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Research Article

Perturbation Solutions of the Quintic Duffing Equation with Strong Nonlinearities

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Abstract
The quintic Duffing equation with strong nonlinearities is considered. Perturbation solutions are constructed using two different techniques: The classical multiple scales method (MS) and the newly developed multiple scales Lindstedt Poincare method (MSLP). The validity criteria for admissible solutions are derived. Both approximate solutions are contrasted with the numerical solutions. It is found that MSLP provides compatible solution with the numerical solution for strong nonlinearities whereas MS solution fail to produce physically acceptable solution for large perturbation parameters.

Keywords: Perturbation Methods, Lindstedt Poincare Method, Multiple Scales Method, Numerical Solutions, Quintic Duffing Equation.

1 Introduction

Perturbation methods are important analytical tools used for over a century to construct approximate analytical solutions. Especially in handling nonlinear differential equations, the methods provided physically acceptable solutions for problems with weak nonlinearities. The requirement of a small perturbation parameter limits the range of validity of the results to weak nonlinearities. For strongly nonlinear systems, the approximate solutions cease to be valid and people usually resort to numerical techniques. Many different perturbation methods developed within time all suffer from this deficiency.
To overcome the difficulty and to validate results for strong nonlinearities, there have been a number of attempts recently. In one of the interesting approaches, Hu and Xiong [1] made a slight modification in the well-known Lindstedt-Poincare method and constructed an approximate solution for the Duffing equation with strong nonlinearity. Their time histories agreed well with the numerical solutions for arbitrarily large perturbation parameters. In a similar paper, the approximate and exact frequencies of the Duffing equation were compared [2]. The case of vanishing restoring force was also treated for the same equation [3]. For large perturbation parameters, the approximate and exact periods were in reasonable agreement.

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It is well known that Lindstedt-Poincare method produces acceptable results for only constant amplitude type steady-state solutions [4]. For transient solutions with variable amplitudes, the method cannot be used. The multiple scales method proved to be effective in such problems [4]. If the formalism given in [1-3] can be integrated to the multiple scales method, a wider range of nonlinear problems could be solved. With this motivation, recently, Pakdemirli *et al.* [5] proposed a new perturbation method combining multiple scales and Lindstedt Poincare methods with a frequency expansion suggested in references [1,2]. The method was named Multiple Scales Lindstedt Poincare method (MSLP), and was applied successfully to several problems: free vibrations of a linear damped oscillator, undamped and damped Duffing oscillators. Exact analytical solution was retrieved by the new method for the linear damped oscillator. For strongly nonlinear undamped and damped Duffing oscillators, results of the new method agreed well with the numerical solutions [5]. The approximate solutions constructed later with MSLP for strong quadratic and cubic nonlinearities were highly encouraging [6]. The method was successfully applied to a forced vibration Duffing problem also [7].

Recently Ramos [8] treated the quintic Duffing equation with artificial parameter method and obtained solutions for strong nonlinearities. Belendez *et al.* [9] used Chebyshev polynomials to expand the restoring force and approximated the original equation to a cubic Duffing equation. They obtained successful approximations to the frequency and response for large perturbation parameters.

In this work, MSLP method is applied for the first time to a vibration problem with quintic nonlinearity. For MS and MSLP solutions, validity criteria are developed. From the validity criterion of MSLP, it is shown that admissible solutions are possible for arbitrarily large perturbation parameters. Both approximate solutions are compared with the numerical ones. While MS produce unphysical solutions in terms of amplitudes and frequencies, results of MSLP are compatible with the numerical ones for strong nonlinearities.

Contrary to the previous work [8, 9], our method is more straightforward, does not require transformations and approximations of the original equation, introduction of artificial parameters and provides admissible solutions compatible with the numerical ones for strong nonlinearities.

2 Multiple Scales (MS) Method

In this section, the Duffing equation with quintic nonlinearity

$$\ddot{u}+\omega_0^2 u+\varepsilon u^5=0 \qquad u(0)=a_0, \quad \dot{u}(0)=0 \tag{2.1}$$

will be solved with the classical multiple scales [4] method. The Duffing equation models are among the class of most fundamental equations that arise in nonlinear oscillations of physical systems. They may appear in various forms with cubic, quintic, cubic and quintic nonlinearities etc. The quintic model considered here arises in free vibrations of restrained beams with intermediate lumped masses. Fast and slow time scales

