Complex-valued travelling wave solutions to the Casimir equation for the Ito system via the first integral method

M. Hajiketabi¹, R. A. Van Gorder², E. Shivanian¹*
(1) Department of Mathematics, Imam Khomeini International University, Ghazvin, 34149-16818, Iran
(2) Department of Mathematics, University of Central Florida, Orlando, Florida 32816, USA

Abstract
The Ito system was previously been shown to admit a reduction to a single nonlinear Casimir equation. In the present paper, we reduce this nonlinear partial differential equation into an ordinary differential equation governing a travelling wave solution. The ordinary differential equation takes the form of a second order nonlinear equation, and the form of this nonlinearity is a rational function. As such, the nonlinearity can become singular. This makes the problem interesting and somewhat challenging to solve for standard methods. Therefore, we make use of the first integral method in order to obtain exact solutions for this equation of second order and thus obtain exact solutions to the Casimir equation for the Ito system. The first integral method actually allows one to account for multiple solutions, and we demonstrate that there indeed can exist multiple complex-valued solutions for this Casimir equation. The results demonstrate the utility of the first integral method, suggesting that the approach can be used to solve many nonlinear differential equations that may admit multiple solutions.

Keywords: Ito system; Nonlinear Casimir equation; First integral method; Nonlinear waves; Multiple solutions.

1 Introduction
The Ito system [1] reads
\[
\begin{aligned}
\frac{\partial U}{\partial t} &= \frac{\partial^3 U}{\partial x^3} + 3U \frac{\partial U}{\partial x} + V \frac{\partial V}{\partial x}, \\
\frac{\partial V}{\partial t} &= \frac{\partial}{\partial x}(UV).
\end{aligned}
\]

(1.1)

Ito showed that the system (1.1) is highly symmetric and possesses infinitely many conservation laws. It is an extension of the KdV equation, with an additional field variable $V$. Olver and Rosenau [2] introduced a dual bi-Hamiltonian system for the Ito system, which admits a Casimir functional and associated Casimir equations. Introducing a stream function for the Casimir equations, they then obtained the single partial differential equation
\[
\frac{\partial^2 w}{\partial t^2} = \left( \frac{\partial w}{\partial t} \right)^2 \frac{\partial}{\partial x} \left[ \left( \frac{\partial w}{\partial t} \right)^2 \left( \frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} \right) \right].
\]

(1.2)

*Corresponding author. Email address: e_shivanian@yahoo.com, Tel: +982833901386
While the Casimir equation (1.2) for the Ito system (1.1) has received relatively little attention in the recent literature, it is an interesting nonlinear partial differential equation related to an extension of the KdV equation. Despite being highly nonlinear, as shown in Van Gorder [3] it has been shown to admit a variety of exact and analytical solutions. Van Gorder [3] obtained analytic solutions to this equation, which consist of explicit exact solutions in some cases and implicit integral relations in others. Then, Hausserman and Van Gorder [4] attempted to classify all possible series solutions to (i) travelling wave reductions of (1.2) and (ii) a class of self-similar reductions to (1.2). Some asymptotic solutions were also given.

The first integral method which is based on the ring theory of commutative algebra was first introduced in Feng [5] for finding the exact 1-soliton solutions of the Burger-KdV equation. This method was further developed by the same author in [6]-[10] and some other mathematicians [11]-[24]. The method is accurate, efficient, trustworthy and does not require complicated and tedious computations. The basic idea of this method is to construct a first integral with polynomial coefficients of an explicit form to an equivalent autonomous planar system by using the division theorem. The method provides exact and explicit solutions. This fact is important when looking for multiple solutions. Often times, numerical methods or analytical approaches will only find a single solution. However, since the first integral method gives us a way to find exact solutions, one will be able to find multiple solutions (if they do exist).

