Communications in Numerical Analysis 2015 No.1 (2015) 1-15

ISPACS International Scientific Publications and Consulting Services

Available online at www.ispacs.com/cna

Volume 2015, Issue 1, Year 2015 Article ID cna-00203, 15 Pages

doi:10.5899/2015/cna-00203

Research Article

Numerical Solution of Fuzzy Differential Equations by Runge-Kutta Verner Method

T. Jayakumar[1]*, T. Muthukumar[1], K. Kanagarajan[1]

(1) *Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore-641 020, Tamilnadu, India*

Abstract
In this paper we study the numerical methods for Fuzzy Differential equations by an application of the Runge-Kutta Verner method for fuzzy differential equations. We prove a convergence result and give numerical examples to illustrate the theory.

Keywords: Fuzzy Differential Equations; Fuzzy Cauchy problem; Runge-Kutta Verner method.

1 Introduction

The concept of fuzzy derivative was first introduced by Chang, Zadeh in [6] it was followed up by Dubois, Prede in [7], who defined and used the extension principle. The fuzzy differential equations (FDEs) and the intial value problem where regularly treated by Kaleva in [18, 19] and by Seikkal in [20]. The numerical method for solving fuzzy differential equations is introduced by Ma, Friedmen, Kandedl in [22] by the standard Eular method and by authors in [1, 2] by Taylor method. Jayakumar et al [13] discussed numerical solution of FDEs of order five. In the last few years many works have been performed by several authours in numerical solutions of fuzzy differential equations [1, 2, 3, 4, 5, 11, 16, 20]. Recently, the numerical solution of FDEs by predictor-corrector method has been stiuded in [5]. In this work we replace the fuzzy differential equation by its parametric form and then solve numerically the new system. Which consider the two classic ordinarry differintial equations with initial condition.
The structure of this paper organizes as follows: In section 2. some besic results on fuzzy numbers and definition of a fuzzy derivative, which have been discussed by Seikkala in [25] are given. In section 3. we define the problem, this is a fuzzy Cauchy problem whose numerical solution is the main interest of this chapter. The numerically solving fuzzy differential equation by the Runge-Kutta Verner method is discussed in section 4. The proposed algorithem is illustrated by solving some examples in section 5. and conclusion is in section 6.

2 Preliminaries

Consider the intial value problem

$$\begin{cases} y'(t) = f(t, y(t)); \quad t_0 \le t \le T, \\ y(a) = \alpha. \end{cases} \tag{2.1}$$

*Corresponding author. Email address: jayakumar.thippan68@gmail.com

The basis of all Runge-Kutta method is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = \sum_{i=1}^{m} w_i k_i, \tag{2.2}$$

where for $i = 1,2,3......,m$ the w_i's are constants and

$$k_i = h.f(t_n + \alpha_i h, y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j). \tag{2.3}$$

Equation (2.2) is to be exact for powers of h through h^m, because it is to be coincident with Taylor series of order m. Therefore, the trunction error T_m, can be written as

$$T_m = \gamma_m h^{m+1} + O(h^{m+2}).$$

The true magnitude of γ_m will generally be much less than the bound of theorem 2.1. Thus, if it the $O(h^{m+2})$ term is small compared with $\gamma_m h^{m+1}$, as we expect, to be so if h is small, then the bound on $\gamma_m h^{m+1}$, will usally be a bound on the error as a whole. The famous nonzero constants α_i, β_{ij} in the Runge Kutta Verner Method are

$$\alpha_1 = 0,\ \alpha_2 = \frac{1}{6},\ \alpha_3 = \frac{4}{15},\ \alpha_4 = \frac{2}{3},\ \ \alpha_5 = \frac{5}{6},\ \alpha_6 = 1,\ \alpha_7 = \frac{1}{15},\ \alpha_8 = 1,$$

$$\beta_{21} = \frac{1}{6},\ \beta_{31} = \frac{4}{15},\ \beta_{32} = \frac{16}{75},\ \beta_{41} = \frac{5}{6},\ \beta_{42} = \frac{-8}{3},\ \beta_{43} = \frac{5}{2},$$

$$\beta_{51} = \frac{-165}{64},\ \beta_{52} = \frac{55}{6},\ \beta_{53} = \frac{-425}{64},\ \beta_{54} = \frac{85}{96},$$

$$\beta_{61} = \frac{12}{5},\ \beta_{62} = -8,\ \beta_{63} = \frac{4015}{612},\ \beta_{64} = \frac{-11}{36},\ \beta_{65} = \frac{86}{225},$$

$$\beta_{71} = \frac{-8263}{15000},\ \beta_{72} = \frac{124}{75},\ \beta_{73} = \frac{-643}{680},\ \beta_{74} = \frac{-84}{250},\ \beta_{75} = \frac{2484}{10625},$$

$$\beta_{81} = \frac{3501}{1720},\ \beta_{82} = \frac{-300}{43},\ \beta_{83} = \frac{297278}{52632},\ \beta_{84} = \frac{-319}{2322},\ \beta_{85} = \frac{24068}{84065},\ \beta_{87} = \frac{3850}{26703},$$

where $m = 8$. Hence we have [see [19]]

$$\begin{aligned}
y_0 &= \alpha, \\
k_1 &= h.f(t_i, y_i), \\
k_2 &= h.f(t_i + \tfrac{1}{6}h, y_i + \tfrac{1}{6}k_1), \\
k_3 &= h.f(t_i + \tfrac{4}{15}h, y_i + \tfrac{4}{75}k_1 + \tfrac{16}{75}k_2), \\
k_4 &= h.f(t_i + \tfrac{2}{3}h, y_i + \tfrac{5}{6}k_1 - \tfrac{8}{3}k_2 + \tfrac{5}{2}k_3), \\
k_5 &= h.f(t_i + \tfrac{5}{6}h, y_i - \tfrac{165}{64}k_1 + \tfrac{55}{6}k_2 - \tfrac{425}{64}k_3 + \tfrac{85}{96}k_4), \\
k_6 &= h.f(t_i + h, y_i + \tfrac{12}{5}k_1 - 8k_2 + \tfrac{4015}{612}k_3 - \tfrac{11}{36}k_4 + \tfrac{88}{255}k_5), \\
k_7 &= h.f(t_i + \tfrac{1}{15}h, y_i - \tfrac{8263}{15000}k_1 + \tfrac{124}{75}k_2 - \tfrac{643}{680}k_3 - \tfrac{81}{250}k_4 + \tfrac{2484}{10625}k_5), \\
k_8 &= h.f(t_i + h, y_i + \tfrac{3501}{1720}k_1 - \tfrac{300}{43}k_2 + \tfrac{297275}{52632}k_3 - \tfrac{319}{2322}k_4 + \tfrac{24068}{84065}k_5 + \tfrac{3850}{26703}k_7), \\
y_{i+1} &= y_i + \tfrac{3}{40}k_1 + \tfrac{875}{2244}k_3 + \tfrac{23}{72}k_4 + \tfrac{264}{1955}k_5 + \tfrac{125}{11592}k_7 + \tfrac{43}{616}k_8,
\end{aligned} \tag{2.4}$$

where

$$a = t_0 \le t_1 \le ... \le t_N = b \text{ and } h = \frac{(b-a)}{N} = t_{i+1} - t_i. \tag{2.5}$$