$$T_0=t, \qquad T_1=\varepsilon t, \qquad T_2=\varepsilon^2 t \tag{2.2}$$

and the time derivatives with respect to these variables are

$$\frac{d}{dt}=D_0+\varepsilon D_1+\varepsilon^2 D_2+\ldots, \qquad \frac{d^2}{dt^2}=D_0^2+2\varepsilon D_0 D_1+\varepsilon^2(D_1^2+2D_0 D_2)+\ldots \tag{2.3}$$

where $D_n=\partial/\partial T_n$. The approximate expansion

$$u=u_0(T_0,T_1,T_2)+\varepsilon u_1(T_0,T_1,T_2)+\varepsilon^2 u_2(T_0,T_1,T_2)+\ldots . \tag{2.4}$$

is inserted into (2.1) and separated at each order of approximation

$$O(1):\ D_0^2 u_0+\omega_0^2 u_0=0, \qquad u_0(0)=a_0, \quad D_0 u_0(0)=0 \tag{2.5}$$

$$O(\varepsilon):\ D_0^2 u_1+\omega_0^2 u_1=-2D_0 D_1 u_0-u_0^5, \qquad u_1(0)=0, \quad (D_0 u_1+D_1 u_0)(0)=0 \tag{2.6}$$

$$O(\varepsilon^2):D_0^2 u_2+\omega_0^2 u_2=-2D_0 D_1 u_1-(D_1^2+2D_0 D_2)u_0-5u_0^4 u_1. \tag{2.7}$$

The solution at the first order can be expressed in terms of real and imaginary forms

$$u_0 = A(T_1,T_2)e^{i\omega_0 T_0} + cc = a\cos(\omega_0 T_0 + \beta), \quad a(0) = a_0, \quad \beta(0) = 0 \tag{2.8}$$

where the connection between the complex and real amplitudes are expressed through the polar form

$$A = \frac{1}{2}ae^{i\beta} \tag{2.9}$$

and cc stands for complex conjugates of the preceding terms. Upon substitution of (2.8) into (2.6) and elimination of secular terms yields

$$D_1 A = \frac{5i}{\omega_0} A^3 \overline{A}^2 . \tag{2.10}$$

The solution at this level of approximation is

$$u_1 = Be^{i\omega_0 T_0} + \frac{1}{24\omega_0^2} A^5 e^{5i\omega_0 T_0} + \frac{5}{8\omega_0^2} A^4 \overline{A} e^{3i\omega_0 T_0} + cc , \tag{2.11}$$

or in terms of real amplitudes and phases

$$u_1 = b\cos(\omega_0 T_0 + \gamma) + \frac{1}{384\omega_0^2} a^5 \cos(5\omega_0 T_0 + 5\beta) + \frac{5}{128\omega_0^2} a^5 \cos(3\omega_0 T_0 + 3\beta) \tag{2.12}$$

where

$$B = \frac{1}{2}be^{i\gamma} . \tag{2.13}$$

The boundary conditions lead to the relations

$$\gamma(0) = 0, \quad D_1 a(0) = 0, \quad b(0) = -\frac{a_0^5}{24\omega_0^2} . \tag{2.14}$$

At the last level of approximation, only the secularities are eliminated

$$2i\omega_0 D_1 B + 2i\omega_0 D_2 A - \frac{55}{6\omega_0^{\ 2}} A^5 \overline{A}^4 + 20A^3 \overline{A}\overline{B} + 30A^2 \overline{A}^2 B = 0 \tag{2.15}$$

with the usage of (2.10). Inserting the polar forms of (2.9) and (2.13) into (2.10) and (2.15), using the boundary conditions when necessary, the approximate solution is found finally

$$u = a_0 \cos(\omega t) + \varepsilon \frac{a_0^5}{384\omega_0^{\ 2}} \{\cos(5\omega t) + 15\cos(3\omega t) - 16\cos(\omega t)\} + O(\varepsilon^2) \tag{2.16}$$

where

$$\omega = \omega_0 + \varepsilon \frac{5}{16\omega_0} a_0^4 - \varepsilon^2 \frac{215}{3072\omega_0^3} a_0^8 . \tag{2.17}$$

For $\omega_0 = 1$, the same result was given in [8] using the classical Lindstedt-Poincare method.