The aim of this paper is to find exact complex-valued solutions of the nonlinear Casimir equation (1.2) by use of the first integral method. The previous analytical or numerical solutions were primarily concerned with real-valued solutions. By using the first integral method, however, we are able to study the existence of complex-valued solutions (which can be harder to find numerically). One benefit to the approach is that we are able to find multiple solutions when they exist. Indeed, we are able to show that the nonlinear Casimir equation (1.2) admits multiple complex solutions for a given parameter set.

2 A description of the first integral method

Consider a general nonlinear partial differential equation (PDE) in the form

\[ P(u, u_x, u_t, u_{xx}, u_{xt}, u_{xxx}, \ldots) = 0. \]  

(2.3)

Assume (2.3) has the traveling wave solution as

\[ u(t,x) = u(\xi), \quad \xi = x - ct, \]  

(2.4)

where \( c \) is wave velocity. Substituting (2.4) into Eq. (2.3) yields into the following ordinary differential equation (ODE):

\[ Q(u, u', u'', u''', \ldots) = 0, \]  

(2.5)

where prime denotes the derivative with respect to the same variable \( \xi \). Next, we introduce a new independent variable

\[ x(\xi) = u(\xi), \quad y(\xi) = u'(\xi), \]  

(2.6)

which change (2.5) to a system of ODEs

\[ \begin{cases} 
  x'(\xi) = y(\xi), \\
  y'(\xi) = f(x(\xi), y(\xi)). 
\end{cases} \]  

(2.7)

According to the qualitative theory of differential equations, if one can find two first integrals to System (2.7) under the same conditions, then analytic solutions to (2.7) can be solved directly. However, in general, it is difficult to realize this even for a single first integral, because for a given plane autonomous system, there is no general theory telling us how to find its first integrals in a systematic way. We shall apply the Division Theorem to obtain one first integral to (2.7) which reduces (2.5) to a first-order integrable ordinary differential equation. An exact solution to (2.6) is then obtained by solving this equation. Now, let us recall the Division Theorem for two variables in the complex domain \( \mathbb{C} \) [5].

Division Theorem. Suppose that \( P(x,y) \) and \( Q(x,y) \) are polynomials of two variables \( x \) and \( y \) in \( \mathbb{C}[x,y] \) and \( P[x,y] \) is irreducible in \( \mathbb{C}[x,y] \). If \( Q[x,y] \) vanishes at all zero points of \( P[x,y] \), then there exists a polynomial \( G[x,y] \) in \( \mathbb{C}[x,y] \) such that \( Q[x,y] = P[x,y]G[x,y] \).
3 Complex-valued solutions for the nonlinear Casimir equation

We will be interested in solving the travelling wave problem for the Casimir equation (1.2). To this end, let us assume a solution of the form $w(x,t) = \phi(z)$ where $z = x - \beta^{-1/2}t$ is the wave variable and $\beta > 0$ is a constant. Then Eq. (1.2) becomes

$$\phi'' = \frac{1}{\beta} \phi'^2 \left[ \phi' (\phi' + \epsilon \phi'') \right]'$$  \hspace{1cm} (3.8)

where prime denotes differentiation with respect to $z$ and either $\epsilon = 1$ or $\epsilon = -1$ depending on the sign ($+$ or $-$) in (1.2). We put $\epsilon = 1$ (this corresponds to the interesting solutions discussed in [4]). Rearranging (3.8) and introducing a new function $f(z) = \phi'(z)$, we have

$$f' = \frac{1}{\beta} \left[ f^2 (f + \epsilon f'') \right]'$$  \hspace{1cm} (3.9)

and performing one integration with respect to $z$, we have

$$\alpha - \frac{\beta}{f} = \left[ f^2 (f + \epsilon f'') \right]$$  \hspace{1cm} (3.10)

where $\alpha$ is a constant of integration. If we rearrange (3.10) in terms of the second derivative, we find

$$f'' = -f' + \alpha f - \beta,$$  \hspace{1cm} (3.11)

which is a second order nonlinear equation. Next, in order to use the first integral method we introduce new independent variables

$$x(z) = f(z), \quad y(z) = f'(z),$$  \hspace{1cm} (3.12)

which change Eq. (3.11) to a system of ODEs

$$\begin{cases}
    x'(z) = y(z), \\
    y'(z) = -x^2(z) + \alpha x(z) - \beta.
\end{cases}$$  \hspace{1cm} (3.13)