Theorem 2.1. *Let $f(t,y)$ belong to $C^8[a,b]$ and let it's partial derivatives are bounded and assume there exists, L,M, positive numbers, such that*

$$|f(t,y)| < M, \quad \left|\frac{\partial^{i+j} f}{\partial t^i \partial y^j}\right| < \frac{L^{i+j}}{M^{j-i}}, i+j \le m,$$

then in the Runge -Kutta Verner method, $y(t_{i+1}) - y_{i+1} \approx \frac{4}{9} h^8 M L^7 + O(h^9)$. A triangular fuzzy number v, is defined by three numbers $a < b < c$ where the graph of $v(x)$, the membership function of the fuzzy number v, is a triangle with base on the intervel $[a,c]$ and vertex at $x = b$. We specify v as $(a/b/c)$.
We will write :
$(1) v > 0 \;\; if a > 0$; $(2) v \ge 0 \;\; if a \ge 0$; $(3) v < 0 \;\; if c < 0$;*and (4)* $v \le 0 \;\; if c \le 0$.
Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded r-level intervals. It means that if $v \in E$ then the r- level set

$$[v]_r = \{s | v(s) \ge r\}, \quad 0 < r \le 1,$$

is a colsed bounded interval which is denoted by

$$[v]_r = [v_1(r), v_2(r)].$$

Let I be a real interval. A mapping $x : I \to E$ is called a fuzzy process and its $r-$level set is denoted by

$$[x(r)]_r = [x_1(t;r), x_2(t;r)], \quad t \in I, r \in (0,1].$$

The derivative $x'(t)$ of a fuzzy process $x(t)$ is defined by

$$[x'(r)]_r = [x'_1(t;r), x'_2(t;r)], \quad t \in I, r \in (0,1].$$

Provided that this equation defines a fuzzy number, as in Seikkala [25].

3 A Fuzzy Cauchy Problem

Consider the fuzzy initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \in I = [0,T], \\ y(0) = y_0, \end{cases} \tag{3.6}$$

where f is a continuous mapping from $R_+ \times R$ into R and $y_0 \in E$ with r- level sets

$$[y_0]_r = [\underline{y}(0;r), \overline{y}(0;r)], \quad r \in (0,1].$$

The extension principle of Zadeh leads to the following definition of $f(t,y)$ when $y = y(t)$ is a fuzzy number

$$f(t,y)(s) = sup\{y(\tau) | s = f(t,\tau)\}, \;\; s \in R.$$

It follows that

$$[f(t,y)]_r = [\underline{f}(t,y;r), \overline{f}(t,y;r)], \quad r \in (0,1],$$

where

$$\left.\begin{aligned} \underline{f}(t,y;r) &= min\{f(t,u) | u \in [\underline{y}(r), \overline{y}(r)]\}, \\ \overline{f}(t,y;r) &= max\{f(t,u) | u \in [\underline{y}(r), \overline{y}(r)]\}. \end{aligned}\right\} \tag{3.7}$$

Theorem 3.1. *Let f satisfy* $|f(t,v)-f(t,\overline{v})| \leq g(t,|v-\overline{v}|), \quad t \geq 0, \quad v,\overline{v} \in R$*, where* $g: R_+ \times R_+$ *is a continuous mapping such that* $r \to g(t,r)$ *is nondecreasing, the initial value problem*

$$u'(t) = g(t,u(t)), \quad u(0) = u_0, \tag{3.8}$$

has a solution on R_+ *for* $u_0 > 0$ *and that* $u(t) = 0$ *is the only solution of (3.8) for* $u_0 = 0$*. Then the fuzzy initial value problem (3.6) has a unique fuzzy solution.*

4 The Runge-Kutta Verner Method

Let the exact solution $[Y(t)]_r = [\underline{Y}(t;r), \overline{Y}(t;r)]$ is approximated by some $[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)]$. From (2.2),(2.3) we define

$$\left.\begin{aligned} \underline{y}(t_{n+1};r) - \underline{y}(t_n;r) &= \sum_{i=1}^{8} w_i \underline{k}_i(t_n, y(t_n;r)), \\ \overline{y}(t_{n+1};r) - \overline{y}(t_n;r) &= \sum_{i=1}^{8} w_i \overline{k}_i(t_n, y(t_n;r)), \end{aligned}\right\} \tag{4.9}$$

where the $w_i's$ are constants and

$$\left.\begin{aligned} &[k_i(t,y(t;r))]_r = \left[\underline{k}_i(t,y(t,r)), \overline{k}_i(t,y(t,r))\right], \qquad i = 1,2,3,4,5,6,7,8 \\ &\underline{k}_i(t,y(t,r)) = h.f(t_n + \alpha_i h, \underline{y}(t_n) + \textstyle\sum_{j=1}^{i-1} \beta_{i,j} \underline{k}_j(t_n, y(t_n;r))), \\ &\overline{k}_i(t,y(t,r)) = h.f(t_n + \alpha_i h, \overline{y}(t_n) + \textstyle\sum_{j=1}^{i-1} \beta_{i,j} \overline{k}_j(t_n, y(t_n;r))), \end{aligned}\right\} \tag{4.10}$$

$$\underline{k}_1(t,y(t;r)) = min\{h.f(t,u) | u \in [\underline{y}(t:r), \overline{y}(t;r)]\},$$

$$\overline{k}_1(t,y(t;r)) = max\{h.f(t,u) | u \in [\underline{y}(t:r), \overline{y}(t;r)]\},$$

$$\underline{k}_2(t,y(t;r)) = min\{h.f(t+h/6,u) | u \in [\underline{z}_1(t,y(t:r)), \overline{z}_1(t,y(t;r))]\},$$

$$\overline{k}_2(t,y(t;r)) = max\{h.f(t+h/6,u) | u \in [\underline{z}_1(t,y(t:r)), \overline{z}_1(t,y(t;r))]\},$$

$$\underline{k}_3(t,y(t;r)) = min\{h.f(t+4h/15,u) | u \in [\underline{z}_2(t,y(t:r)), \overline{z}_2(t,y(t;r))]\},$$