3 Multiple Scales Lindstedt Poincare (MSLP) Method

The same problem will be treated with the recently developed multiple scales Lindstedt Poincare method [5, 6]. Similar to the Lindstedt Poincare method, a time transformation is required first

$$\tau = \omega t \tag{3.18}$$

$$\omega^2 u'' + \omega_0^{\ 2} u + \varepsilon u^5 = 0 \tag{3.19}$$

where prime represents derivative with respect to the new time variable τ. Fast and slow time scales are

$$T_0 = \tau, \qquad T_1 = \varepsilon\tau, \qquad T_2 = \varepsilon^2\tau . \tag{3.20}$$

Using

$$\frac{d^2}{d\tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) + \dots \tag{3.21}$$

where $D_n = \partial / \partial T_n$ and substituting the expansions

$$u = u_0(T_0, T_1, T_2) + \varepsilon u_1(T_0, T_1, T_2) + \varepsilon^2 u_2(T_0, T_1, T_2) + \dots \tag{3.22}$$

$$\omega_0^2 = \omega^2 - \varepsilon\omega_1 - \varepsilon^2 \omega_2 \tag{3.23}$$

into (3.19) yields after separation

$$O(1):\ \omega^2 D_0^2 u_0 + \omega^2 u_0 = 0, \qquad u_0(0) = a_0, \quad D_0 u_0(0) = 0 \tag{3.24}$$

$$O(\varepsilon):\ \omega^2 D_0^2 u_1 + \omega^2 u_1 = -2\omega^2 D_0 D_1 u_0 + \omega_1 u_0 - u_0^5, \quad u_1(0) = 0, (D_0 u_1 + D_1 u_0)(0) = 0 \tag{3.25}$$

$$O(\varepsilon^2):\ \omega^2 D_0^2 u_2 + \omega^2 u_2 = -2\omega^2 D_0 D_1 u_1 - \omega^2 (D_1^2 + 2D_0 D_2) u_0 + \omega_1 u_1 + \omega_2 u_0 - 5u_0^4 u_1. \tag{3.26}$$

Note that following [1,2], in equation (3.19), instead of the transformation frequency, the natural frequency is expanded. The solution at the first order is

$$u_0 = Ae^{iT_0} + cc = a\cos(T_0 + \beta), \quad a(0) = a_0, \quad \beta(0) = 0. \tag{3.27}$$

This solution is substituted into the right hand side of the next level of approximation and secular terms are eliminated

$$-2i\omega^2 D_1 A + \omega_1 A - 10A^3 \overline{A}^2 = 0. \tag{3.28}$$

In this method, there are two mechanisms to eliminate secularities, one coming from MS and the other from LP [5]. First choice is to select D_1A=0 and check whether the frequency correction is real. If ω_1 turns out to be complex, then $D_1A \neq 0$ which implies ω_1=0 and secularities are eliminated by choosing D_1A. A complex ω_1 implies that there is amplitude variation and LP method fails to produce physical solutions [4]. The method combines the potentials of MS and LP type of eliminating secularities thereby introducing more flexibility. For equation (3.28), selection of

$$D_1 A = 0 \Rightarrow \quad A = A(T_2) \tag{3.29}$$

produces

$$\omega_1 = 10A^2 \overline{A}^2 = \frac{5}{8} a^4 \tag{3.30}$$

which is a suitable choice because ω_1 turns out to be real. The solution at order ε is

$$u_1 = Be^{iT_0} + \frac{1}{24\omega^2} A^5 e^{5iT_0} + \frac{5}{8\omega^2} A^4 \overline{A} e^{3iT_0} + cc \tag{3.31}$$

or

$$u_1 = b\cos(T_0 + \gamma) + \frac{1}{384\omega^2} a^5 \cos(5T_0 + 5\beta) + \frac{5}{128\omega^2} a^5 \cos(3T_0 + 3\beta). \tag{3.32}$$

The boundary conditions lead to the following conditions

$$\gamma(0) = 0, \quad b(0) = -\frac{a_0^5}{24\omega^2}. \tag{3.33}$$

At the last level of approximation, secular terms are eliminated

$$-2i\omega^2 D_1 B - 2i\omega^2 D_2 A + \omega_1 B + \omega_2 A - \frac{95}{6\omega^2} A^5 \overline{A}^4 - 20A^3 \overline{A}\overline{B} - 30A^2 \overline{A}^2 B = 0. \tag{3.34}$$

Upon assuming $D_1 B = 0$ and $D_2 A = 0$, ω_2 turns out to be real

$$\omega_2 = -\frac{65}{1536} \frac{a^8}{\omega^2} \tag{3.35}$$

which is admissible. Substituting (3.35) and (3.30) into (3.23) yields

$$\omega^2 = \omega_0^2 + \varepsilon \frac{5}{8} a_0^4 - \varepsilon^2 \frac{65}{1536} \frac{a_0^8}{\omega^2}. \tag{3.36}$$