Making the transformation

$$d\eta = \frac{dz}{x'(z)},$$  \hspace{1cm} (3.14)

system (3.13) becomes

$$\begin{cases}
    \frac{dx}{d\eta} = \frac{dy}{d\eta} = y(z)x^3(z), \\
    \frac{dy}{d\eta} = \frac{dz}{d\eta} = -x^4(z) + \alpha x(z) - \beta.
\end{cases}$$  \hspace{1cm} (3.15)

Now, we apply the Division Theorem to seek the first integral to system (3.15). Suppose that $x = x(\eta)$ and $y = y(\eta)$ are the nontrivial solutions to (3.15), and $q(x,y) = \sum_{i=0}^{m} a_i(x) y^i$ is an irreducible polynomial in $\mathbb{C}[x,y]$, such that

$$q(x(\eta),y(\eta)) = \sum_{i=0}^{m} a_i(x(\eta)) y^i(\eta) = 0,$$  \hspace{1cm} (3.16)

where $a_i(x)(i = 0, 1, \ldots, m)$, are polynomials in $x$ and $a_m(x) \neq 0$ for all $m$. Eq. (3.16) is called the first integral to system (3.15). Suppose that $m = 1$ in Eq. (3.16). According to the Division Theorem, there exists a polynomial $G(x,y) = g(x) + h(x)y$ in $\mathbb{C}[x,y]$ such that

$$\frac{dq}{d\eta} \bigg|_{(3.15)} = \left( \frac{dq}{dx} \frac{dx}{d\eta} + \frac{dq}{dy} \frac{dy}{d\eta} \right) \bigg|_{(3.15)}$$

$$= \left( \sum_{i=0}^{1} a'_i(x) y^i \right) (xy^3) + \left( \sum_{i=0}^{1} ia_i(x) y^{i-1} \right) (-x^4 + \alpha x - \beta)$$  \hspace{1cm} (3.17)

$$= (g(x) + h(x)y) \left( \sum_{i=0}^{1} a_i(x) y^i \right).$$
where prime denotes differentiation with respect to the variable $x$. By comparing with the coefficients of $y^i$ ($i = 2, 1, 0$) of both sides of (3.17), we have

$$a_1'(x)x^3 = a_1(x)h(x),$$  
(3.18)

$$a_0'(x)x^3 = g(x)a_1(x) + h(x)a_0(x),$$  
(3.19)

$$a_1(x)(-x^4 + \alpha x - \beta) = g(x)a_0(x).$$  
(3.20)

Since $a_i(x)$ ($i = 0, 1$) and $h(x)$ are polynomials, then from Eq. (3.18) different modes for $h(x)$ and $a_1(x)$ is possible. We discuss these in three cases, below.

**Case I:** From Eq. (3.18) we can deduce that $a_1(x)$ is constant and $h(x) = 0$. For simplicity take $a_1(x) = 1$. Balancing the degrees of $g(x)$ and $a_0(X)$ in Eqs. (3.19) and (3.20), we conclude that $\deg (g(x)) = 3$, $\deg (a_0(x)) = 1$, only. Suppose that

$$g(x) = B_0 + B_1x + B_2x^2 + B_3x^3, \quad (B_3 \neq 0), \quad a_0(x) = A_0 + A_1x, \quad (A_1 \neq 0),$$  
(3.21)

where $B_0, B_1, B_2, B_3, A_0$ and $A_1$ are all constants to be determined.