$$\overline{k}_3(t,y(t;r)) = max\{h.f(t+4h/15,u) | u \in [\underline{z}_2(t,y(t:r)), \overline{z}_2(t,y(t;r))]\}$$

$$\underline{k}_4(t,y(t;r)) = min\{h.f(t+2h/3,u) | u \in [\underline{z}_3(t,y(t:r)), \overline{z}_3(t,y(t;r))]\},$$

$$\overline{k}_4(t,y(t;r)) = max\{h.f(t+2h/3,u) | u \in [\underline{z}_3(t,y(t:r)), \overline{z}_3(t,y(t;r))]\}$$

$$\underline{k}_5(t,y(t;r)) = min\{h.f(t+5h/6,u) | u \in [\underline{z}_4(t,y(t:r)), \overline{z}_4(t,y(t;r))]\},$$

$$\overline{k}_5(t,y(t;r)) = max\{h.f(t+5h/6,u) | u \in [\underline{z}_4(t,y(t:r)), \overline{z}_4(t,y(t;r))]\}$$

$$\underline{k}_6(t,y(t;r)) = min\{h.f(t+h,u) | u \in [\underline{z}_5(t,y(t:r)), \overline{z}_5(t,y(t;r))]\},$$

$$\overline{k}_6(t,y(t;r)) = max\{h.f(t+h,u) | u \in [\underline{z}_5(t,y(t:r)), \overline{z}_5(t,y(t;r))]\}$$

$$\underline{k}_7(t,y(t;r)) = min\{h.f(t+h/15,u) | u \in [\underline{z}_6(t,y(t:r)), \overline{z}_6(t,y(t;r))]\},$$

$$\overline{k}_7(t,y(t;r)) = max\{h.f(t+h/15,u) | u \in [\underline{z}_6(t,y(t:r)), \overline{z}_6(t,y(t;r))]\}$$

$$\underline{k}_8(t,y(t;r)) = min\{h.f(t+h,u)|u \in [\underline{z}_7(t,y(t:r)),\overline{z}_7(t,y(t;r))]\},$$

$$\overline{k}_8(t,y(t;r)) = max\{h.f(t+h,u)|u \in [\underline{z}_7(t,y(t:r)),\overline{z}_7(t,y(t;r))]\}.$$

Now we define,

$$\underline{z}_1(t,y(t;r)) = \underline{y}(t;r) + \frac{1}{6}\underline{k}_1(t,y(t;r)),$$

$$\overline{z}_1(t,y(t;r)) = \overline{y}(t;r) + \frac{1}{6}\overline{k}_1(t,y(t;r)),$$

$$\underline{z}_2(t,y(t;r)) = \underline{y}(t;r) + \frac{4}{75}\underline{k}_1(t,y(t;r)) + \frac{16}{75}\underline{k}_2(t,y(t;r)),$$

$$\overline{z}_2(t,y(t;r)) = \overline{y}(t;r) + \frac{4}{75}\overline{k}_1(t,y(t;r)) + \frac{16}{75}\overline{k}_2(t,y(t;r)),$$

$$\underline{z}_3(t,y(t;r)) = \underline{y}(t;r) + \frac{5}{6}\underline{k}_1(t,y(t;r)) - \frac{8}{3}\underline{k}_2(t,y(t;r)) + \frac{5}{2}\underline{k}_3(t,y(t;r),$$

$$\overline{z}_3(t,y(t;r)) = \overline{y}(t;r) + \frac{5}{6}\overline{k}_1(t,y(t;r)) - \frac{8}{3}\overline{k}_2(t,y(t;r)) + \frac{5}{2}\overline{k}_3(t,y(t;r),$$

$$\underline{z}_4(t,y(t;r) = \underline{y}(t;r) - \frac{165}{64}\underline{k}_1(t,y(t;r)) + \frac{55}{6}\underline{k}_2(t,y(t;r))$$
$$- \frac{425}{64}\underline{k}_3(t,y(t;r) + \frac{85}{96}\underline{k}_4(t,y(t;r),$$

$$\overline{z}_4(t,y(t;r)) = \overline{y}(t;r) - \frac{165}{64}\overline{k}_1(t,y(t;r)) + \frac{55}{6}\overline{k}_2(t,y(t;r))$$
$$- \frac{425}{64}\overline{k}_3(t,y(t;r) + \frac{85}{96}\overline{k}_4(t,y(t;r),$$

$$\underline{z}_5(t,y(t;r)) = \underline{y}(t;r) + \frac{12}{5}\underline{k}_1(t,y(t;r)) - 8\underline{k}_2(t,y(t;r))$$
$$+ \frac{4015}{612}\underline{k}_3(t,y(t;r) - \frac{11}{36}\underline{k}_4(t,y(t;r)) + \frac{88}{255}\underline{k}_5(t,y(t;r),$$

$$\overline{z}_5(t,y(t;r)) = \overline{y}(t;r) + \frac{12}{5}\overline{k}_1(t,y(t;r)) - 8\overline{k}_2(t,y(t;r))$$
$$+ \frac{4015}{612}\overline{k}_3(t,y(t;r) - \frac{11}{36}\overline{k}_4(t,y(t;r)) + \frac{88}{255}\overline{k}_5(t,y(t;r),$$

$$\underline{z}_6(t,y(t;r)) = \underline{y}(t;r) - \frac{8263}{15000}\underline{k}_1(t,y(t;r)) + \frac{124}{75}\underline{k}_2(t,y(t;r))$$
$$- \frac{643}{680}\underline{k}_3(t,y(t;r) + \frac{81}{250}\underline{k}_4(t,y(t;r)) + \frac{2484}{10625}\underline{k}_5(t,y(t;r),$$

$$\overline{z}_6(t,y(t;r)) = \overline{y}(t;r) - \frac{8263}{15000}\overline{k}_2(t,y(t;r)) + \frac{124}{75}\overline{k}_2(t,y(t;r))$$
$$- \frac{643}{680}\overline{k}_3(t,y(t;r) + \frac{81}{250}\overline{k}_4(t,y(t;r)) + \frac{2484}{10625}\overline{k}_5(t,y(t;r),$$

$$\underline{z}_7(t,y(t;r)) = \underline{y}(t;r) + \frac{3501}{1720}\underline{k}_1(t,y(t;r)) - \frac{300}{43}\underline{k}_2(t,y(t;r))$$
$$+\frac{297275}{52632}\underline{k}_3(t,y(t;r) - \frac{319}{2322}\underline{k}_4(t,y(t;r))$$
$$+\frac{24068}{84065}\underline{k}_5(t,y(t;r) + \frac{3850}{26703}\underline{k}_7(t,y(t;r)),$$
$$\overline{z}_7(t,y(t;r)) = \overline{y}(t;r) + \frac{3501}{1720}\overline{k}_1(t,y(t;r)) - \frac{300}{43}\overline{k}_2(t,y(t;r))$$
$$+\frac{297275}{52632}\overline{k}_3(t,y(t;r) - \frac{319}{2322}\overline{k}_4(t,y(t;r))$$
$$+\frac{24068}{84065}\overline{k}_5(t,y(t;r)) + \frac{3850}{26703}\overline{k}_7(t,y(t;r)).$$

