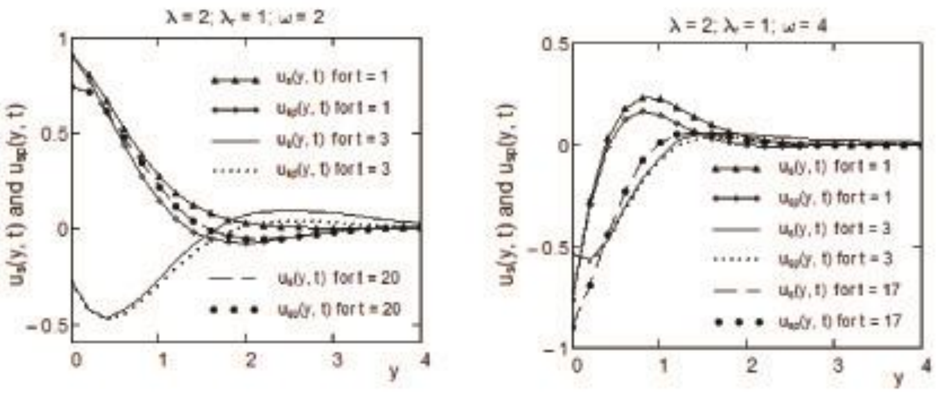
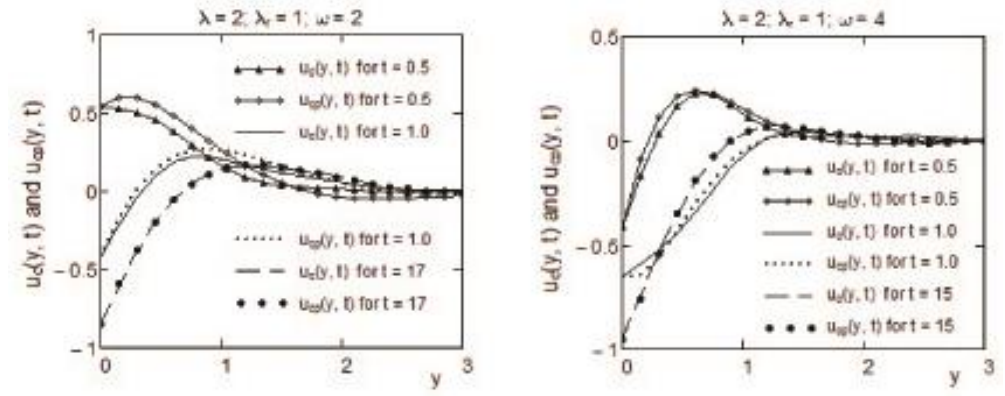


Figure 2: Profiles of the velocity u(y,t) given by Eq. (4.29) for different values of t.



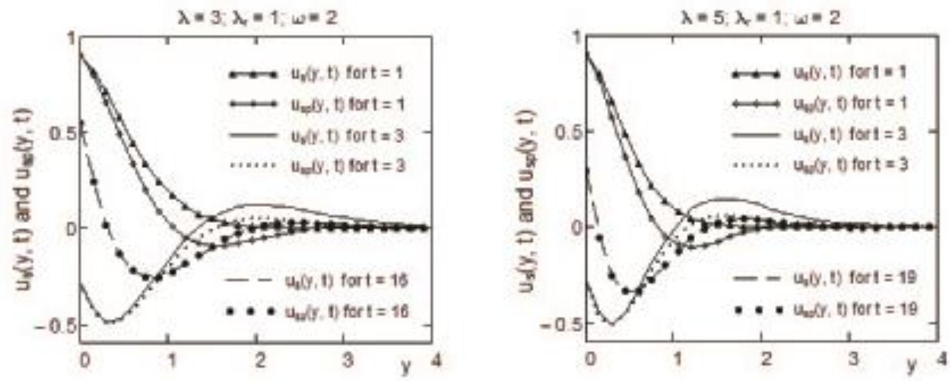
a. b.

Figure 3: Required time to reach the steady-state for the motion of an Oldroyd-B fluid due to the sine oscillations of the plate for ω=2 and ω=4.