Substituting $a_0(x)$, $a_1(x)$ and $h(x)$ in Eqs. (3.19) and (3.20), and setting all the coefficients of powers $x$ to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain the following solutions:

$$B_0 = B_1 = B_2 = \alpha = \beta = 0, \quad A_1 = B_3 = 1,$$  
(3.22)

and

$$B_0 = B_1 = B_2 = \alpha = \beta = 0, \quad A_1 = B_3 = -1,$$  
(3.23)

where $i = \sqrt{-1}$. Since $\beta > 0$, this solution is unacceptable.

**Remark 3.1.** If $a_1(x)$ is a non-constant polynomial, according to Equation (3.18), $h(x)$ is a polynomial of the second degree, only. Due to this fact, we consider the following scenarios.

**Case II:** If $a_1(x)$ is a polynomial of the first order, according to the above remark $h(x)$ is a quadratic polynomial. Let us assume that

$$a_1(x) = C_0 + C_1x, \quad (C_1 \neq 0), \quad h(x) = D_0 + D_1x + D_2x^2, \quad (D_2 \neq 0),$$  
(3.24)

where $D_0, D_1, D_2, D_3, C_0$ and $C_1$ are all constants to be determined. By letting $a_1(x)$ and $h(x)$ in Eq. (3.18), we obtain

$$h(x) = x^2, \quad a_1(x) = C_1x, \quad (C_1 \neq 0).$$  
(3.25)

Now, if $g(x)$ is a constant, we assume that $g(x) = B_0$. From Eq. (3.20), it follows that $B_0 \neq 0$ and $a_0(x)$ is a polynomial of order five. Suppose that

$$a_0(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5, \quad (A_5 \neq 0).$$  
(3.26)

Substituting $a_0(x)$ and $g(x)$ in Eq. (3.19), and setting all the coefficients of powers $x$ to be zero, we conclude $A_5 = 0, B_0C_1 = 0$, which is impossible. So $g(x)$ can not be a constant.

If $g(x)$ is a polynomial of the first order, assume that $g(x) = B_0 + B_1x$, that $B_1 \neq 0$. From Eq. (3.20), it follows that $a_0(x)$ is a polynomial of order four. Suppose that

$$a_0(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4, \quad (A_4 \neq 0).$$  
(3.27)

Again, put the $a_0(x)$ and $g(x)$ in Eq. (3.19) and setting all the coefficients of powers $x$ to be zero, we obtain $A_4 = 0$, which is impossible. Thus $g(x)$ can not be a polynomial of the first order.

If $g(x)$ is a polynomial of the two order, assume $g(x) = B_0 + B_1x + B_2x^2$, that $B_2 \neq 0$. From Eq. (3.20), it follows that $a_0(x)$ is a polynomial of order three. Suppose that

$$a_0(x) = A_0 + A_1x + A_2x^2 + A_3x^3, \quad (A_3 \neq 0).$$  
(3.28)
Substituting \( a_0(x) \), and \( g(x) \) in Eqs. (3.19) and (3.20), and setting all the coefficients of powers \( x \) to be zero, we conclude \( C_1 = 0 \), which is impossible. Thus \( g(x) \) can not be a polynomial of order two.

Suppose \( g(x) \) is a polynomial of the third degree, set

\[
g(x) = B_0 + B_1x + B_2x^2 + B_3x^3, \quad (B_3 \neq 0),
\]

(3.29)

then, from Eq. (3.20), we conclude \( a_0(x) \) is a polynomial of degree two. Set

\[
a_0(x) = A_0 + A_1x + A_2x^2, \quad (A_2 \neq 0).
\]

(3.30)

Substituting \( a_0(x) \) and \( g(x) \) in Eqs. (3.19) and (3.20), and setting all the coefficients of powers \( x \) to be zero, we obtain

\[
A_0 = \sqrt{B}C_1, A_1 = 0, A_2 = -iC_1, B_0 = B_2 = 0, B_1 = -\sqrt{B}B_3 = -i, \alpha = 0,
\]