Next we define,

$$\left.\begin{aligned} F[t,y(t:r)] &= \frac{3}{40}\underline{k}_1(t,y(t;r)) + \frac{875}{2244}\underline{k}_3(t,y(t;r)) + \frac{23}{72}\underline{k}_4(t,y(t;r)) \\ &+\frac{264}{1955}\underline{k}_5(t,y(t;r)) + \frac{125}{11592}\underline{k}_7(t,y(t;r)) + \frac{43}{616}\underline{k}_8(t,y(t;r)), \\ G[t,y(t:r)] &= \frac{3}{40}\overline{k}_1(t,y(t;r)) + \frac{875}{2244}\overline{k}_3(t,y(t;r)) + \frac{23}{72}\overline{k}_4(t,y(t;r)) \\ &+\frac{264}{1955}\overline{k}_5(t,y(t;r)) + \frac{125}{11592}\overline{k}_7(t,y(t;r)) + \frac{43}{616}\overline{k}_8(t,y(t;r)) \end{aligned}\right\} \tag{4.11}$$

The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted by $[Y(t_n]_r = [\underline{Y}(t_n;r), \overline{Y}(t_n,r)]$ and $[\underline{y}(t_n;r), \overline{y}(t_n;r)]$, respectively. The solution is calculated by grid points at (3.6),(4.10) we have

$$\left.\begin{aligned} \underline{Y}(t_{n+1};r) &\approx \underline{Y}(t_n;r) + F[t_n, Y(t_n;r)], \\ \overline{Y}(t_{n+1};r) &\approx \overline{Y}(t_n;r) + G[t_n, Y(t_n;r)]. \end{aligned}\right\} \tag{4.12}$$

We define

$$\left.\begin{aligned} \underline{y}(t_{n+1};r) &= \underline{y}(t_n;r) + F[t_n, Y(t_n;r)], \\ \overline{y}(t_{n+1};r) &= \overline{y}(t_n;r) + G[t_n, Y(t_n;r)]. \end{aligned}\right\} \tag{4.13}$$

The following lemmas will be applied to show convergence of these approximates i.e.,

$$\lim_{h\to 0}\underline{y}(t;r) = \underline{Y}(t;r), \quad \lim_{h\to 0}\overline{y}(t;r) = \overline{Y}(t;r).$$

Lemma 4.1. *Let the sequence of numbers* $\{W_n\}_{n=0}^{N}$ *satisfy*

$$|W_{n+1}| \leq A|W_n| + B, \qquad 0 \leq n \leq N-1,$$

for some given positive constants A and B . Then

$$|W_n| \leq A^n|W_0| + B\frac{A^n-1}{A-1}, \qquad 0 \leq n \leq N$$

The proof, using mathematical induction is straight forward.

Lemma 4.2. *Let the sequence of numbers* $\{W_n\}_{n=0}^N, \{V_n\}_{n=0}^N$ *satisfy*

$$\left.\begin{aligned} |W_{n+1}| &\leq |W_n| + A.max\{|W_n|, |V_n|\} + B, \\ |V_{n+1}| &\leq |V_n| + A.max\{|W_n|, |V_n|\} + B. \end{aligned}\right\} \tag{4.14}$$

For some given positive constants A and B, and denote

$$U_n = |W_n| + |V_n|, \qquad 0 \leq n \leq N.$$

Then

$$U_n \leq \overline{A}^n U_0 + \overline{B}\frac{\overline{A}^{n-1}}{\overline{A}-1}, \qquad 0 \leq n \leq N$$

where $\overline{A} = 1 + 2A$ *and* $\overline{B} = 2B$.

Proof. By virtue of equation (4.13) we get

$$\begin{aligned} |W_{n+1}| + |V_{n+1}| &\leq |W_n| + |V_n| + 2A(|W_n| + |V_n|) + 2B \\ &= (1+2A)(|W_n| + |V_n|) + 2B \end{aligned}$$

and by applying Lemma 4.1 for $U_n,\ 0 \leq n \leq N$ we conclude equation (4.12) is valid.
Our next result determines the pointwise convergence of the Runge-Kutta Verner approximates to the exact solution. Let $F(t,u,v)$ and $G(t,u,v)$ are obtained by substituting $[y(t)]_r = [u,v]$ in (4.10),

$$\begin{aligned} F[t, y(t:r)] = {} & \frac{3}{40}\underline{k}_1(t, y(t;r)) + \frac{875}{2244}\underline{k}_3(t, y(t;r)) + \frac{23}{72}\underline{k}_4(t, y(t;r)) \\ & + \frac{264}{1955}\underline{k}_5(t, y(t;r)) + \frac{125}{11592}\underline{k}_7(t, y(t;r)) + \frac{43}{616}\underline{k}_8(t, y(t;r)), \\ G[t, y(t:r)] = {} & \frac{3}{40}\overline{k}_1(t, y(t;r)) + \frac{875}{2244}\overline{k}_3(t, y(t;r)) + \frac{23}{72}\overline{k}_4(t, y(t;r)) \\ & + \frac{264}{1955}\overline{k}_5(t, y(t;r)) + \frac{125}{11592}\overline{k}_7(t, y(t;r)) + \frac{43}{616}\overline{k}_8(t, y(t;r)) \end{aligned}$$

The domain where F and G are defined is therefore

$$K = \{(t,u,v) | 0 \leq t \leq T, \quad -\infty < v < \infty, \ -\infty < u < v\}$$

□

Theorem 4.1. *Let F(t,u,v) and G(t,u,v) belong to* $C^8(K)$ *and let the partial derivatives of F and G be bounded over K. Then, for arbitary fixed* $r, 0 \leq r \leq 1$, *the approximately solutions (4.12) converge to the exact solutions* $\underline{Y}(t;r)$ *and* $\overline{Y}(t;r)$ *uniformly in t.*