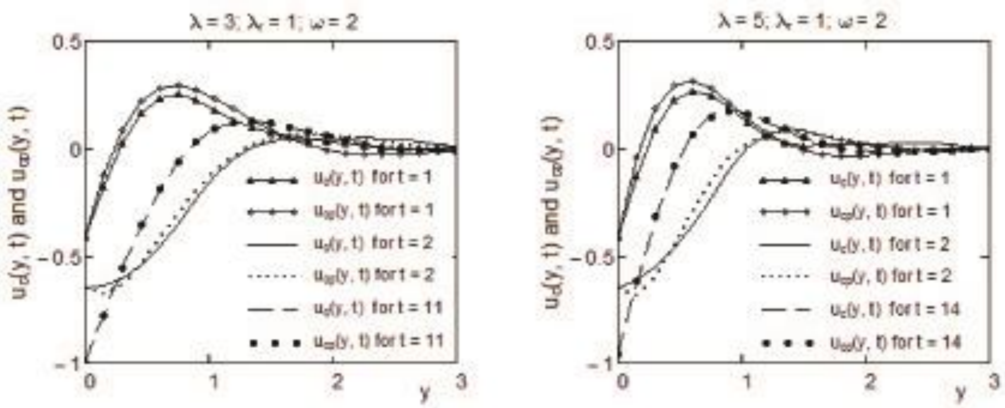
a. b.

Figure 4: Required time to reach the steady-state for the motion of an Oldroyd-B fluid due to the cosine oscillations of the plate for ω=2 and ω=4.



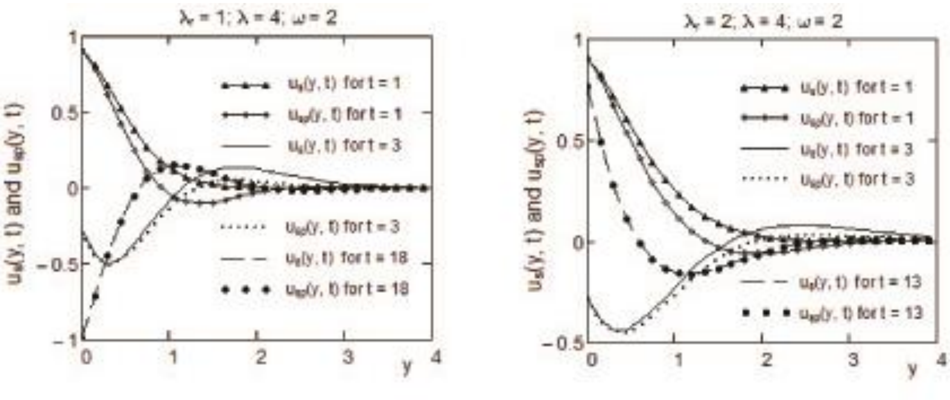
a. b.

Figure 5: Required time to reach the steady-state for the motion of an Oldroyd-B fluid due to the sine oscillations of the plate for λ=3 and λ=5.



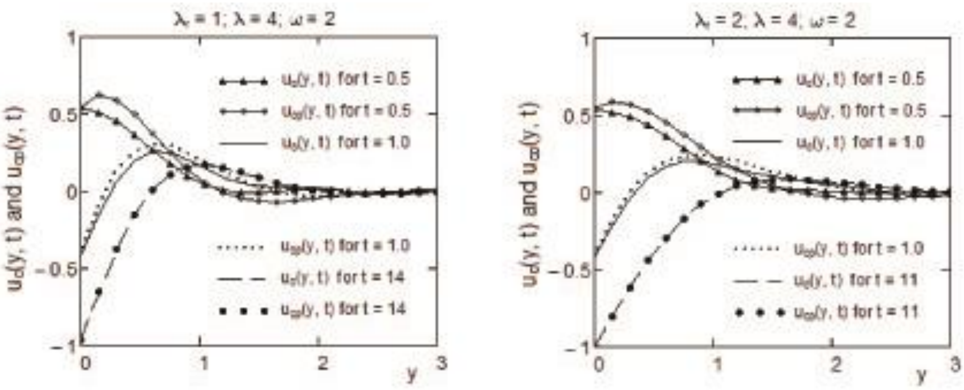
a. b.

Figure 6: Required time to reach the steady-state for the motion of an Oldroyd-B fluid due to the cosine oscillations of the plate for λ=3 and λ=5.



a. b.

Figure 7: Required time to reach the steady-state for the motion of an Oldroyd-B fluid due to the sine oscillations of the plate for $\lambda_r=1$ and $\lambda_r=2$.



a. b.

Figure 7: Required time to reach the steady-state for the motion of an Oldroyd-B fluid due to the cosine oscillations of the plate for $\lambda_r=1$ and $\lambda_r=2$.

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