(3.31)

\[
A_0 = -\sqrt{B}C_1, A_1 = 0, A_2 = -iC_1, B_0 = B_2 = 0, B_1 = \sqrt{B}, B_3 = -i, \alpha = 0,
\]

(3.32)

\[
A_0 = \sqrt{B}C_1, A_1 = 0, A_2 = iC_1, B_0 = B_2 = 0, B_1 = -\sqrt{B}, B_3 = i, \alpha = 0,
\]

(3.33)

\[
A_0 = -\sqrt{B}C_1, A_1 = 0, A_2 = iC_1, B_0 = B_2 = 0, B_1 = \sqrt{B}, B_3 = i, \alpha = 0.
\]

(3.34)

Setting Eqs. (3.31)-(3.34) in Eq. (3.16), we obtain

\[
-ic_1x^2 + \sqrt{B}C_1 + C_1xy = 0,
\]

(3.35)

\[
-ic_1x^2 - \sqrt{B}C_1 + C_1xy = 0,
\]

(3.36)

\[
ic_1x^2 + \sqrt{B}C_1 + C_1xy = 0,
\]

(3.37)

\[
ic_1x^2 - \sqrt{B}C_1 + C_1xy = 0,
\]

(3.38)

respectively. Combining these first integrals with Eqs. (3.15), (3.14) and (3.12), the second order differential Eq. (3.11) can be reduced to

\[
-ic_1(f(z))^2 + \sqrt{B}C_1 + C_1(f(z))^2f'(z) = 0,
\]

(3.39)

\[
-ic_1(f(z))^2 - \sqrt{B}C_1 + C_1(f(z))^2f'(z) = 0,
\]

(3.40)

\[
ic_1(f(z))^2 + \sqrt{B}C_1 + C_1f(z)f'(z) = 0,
\]

(3.41)

\[
ic_1(f(z))^2 - \sqrt{B}C_1 + C_1(f(z))^2f'(z) = 0.
\]

(3.42)

Solving Eq. (3.39), we have

\[
f_{1,2}(z) = \pm \sqrt{-e^{2iz+2c_1} - i\sqrt{B}},
\]

(3.43)

\[
f_{3,4}(z) = \pm \sqrt{e^{2iz+2c_1} - i\sqrt{B}},
\]

(3.44)

where \( c_1 \) is an arbitrary constant. Solving Eq. (3.40), we have

\[
f_{5,6}(z) = \pm \sqrt{-e^{2iz+2c_2} + i\sqrt{B}},
\]

(3.45)

\[
f_{7,8}(z) = \pm \sqrt{e^{2iz+2c_2} + i\sqrt{B}},
\]

(3.46)

where \( c_2 \) is an arbitrary constant. Solving Eq. (3.41), we have

\[
f_{9,10}(z) = \pm \sqrt{e^{-2iz+2c_3} + i\sqrt{B}},
\]

(3.47)

\[
f_{11,12}(z) = \pm \sqrt{-e^{-2iz+2c_3} + i\sqrt{B}},
\]

(3.48)
where $c_3$ is an arbitrary constant. Solving Eq. (3.42), we have

$$f_{13,14}(z) = \pm \sqrt{e^{-2iz+2c_4} - i\sqrt{\beta}},$$  

(3.49)  

$$f_{15,16}(z) = \pm \sqrt{e^{-2iz+2c_4} - i\sqrt{\beta}},$$  

(3.50)  

where $c_4$ is an arbitrary constant. Note that,

$$f_{1,2}(z) = if_{7,8}(z), f_{3,4}(z) = if_{5,6}(z), f_{9,10}(z) = if_{15,16}(z), f_{11,12}(z) = if_{13,14}(z).$$  

(3.51)  