Proof. It is sufficient to show

$$\lim_{h\to 0}\underline{y}(t_N;r) = \underline{Y}(t_N;r),$$
$$\lim_{h\to 0}\overline{y}(t_N;r) = \overline{Y}(t_N;r).$$

where $t_N = T$. For $n = 0, 1, \ldots.., N-1$, by using Taylor theorem we get

$$\left.\begin{aligned} \underline{Y}(t_{n+1};r) &= \underline{Y}(t_n;r) + F[(t_n, \underline{Y}(t_n;r), \overline{Y}(t_n;r)] + \frac{4h^8}{9}ML^6 + O(h^9), \\ \overline{Y}(t_{n+1};r) &= \overline{Y}(t_n;r) + G[(t_n, \underline{Y}(t_n;r), \overline{Y}(t_n;r)] + \frac{4h^8}{9}ML^6 + O(h^9), \end{aligned}\right\} \tag{4.15}$$

denote

$$W_n = \underline{Y}(t_n;r) - \underline{y}(t_n;r), V_n = \overline{Y}(t_n;r) - \overline{y}(t_n;r).$$

Hence from (4.12) and (4.14)

$$W_{n+1} = W_n + \{F[(t_n, \underline{Y}(t_n;r), \overline{Y}(t_n;r)] - F[(t_n, \underline{y}(t_n;r), \overline{y}(t_n;r)]\} + \frac{4h^8}{9}ML^6 + O(h^9),$$

$$V_{n+1} = V_n + \{G[(t_n, \underline{Y}(t_n;r), \overline{Y}(t_n;r)] - G[(t_n, \underline{y}(t_n;r), \overline{y}(t_n;r)]\} + \frac{4h^8}{9}ML^6 + O(h^9),$$

Then

$$|W_{n+1}| \leq |W_n| + 2Lh.max\{|W_n, |V_n|\} + \frac{4}{9}h^8ML^7 + O(h^9),$$

$$|V_{n+1}| \leq |V_n| + 2Lh.max\{|W_n, |V_n|\} + \frac{4}{9}h^8ML^8 + O(h^9),$$

for $t \in [0,T]$ and $L > 0$ is bound for the partial derivatives of F and G. Thus by lemma 4.2

$$|W_n| \leq (1+4Lh)^n|U_0| + \left(\tfrac{8}{9}h^8ML^8 + O(h^9)\right)\tfrac{(1+4Lh)^n-1}{4Lh},$$

$$|V_n| \leq (1+4Lh)^n|U_0| + \left(\tfrac{8}{9}h^8ML^8 + O(h^9)\right)\tfrac{(1+4Lh)^n-1}{4Lh},$$

where $|U_0| = |W_0| + |V_0|$. In particular

$$|W_n| \leq (1+4Lh)^N|U_0| + \left(\tfrac{4}{3}h^8ML^7 + O(h^8)\right)\tfrac{(1+4Lh)^{\frac{T}{h}}-1}{4Lh},$$

$$|V_n| \leq (1+4Lh)^N|U_0| + \left(\tfrac{4}{3}h^8ML^7 + O(h^8)\right)\tfrac{(1+4Lh)^{\frac{T}{h}}-1}{4Lh},$$

Since $W_0 = V_0 = 0$, we obtain

$$|W_N| \leq \left(\frac{4}{3}h^7ML^8 + O(h)\right)\frac{e^{4LT}-1}{L}h^6 + O(h^8),$$

$$|V_N| \leq \left(\frac{4}{3}h^7ML^8 + O(h)\right)\frac{e^{4LT}-1}{L}h^6 + O(h^8),$$

and if $h \to 0$ we get $W_N \to 0$ and $V_N \to 0$ which completes the proof. □

5 Numerical Examples

Example 5.1. *Consider the fuzzy initial value problem,*

$$\begin{cases} y'(t) = y(t), & t \in I = [0,1], \\ y(0) = (0.75 + 0.25r, 1.125 - 0.125r), & 0 < r \leq 1. \end{cases}$$

By using the Runge-Kutta Verner method, we have

$$\underline{y}(t_{n+1};r) = \underline{y}(t_n;r)\left[1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \frac{h^5}{120} + \frac{h^6}{720} + \frac{h^7}{5400}\right]$$

$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r)\left[1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \frac{h^5}{120} + \frac{h^6}{720} + \frac{h^7}{5400}\right]$$

The exact solution is given by $\underline{Y}(t;r)=\underline{y}(0;r)e^{t}, \quad \overline{Y}(t;r)=\overline{y}(0;r)e^{t}$, *which at* $t=1$,

$$Y(1;r)=[(0.75+0.25r)e,(1.125-0.125r)e], \ \ 0<r\leq 1$$

The exact and approximate solutions by Runge-Kutta Verner method plotted at $t=1$ *in figure 1*

Table 1

r	RK-Order five		RK-Verner		Exact Solution	
	$\underline{y}(t_i;r)$	$\overline{y}(t_i;r)$	$\underline{y}(t_i;r)$	$\overline{y}(t_i;r)$	$\underline{Y}(t_i;r)$	$\overline{Y}(t_i;r)$
0.1	2.106497822	3.023843648	2.106667891	3.024087779	2.106668417	3.024088534
0.2	2.174449365	2.989867880	2.174624920	2.990109265	2.174625463	2.990110011
0.3	2.242400908	2.955892105	2.242581948	2.956130750	2.242582508	2.956131488
0.4	2.310352450	2.921916334	2.310538977	2.922152236	2.310539554	2.922152966
0.5	2.378303992	2.887940563	2.378496006	2.888173721	2.378496600	2.888174443
0.6	2.446255535	2.853964791	2.446453035	2.854195207	2.446453646	2.854195920
0.7	2.514207078	2.819989020	2.514410063	2.820216693	2.514410691	2.820217397
0.8	2.582158620	2.786013249	2.582367092	2.786238178	2.582367737	2.786238874
0.9	2.650110163	2.752037477	2.650324121	2.752259664	2.650324783	2.752260351
1.0	2.718061706	2.718061706	2.718281150	2.718281150	2.718281828	2.718281828

Table 2

r	Error in RK-Order five		Error in RK-Verner	
	$\underline{y}(t_i;r)$	$\overline{y}(t_i;r)$	$\underline{Y}(t_i;r)$	$\overline{Y}(t_i;r)$
0.1	1.86×10^{-4}	2.45×10^{-4}	7.55×10^{-7}	7.55×10^{-7}
0.2	1.76×10^{-4}	2.42×10^{-4}	5.43×10^{-7}	7.46×10^{-7}
0.3	1.82×10^{-4}	2.39×10^{-4}	5.60×10^{-7}	7.38×10^{-7}
0.4	1.87×10^{-4}	2.37×10^{-4}	5.67×10^{-7}	7.30×10^{-7}
0.5	1.92×10^{-4}	2.34×10^{-4}	5.94×10^{-7}	7.22×10^{-7}
0.6	1.98×10^{-4}	2.31×10^{-4}	6.11×10^{-7}	7.13×10^{-7}
0.7	2.04×10^{-4}	2.28×10^{-4}	6.28×10^{-7}	7.04×10^{-7}
0.8	2.09×10^{-4}	2.26×10^{-4}	6.45×10^{-7}	6.96×10^{-7}
0.9	2.15×10^{-4}	2.23×10^{-4}	6.62×10^{-7}	6.87×10^{-7}
1.0	2.20×10^{-4}	2.20×10^{-4}	6.781×10^{-7}	6.78×10^{-7}