Since ω exists on both sides, solving the algebraic equation gives the final frequency equation

$$\omega = \sqrt{\frac{1}{2}\left(\omega_0^2 + \frac{5}{8}\varepsilon a_0^4\right) + \frac{1}{2}\sqrt{\omega_0^4 + \frac{5}{4}\varepsilon a_0^4 \omega_0^2 + \frac{85}{384}\varepsilon^2 a_0^8}}. \tag{3.37}$$

The approximate solution in terms of this frequency is finally

$$u = a_0 \cos(\omega t) + \varepsilon \frac{a_0^5}{384\omega^2}\{\cos(5\omega t) + 15\cos(3\omega t) - 16\cos(\omega t)\} + O(\varepsilon^2). \tag{3.38}$$

Note that for $\omega_0 = 1$, Ramos [8] gave

$$\omega^2 = 1 + \varepsilon \frac{5}{8} a_0^4 - \varepsilon^2 \frac{65}{1536} \frac{a_0^8}{\left(1 + \varepsilon \frac{5}{8} a_0^4\right)} \tag{3.39}$$

as the frequency. Comparing (3.39) with (3.36), one recognizes that ω in the last term in (3.36) was approximated to a first order in (3.39) which will introduce problems for large perturbation parameters. As the perturbation parameter becomes sufficiently large, imaginary frequencies will be obtained for (3.39) whereas for (3.37) there will be no such problem.

4 Validity Criteria and the Numerical Solutions

For perturbation solutions to be valid, the correction term should be much smaller than the leading term. For both methods, assuming the functional variations to be of $O(1)$, the necessary conditions are

$$\frac{\varepsilon a_0^4}{24\omega_0^2} << 1 \qquad \text{(MS)} \tag{4.40}$$

$$\frac{\varepsilon a_0^4}{24\omega^2} << 1 \qquad \text{(MSLP)}. \tag{4.41}$$

The only difference in the criteria is the replacement of ω instead of ω_0 in the MSLP method. For strong nonlinearities, ε may be arbitrarily large. For MS, taking the limit as the perturbation parameter tends to infinity

$$\lim_{\varepsilon \to \infty} \frac{\varepsilon a_0^4}{24\omega_0^2} = \infty \tag{4.42}$$

yields infinity as expected. Hence MS solution cannot be valid for large values of ε. However, for MSLP, the corresponding limit is

$$\lim_{\varepsilon \to \infty} \frac{\varepsilon a_0^4}{24\omega^2} = \frac{\varepsilon a_0^4}{24\left(\frac{1}{2}(\omega_0^2 + \frac{5}{8}\varepsilon a_0^4) + \frac{1}{2}\sqrt{\omega_0^4 + \frac{5}{4}\varepsilon a_0^4 \omega_0^2 + \frac{85}{384}\varepsilon^2 a_0^8}\right)} \cong 0.076 << 1 \tag{4.43}$$

which satisfies the validity criterion for arbitrarily large perturbation parameters. Therefore, it should be expected that numerical solutions and the MSLP solutions match with each other for arbitrarily large perturbation parameters. A similar mechanism inherited in analytical solutions of [8, 9] provides compatible solutions with numerical ones for strong nonlinearities.

Numerical solutions of the original equation are contrasted with both methods in Figures 1-5. In all figures a_0=1 and $\omega_0=\pi$. In Figure 1, for ε=4 all solutions match with each other. A deviation in frequency is observed for MS for ε=20 with MSLP and numerical solutions being in excellent agreement (Figure 2). The qualitative behavior of MS solutions do not represent the real physics as ε is increased further (Figure 3, ε=50). In

Figure 4, the erroneous solutions of MS became more apparent for ε=100 while MSLP still matches with the numerical ones. Finally, the amplitude errors and the error in qualitative behavior are amplified in Figure 5 for ε=1000.