According to

$$f(z) = \phi'(z),$$  

(3.52)  

with the integration of Eqs. (3.43), (3.44), (3.47) and (3.48) we have

$$\phi_{1,2}(z) = \pm \left(-i\sqrt{-e^{2iz+2c_1} - i\sqrt{\beta}} - (-1)^{\frac{1}{2}}\beta^{\frac{1}{2}}\right) \times \arctan \left[\frac{(-1)^{\frac{1}{2}} \sqrt{-e^{2iz+2c_1} - i\sqrt{\beta}}}{\beta^{\frac{1}{2}}}\right] + d_1,$$  

(3.53)  

$$\phi_{3,4}(z) = \pm \left(-i\sqrt{e^{2iz+2c_1} - i\sqrt{\beta}} - (-1)^{\frac{1}{2}}\beta^{\frac{1}{2}}\right) \times \arctan \left[\frac{(-1)^{\frac{1}{2}} \sqrt{e^{2iz+2c_1} - i\sqrt{\beta}}}{\beta^{\frac{1}{2}}}\right] + d_2,$$  

(3.54)  

$$\phi_{9,10}(z) = \pm \sqrt{e^{-2iz+2c_1} + i\sqrt{\beta}} \left(-1\right)^{\frac{1}{4}} \beta^{\frac{1}{4}} \arccot \left[\frac{(-1)^{\frac{1}{2}} e^{-iz} \sqrt{e^{2c_1} + ie^{2iz} \sqrt{\beta}}}{\beta^{\frac{1}{2}}} \right] + d_3,$$  

(3.55)  

$$\phi_{11,12}(z) = \pm \frac{\sqrt{-e^{-2iz+2c_1} + i\sqrt{\beta}}}{\sqrt{ie^{2c_1} + e^{2iz} \sqrt{\beta}}} \left(i\sqrt{ie^{2c_1} + e^{2iz} \sqrt{\beta} + e^{i2\pi z} \beta^{\frac{1}{2}} - ie^{i2\pi z} \beta^{\frac{1}{2}} \log \left[e^{i2\pi z} \beta^{\frac{1}{2}} + \sqrt{\beta} \right]} + d_4, 

(3.56)  

where $d_1, d_2, d_3$ and $d_4$ are constants of integration. From Eqs. (3.52) and (3.51) we obtain

$$\phi_{7,8}(z) = -i\phi_{1,2}(z) + d_5, \quad \phi_{5,6}(z) = -i\phi_{3,4}(z) + d_6, \quad \phi_{15,16}(z) = -i\phi_{9,10}(z) + d_7, \quad \phi_{13,14}(z) = -i\phi_{11,12}(z) + d_8.$$  

(3.57)
where $d_s, d_6, d_7$ and $d_8$ are arbitrary constants. $\phi_i(z), (i = 1, \ldots, 16)$ are solutions for Eq. (3.8) for $\epsilon = 1$. Therefore, the exact solutions to the Casimir equation (1.2) can be written as

$$w_{1,2}(x,t) = \pm \left( -i \sqrt{-e^{2i\left(\frac{1}{\sqrt{\beta'}}\right) + 2c_1} - i \sqrt{\beta'}} \right) - (1) \frac{3}{2} \beta^\frac{1}{2} \arctan \left[ \frac{(-1)^{\frac{3}{2}} \sqrt{-e^{2i\left(\frac{1}{\sqrt{\beta'}}\right) + 2c_1} - i \sqrt{\beta'}}}{\beta^\frac{1}{2}} \right] + d_1, \quad (3.58)$$

$$w_{3,4}(x,t) = \pm \left( -i \sqrt{e^{2i\left(\frac{1}{\sqrt{\beta'}}\right) + 2c_1} - i \sqrt{\beta'}} \right) - (1) \frac{3}{2} \beta^\frac{1}{2} \arctan \left[ \frac{(-1)^{\frac{3}{2}} \sqrt{e^{2i\left(\frac{1}{\sqrt{\beta'}}\right) + 2c_1} - i \sqrt{\beta'}}}{\beta^\frac{1}{2}} \right] + d_2, \quad (3.59)$$