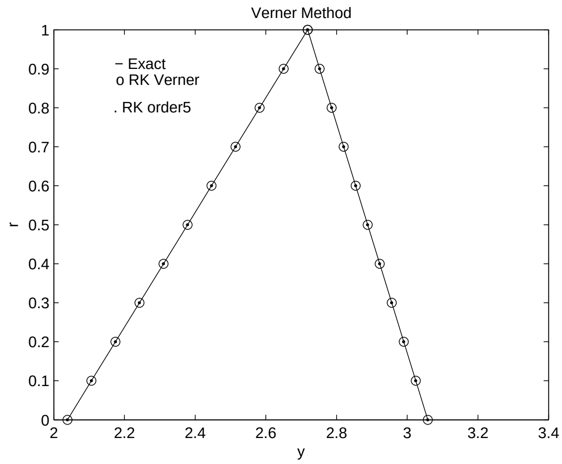

Figure 1: h=0.5

Example 5.2. *Consider the fuzzy intial value problem,*

$$y'(t) = c_1 y^2(t) + c_2, \qquad y(0) = 0,$$

Where $c_i > 0$, for $i = 1,2$ are triangular fuzzy numbers, The exact solution is given by

$$Y_1(t;r) = l_1(r)tan(w_1(r)t), \;\; Y_2(t;r) = l_2(r)tan(w_2(r)t),$$

with

$$l_1(r) = \sqrt{c_{2,1}(r)/c_{1,1}(r)}, \;\; l_2(r) = \sqrt{c_{2,2}(r)/c_{1,2}(r)}$$

$$w_1(r) = \sqrt{c_{1,1}(r)/c_{2,1}(r)}, \;\; w_2(r) = \sqrt{c_{1,2}(r)/c_{2,2}(r)}$$

$$[c_1]_r = [0.5 + 0.5r, \;\; 1.5 - 0.5r] and \; [c_2]_r = [0.75 + 0.25r, \;\; 1.25 - 0.25r]$$

The $r-$level sets of $y'(t)$ are

$$Y_1'(t;r) = c_{2,1}(r)sec^2(w_1(r)t), \quad Y_2'(t;r) = c_{2,2}(r)sec^2(w_2(r)t),$$

which defines a fuzzy number. We have

$$f_1(t,y;r) = min\{c_1u^2 + c_2 | u \in [y_1(t;r), y_2(t;r)], \;\; c_1 \in [c_{1,1}(r), \; c_{1,2}(r)], \;\; c_2 \in [c_{2,1}(r), \; c_{2,2}(r)]\}$$

$$f_2(t,y;r) = max\{c_1u^2 + c_2 | u \in [y_1(t;r), y_2(t;r)], \;\; c_1 \in [c_{1,1}(r), \; c_{1,2}(r)], \;\; c_2 \in [c_{2,1}(r), \; c_{2,2}(r)]\}$$

By Runge-Kutta Verner method at t_n, $0 \leq n \leq N$

$$\underline{k}_1(t_n;r) = h(\underline{c}_1(r).\underline{y}_1^2(t_n;r) + \underline{c}_2(r)), \quad \overline{k}_1(t_n;r) = h(\overline{c}_1(r).\overline{y}_1^2(t_n;r) + \overline{c}_2(r)),$$

$$\underline{k}_2(t_n;r) = h(\underline{c}_1(r).\underline{z}_1^2(t_n;r) + \underline{c}_2(r)), \quad \overline{k}_2(t_n;r) = h(\overline{c}_1(r).\overline{z}_1^2(t_n;r) + \overline{c}_2(r)),$$

$$\underline{k}_3(t_n;r) = h(\underline{c}_1(r).\underline{z}_2^2(t_n;r) + \underline{c}_2(r)), \quad \overline{k}_3(t_n;r) = h(\overline{c}_1(r).\overline{z}_2^2(t_n;r) + \overline{c}_2(r)),$$

$$\underline{k}_4(t_n;r) = h(\underline{c}_1(r).\underline{z}_3^2(t_n;r) + \underline{c}_2(r)), \quad \overline{k}_4(t_n;r) = h(\overline{c}_1(r).\overline{z}_3^2(t_n;r) + \overline{c}_2(r)),$$

$$\underline{k}_5(t_n;r) = h(\underline{c}_1(r).\underline{z}_4^2(t_n;r) + \underline{c}_2(r)), \quad \overline{k}_5(t_n;r) = h(\overline{c}_1(r).\overline{z}_4^2(t_n;r) + \overline{c}_2(r)),$$

$$\underline{k}_6(t_n;r) = h(\underline{c}_1(r).\underline{z}_5^2(t_n;r) + \underline{c}_2(r)), \quad \overline{k}_6(t_n;r) = h(\overline{c}_1(r).\overline{z}_5^2(t_n;r) + \overline{c}_2(r)),$$

$$\underline{k}_7(t_n;r) = h(\underline{c}_1(r).\underline{z}_6^2(t_n;r) + \underline{c}_2(r)), \quad \overline{k}_7(t_n;r) = h(\overline{c}_1(r).\overline{z}_6^2(t_n;r) + \overline{c}_2(r)),$$

$$\underline{k}_8(t_n;r) = h(\underline{c}_1(r).\underline{z}_7^2(t_n;r) + \underline{c}_2(r)), \quad \overline{k}_8(t_n;r) = h(\overline{c}_1(r).\overline{z}_7^2(t_n;r) + \overline{c}_2(r))$$

where

$$\underline{z}_1(t,y(t;r)) = \underline{y}(t;r) + \frac{1}{6}\underline{k}_1(t,y(t;r)), \quad \overline{z}_1(t,y(t;r)) = \overline{y}(t;r) + \frac{1}{6}\overline{k}_1(t,y(t;r)),$$

$$\underline{z}_2(t,y(t;r)) = \underline{y}(t;r) + \frac{4}{75}\underline{k}_1(t,y(t;r)) + \frac{16}{75}\underline{k}_2(t,y(t;r)),$$

$$\overline{z}_2(t,y(t;r)) = \overline{y}(t;r) + \frac{4}{75}\overline{k}_1(t,y(t;r)) + \frac{16}{75}\overline{k}_2(t,y(t;r)),$$