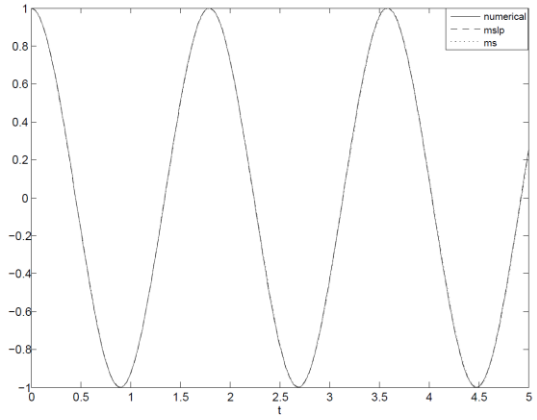

Figure 1: Comparison of the time histories of the MS, MSLP methods and numerical simulations (ε=4, a_0=1, ω_0=π)

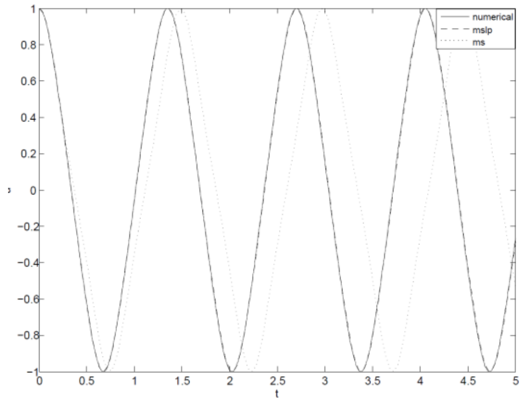

Figure 2: Comparison of the time histories of the MS, MSLP methods and numerical simulations (ε=20, a_0=1, ω_0=π)

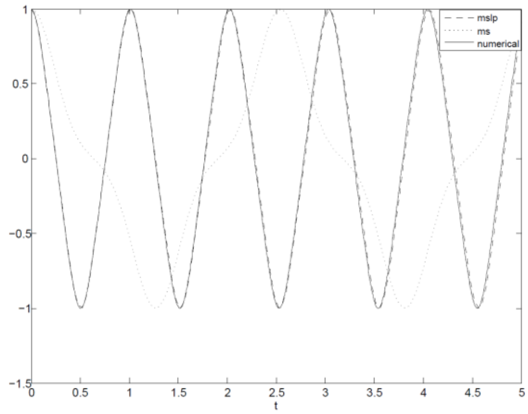

Figure 3: Comparison of the time histories of the MS, MSLP methods and numerical simulations (ε=50, a_0=1, ω_0=π)

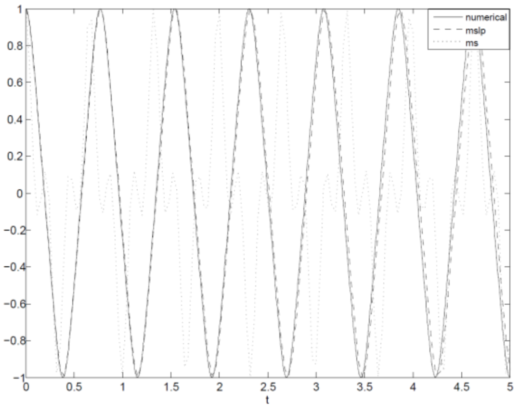

Figure 4: Comparison of the time histories of the MS, MSLP methods and numerical simulations (ε=100, a_0=1, ω_0=π)

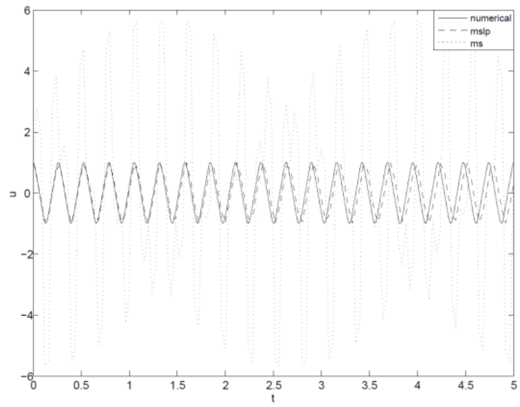

Figure 5: Comparison of the time histories of the MS, MSLP methods and numerical simulations (ε=1000, a_0=1, ω_0=π)

Note that MSLP solutions are in reasonable agreement with the numerical solutions for such large perturbation parameters. One of the reasons of the deficiency of MS solutions is that the frequencies turns out to be negative for large perturbation parameters (see (2.17)). However, the frequency remains always positive in case of MSLP (see (3.37)).

5 Concluding Remarks

Approximate analytical solutions are constructed for strong quintic nonlinearities for the first time using a combination of multiple scales and Lindstedt Poincare methods (MSLP). Solutions of the classical multiple scales and the multiple scales Lindstedt Poincare method are contrasted with numerical solutions and validity criteria are given for both methods. It is found that MSLP is capable of producing reliable solutions for strong quintic nonlinearities.

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