$$w_{9,10}(x,t) = \pm \sqrt{e^{-2i\left(\frac{1}{\sqrt{\beta'}}\right) + 2c_1} + i \sqrt{\beta'}} \left( i \right) \beta^\frac{1}{2} \arccot \left[ \frac{(-1)^{\frac{3}{2}} e^{-i\left(\frac{1}{\sqrt{\beta'}}\right)} \sqrt{e^{2c_1} + i e^{2i\left(\frac{1}{\sqrt{\beta'}}\right)} \sqrt{\beta}}}{\beta^\frac{1}{2}} \right] + d_3, \quad (3.60)$$

$$w_{11,12}(x,t) = \pm \sqrt{e^{-2i\left(\frac{1}{\sqrt{\beta'}}\right) + 2c_1} + i \sqrt{\beta'}} \left( i \right) e^{-i\left(\frac{1}{\sqrt{\beta'}}\right) \sqrt{e^{2c_1} + i e^{2i\left(\frac{1}{\sqrt{\beta'}}\right)} \sqrt{\beta}}} \sqrt{e^{2c_1} + i e^{2i\left(\frac{1}{\sqrt{\beta'}}\right)} \sqrt{\beta}} \left( x - \frac{1}{\sqrt{\beta'}} \right) \beta^\frac{1}{2} - i e^{-i\left(\frac{1}{\sqrt{\beta'}}\right) \sqrt{e^{2c_1} + i e^{2i\left(\frac{1}{\sqrt{\beta'}}\right)} \sqrt{\beta}}} \beta^\frac{1}{2} \times \log \left[ e^{-i\left(\frac{1}{\sqrt{\beta'}}\right) \sqrt{e^{2c_1} + i e^{2i\left(\frac{1}{\sqrt{\beta'}}\right)} \sqrt{\beta} + \sqrt{\beta}}} \right] + d_4. \quad (3.61)$$

From Eq. (3.57) we have

$$w_{7,8}(x,t) = -iw_{1,2}(x,t) + d_5, \quad w_{5,6}(x,t) = -iw_{3,4}(x,t) + d_6,$$

$$w_{15,16}(x,t) = -iw_{9,10}(x,t) + d_7, \quad w_{13,14}(x,t) = -iw_{11,12}(x,t) + d_8. \quad (3.62)$$
Finally, by balancing the degrees of polynomials Eqs. (3.19) and (3.20), we conclude that

\[ g(x) = A_0 + A_1x, \quad (A_1 \neq 0), \quad (3.63) \]

then, put \( g(x) \) and \( a_0(x) \) in Eq. (3.19), and setting all the coefficients of powers \( x \) to be zero, we conclude \( B_4 = 0 \), that it’s impossible. Thus \( g(x) \) can not be a polynomial of order four.

Finally, by balancing the degrees of polynomials Eqs. (3.19) and (3.20), we conclude that \( g(x) \) can not be a polynomial of order five.

**Case III:** If \( a_1(x) \) is a polynomial of order two, set

\[ a_1(x) = C_0 + C_1x + C_2x^2, \quad (C_2 \neq 0). \quad (3.64) \]

Then, substituting \( a_1(x) \) and \( h(x) \) (from (3.24)) in Eq. (3.18), and setting all the coefficients of powers \( x \) to be zero, we obtain

\[ a_1(x) = C_1x + C_2x^2, \quad (C_2 \neq 0), \quad h(x) = 2x^2 - \frac{C_1}{C_2}x. \quad (3.65) \]