$$\underline{z}_3(t,y(t;r)) = \underline{y}(t;r) + \frac{5}{6}\underline{k}_1(t,y(t;r)) - \frac{8}{3}\underline{k}_2(t,y(t;r)) + \frac{5}{2}\underline{k}_3(t,y(t;r),$$

$$\overline{z}_3(t,y(t;r)) = \overline{y}(t;r) + \frac{5}{6}\overline{k}_1(t,y(t;r)) - \frac{8}{3}\overline{k}_2(t,y(t;r)) + \frac{5}{2}\overline{k}_3(t,y(t;r),$$

$$\underline{z}_4(t,y(t;r) = \underline{y}(t;r) - \frac{165}{64}\underline{k}_1(t,y(t;r)) + \frac{55}{6}\underline{k}_2(t,y(t;r)) - \frac{425}{64}\underline{k}_3(t,y(t;r) + \frac{85}{96}\underline{k}_4(t,y(t;r),$$

$$\overline{z}_4(t,y(t;r)) = \overline{y}(t;r) - \frac{165}{64}\overline{k}_1(t,y(t;r)) + \frac{55}{6}\overline{k}_2(t,y(t;r)) - \frac{425}{64}\overline{k}_3(t,y(t;r) + \frac{85}{96}\overline{k}_4(t,y(t;r),$$

$$\underline{z}_5(t,y(t;r)) = \underline{y}(t;r) + \frac{12}{5}\underline{k}_1(t,y(t;r)) - 8\underline{k}_2(t,y(t;r)) + \frac{4015}{612}\underline{k}_3(t,y(t;r) - \frac{11}{36}\underline{k}_4(t,y(t;r)) + \frac{88}{255}\underline{k}_5(t,y(t;r),$$

$$\overline{z}_5(t,y(t;r)) = \overline{y}(t;r) + \frac{12}{5}\overline{k}_1(t,y(t;r)) - 8\overline{k}_2(t,y(t;r)) + \frac{4015}{612}\overline{k}_3(t,y(t;r) - \frac{11}{36}\overline{k}_4(t,y(t;r)) + \frac{88}{255}\overline{k}_5(t,y(t;r),$$

$$\underline{z}_6(t,y(t;r)) = \underline{y}(t;r) - \frac{8263}{15000}\underline{k}_1(t,y(t;r)) + \frac{124}{75}\underline{k}_2(t,y(t;r)) - \frac{643}{680}\underline{k}_3(t,y(t;r)$$
$$+\frac{81}{250}\underline{k}_4(t,y(t;r)) + \frac{2484}{10625}\underline{k}_5(t,y(t;r),$$

$$\bar{z}_6(t,y(t;r)) = \bar{y}(t;r) - \frac{8263}{15000}\bar{k}_2(t,y(t;r)) + \frac{124}{75}\bar{k}_2(t,y(t;r)) - \frac{643}{680}\bar{k}_3(t,y(t;r)$$
$$+\frac{81}{250}\bar{k}_4(t,y(t;r)) + \frac{2484}{10625}\bar{k}_5(t,y(t;r),$$

$$\underline{z}_7(t,y(t;r)) = \underline{y}(t;r) + \frac{3501}{1720}\underline{k}_1(t,y(t;r)) - \frac{300}{43}\underline{k}_2(t,y(t;r)) + \frac{297275}{52632}\underline{k}_3(t,y(t;r) - \frac{319}{2322}\underline{k}_4(t,y(t;r))$$
$$+\frac{24068}{84065}\underline{k}_5(t,y(t;r) + \frac{3850}{26703}\underline{k}_7(t,y(t;r)),$$

$$\bar{z}_7(t,y(t;r)) = \bar{y}(t;r) + \frac{3501}{1720}\bar{k}_1(t,y(t;r)) - \frac{300}{43}\bar{k}_2(t,y(t;r)) + \frac{297275}{52632}\bar{k}_3(t,y(t;r) - \frac{319}{2322}\bar{k}_4(t,y(t;r))$$
$$+\frac{24068}{84065}\bar{k}_5(t,y(t;r)) + \frac{3850}{26703}\bar{k}_7(t,y(t;r)).$$

Table 3

r	RK-Order five		RK-Verner		Exact Solution	
	$\underline{y}(t_i;r)$	$\bar{y}(t_i;r)$	$\underline{y}(t_i;r)$	$\bar{y}(t_i;r)$	$\underline{Y}(t_i;r)$	$\overline{Y}(t_i;r)$
0.1	0.907765412	3.51404262	0.907804578	3.788167313	0.907804620	3.788162443
0.2	0.958432140	3.12243215	0.958507321	3.404672221	0.958503811	3.285743458
0.3	1.012764215	2.80144321	1.021876521	2.899150750	1.012872911	2.899145990
0.4	1.071436752	2.53145312	1.071542988	2.591950700	1.071539268	2.591943937
0.5	1.134446235	2.30385428	1.134835440	2.341537913	1.134831610	2.341533263
0.6	1.203435760	2.10964426	1.203810023	2.133147941	1.203806083	2.133143341
0.7	1.278467850	1.94187396	1.279377801	1.956718228	1.279373751	1.956713678
0.8	1.361046321	1.79567350	1.362504321	1.805159032	1.362500161	1.805154542
0.9	1.452445321	1.66745431	1.454513787	1.673688345	1.454509517	1.673683905
1.0	1.553740225	1.553740225	1.557412106	1.557412106	1.557407725	1.557407725

Table 4

r	Error in RK-Order five		Error in RK-Verner	
	$\underline{y}(t_i;r)$	$\overline{y}(t_i;r)$	$\underline{Y}(t_i;r)$	$\overline{Y}(t_i;r)$
0.1	3.92×10^{-5}	2.74×10^{-1}	3.40×10^{-6}	4.87×10^{-6}
0.2	1.04×10^{-4}	1.63×10^{-1}	3.51×10^{-6}	4.82×10^{-6}
0.3	1.08×10^{-4}	9.77×10^{-2}	3.61×10^{-6}	4.76×10^{-6}
0.4	1.12×10^{-4}	6.05×10^{-2}	3.72×10^{-6}	4.71×10^{-6}
0.5	1.84×10^{-4}	3.76×10^{-2}	3.83×10^{-6}	4.65×10^{-6}
0.6	3.85×10^{-4}	2.35×10^{-2}	3.94×10^{-6}	4.60×10^{-6}
0.7	3.05×10^{-4}	1.48×10^{-2}	4.05×10^{-6}	4.55×10^{-6}
0.8	4.10×10^{-4}	9.45×10^{-3}	4.16×10^{-6}	4.49×10^{-6}
0.9	2.06×10^{-3}	6.28×10^{-3}	4.27×10^{-6}	4.44×10^{-6}
1.0	3.70×10^{-3}	3.70×10^{-3}	4.38×10^{-6}	4.38×10^{-6}

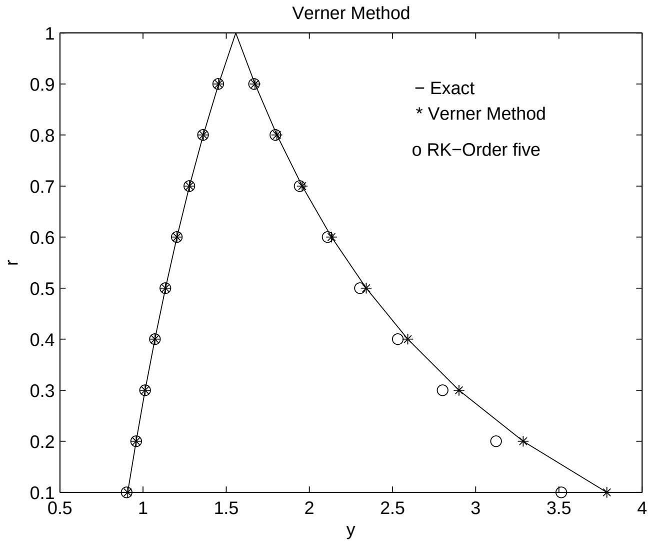

Figure 2: h=0.5

6 Conclusion

In this paper we have applied iterative solution of *Runge-Kutta Verner method* for numerical solution of fuzzy differential equations. It is clear that the method introduced in this paper with $O(h^8)$ performs better than *Runge-Kutta method of order five* $O(h^5)$

References

[1] S. Abbasbandy, T. Allahviranloo, Numerical solution of fuzzy differential equation by Taylor method, Journal of Computational Methods in Applied mathematics, 2 (2002) 113-124.
http://dx.doi.org/10.2478/cmam-2002-0006

[2] S. Abbasbandy, T. Allahviranloo, Numerical solution of fuzzy differential equation, Mathematical and Computational Applications, 7 (2002) 41-52.

[3] S. Abbasbandy, T. Allahviranloo, Numerical Solution of fuzzy differential equation by Runge-Kutta Method, Nonliner Studies, 11 (2004) 117-129.

[4] T. Allahviranloo, Numerical solution of fuzzy differential equations by Adams-Bashforth two-step method, Journal of Applied Mathematics Islamic Azad University Lahijan, (2004) 36-47.

[5] T. Allahviranloo, T. Ahmady, E. Ahmady, Numerical solution of fuzzy differential equations by Predictor-Corrector method, Information Sciences, 177 (2007) 1633-1647.
http://dx.doi.org/10.1016/j.ins.2006.09.015

[6] S. L. Chang, L. A. Zadeh, On fuzzy mapping and control, IEEE Trans.systems Man Cybernet, 2 (1972) 30-34.

[7] D. Dubois, H. Prade, Towards fuzzy differential calculus, Part 3.Differention, Fuzzy Sets and System, 8 (1982) 225-233.
http://dx.doi.org/10.1016/S0165-0114(82)80001-8

[8] M. L. Puri, D. A. Ralescu, Differential of fuzzy function, J. Math. Anal. Appl, 9 (1983) 321-325.

[9] R. Goetschel, Woxman, Elementary fuzzy calculus, Fuzzy Sets and Systems, 18 (1986) 31-43.
http://dx.doi.org/10.1016/0165-0114(86)90026-6

[10] T. Jayakumar, K. Kanakarajan, Numerical solution for hybrid fuzzy system by improved Euler method, International Journal of Applied Mathematical Science, 38 (2012) 1847-1862.

[11] T. Jayakumar, K. Kanakarajan, S. Indrakumar, Numerical solution of N^{th}-order fuzzy differential equation by Runge-Kutta Nystrom method, International Journal of Mathematical Engineering and Science, 1 (5) (2012) 1-13.

[12] T. Jayakumar, K. Kanakarajan, S. Indrakumar, Numerical Solution of N^{th}-Order Fuzzy Differential Equation by Runge-KuttaMethod of Order Five, International Journal of Mathematical Analysics, 6 (58) (2012) 2885-2896.

[13] T. Jayakumar, D. Maheshkumar, K. Kanagarajan, Numerical solution of fuzzy differential equations by Runge-Kutta method of order five, International Journal of Applied Mathematical Science, 6 (2012) 2989-3002.

[14] T. Jayakumar, K. Kanagarajan, Numerical solution for hybrid fuzzy system by Runge-Kutta method of order five, International Journal of Applied Mathematical Science, 6 (2012) 3591-3606.

[15] T. Jayakumar, K. Kanagarajan, Numerical solution for hybrid fuzzy system by Runge-Kutta Fehberg method, International Journal of Mathematical Analysics, 6 (53) (2012) 2619-2632.

[16] T. Jayakumar, K. Kanakarajan, S. Indrakumar, Numerical solution for hybrid fuzzy systems by Runge-Kutta Heun method, Far East Journal of Mathematical Sciences, 76 (2013) 205-222.

[17] T. Jayakumar, K. Kanagarajan, Numerical solution for hybrid fuzzy system by Milne's fourth order predictor-corrector method, International Journal of Mathematical fourm, 9 (6) (2014) 273-289.

[18] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems, 24 (1987) 301-317.
http://dx.doi.org/10.1016/0165-0114(87)90029-7

[19] O. Kaleva, The Cauchy problem for fuzzy differential equations, Fuzzy Sets and Systems, 35 (1990) 389-386.
http://dx.doi.org/10.1016/0165-0114(90)90010-4

[20] K. Kanagarajan, M. Sambath, Numerical solution for fuzzy differential equations by third order Runge-Kutta method, International Journal of Applied Mathematics and Computation, 2 (4) (2010) 1-10.

[21] V. Lakshmikantham, R. N. Mohapatra, Theory of fuzzy differential equations and inclusions, Taylor and Francis, United Kingdom, (2003).
http://dx.doi.org/10.1201/9780203011386

[22] M. Ma, M. Friedman, A. Kandel, Numerical solutions of fuzzy differential equations, Fuzzy Sets and Systems, 105 (1999) 133-138.
http://dx.doi.org/10.1016/S0165-0114(97)00233-9

[23] J. J. Nieto, R. Rodriguez-lopez, Bounded solutions for fuzzy differential and integral equations, chaos, Solitons and Fractals, 27 (2006) 1376-1386.
http://dx.doi.org/10.1016/j.chaos.2005.05.012

[24] A. Ralston, P. Rabinowitz, First course in numerical analysis, (1978).

[25] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems, 24 (1987) 319-330.
http://dx.doi.org/10.1016/0165-0114(87)90030-3