However, as in the second case, after a study of all possible forms for \( g(x) \) and \( a_0(x) \) in Eqs. (3.19) and (3.20), we have reached the conclusion that \( g(x) \) and \( a_0(x) \) can be third-degree polynomials. Assume

\[ a_0(x) = A_0 + A_1x + A_2x^2 + A_3x^3, \quad (A_3 \neq 0), \quad g(x) = B_0 + B_1x + B_2x^2 + B_3x^3. \quad (3.66) \]

Put \( g(x) \), \( a_0(x) \), \( h(x) \) and \( a_1(x) \) in Eqs. (3.19) and (3.20), and setting all the coefficients of powers \( x \) to be zero we obtain

\[
\begin{align*}
A_0 & = 0, \quad A_1 = \sqrt{B}C_2, \quad A_2 = 0, \quad A_3 = -iC_2, \quad C_1 = 0, \quad B_0 = 0, \quad B_1 = -\sqrt{B}, \quad B_2 = 0, \quad B_3 = -i, \quad \alpha = 0. \quad (3.67) \\
A_0 & = 0, \quad A_1 = -\sqrt{B}C_2, \quad A_2 = 0, \quad A_3 = -iC_2, \quad C_1 = 0, \quad B_0 = 0, \quad B_1 = \sqrt{B}, \quad B_2 = 0, \quad B_3 = -i, \quad \alpha = 0. \quad (3.68) \\
A_0 & = 0, \quad A_1 = \sqrt{B}C_2, \quad A_2 = 0, \quad A_3 = iC_2, \quad C_1 = 0, \quad B_0 = 0, \quad B_1 = -\sqrt{B}, \quad B_2 = 0, \quad B_3 = i, \quad \alpha = 0. \quad (3.69) \\
A_0 & = 0, \quad A_1 = -\sqrt{B}C_2, \quad A_2 = 0, \quad A_3 = iC_2, \quad C_1 = 0, \quad B_0 = 0, \quad B_1 = \sqrt{B}, \quad B_2 = 0, \quad B_3 = i, \quad \alpha = 0. \quad (3.70)
\end{align*}
\]

Setting Eqs. (3.67)-(3.70) in Eq. (3.16), we obtain

\[
\begin{align*}
-iC_2x^3 + \sqrt{B}C_2x + C_2x^2y &= 0, \quad (3.71) \\
-iC_2x^3 - \sqrt{B}C_2x + C_2x^2y &= 0, \quad (3.72) \\
iC_2x^3 + \sqrt{B}C_2x + C_2x^2y &= 0, \quad (3.73) \\
iC_2x^3 - \sqrt{B}C_2x + C_2x^2y &= 0. \quad (3.74)
\end{align*}
\]

Combining these equations with Eqs. (3.15), (3.14) and (3.12), we find solutions which were obtained in Case II. Therefore, the solutions found under Case II are the possible solutions to Eq. (1.2).

**Remark 3.2.** All solutions obtained in this section have been checked with Mathematica 8.0 by plugging each solution back into the original equation (1.2) for the (+) sign case.
4 Discussion and Conclusions

In this paper, we obtained exact solutions for the nonlinear Casimir equation for the Ito system by using the first integral method. Unlike in other works, we were able to obtain complex-valued solutions, since the method allows one to obtain exact solutions. Typically, exact solution methods work best for studying complex-valued solutions to nonlinear differential equations. The first integral method is therefore reliable for obtaining solutions to these types of nonlinear differential equations, and can potentially be used to study solutions to a number of interesting equations related to mathematical physics.

Another important feature is that the first integral method allows one to obtain multiple exact solutions, provided of course that multiple solutions exist. For the present problem, we find that a large number of multiple solutions exist. Each of these solutions corresponds to a complex root of the polynomial equations which describe the first integral for the travelling wave form of the Casimir equation. If the parameters (such as wave-speed) are selected so that some of these roots are real, then real-valued solutions can be found. Such solutions would then allow one to recover a few of the solutions found in [4].

Acknowledgements

The author is very grateful to anonymous reviewer for carefully reading the paper and for his/her comments and suggestions which have improved the paper.

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