Communications in Numerical Analysis 2014 (2014) 1-7

Available online at www.ispacs.com/cna

Volume 2014, Year 2014 Article ID cna-00216, 7 Pages

doi:10.5899/2014/cna-00216

Research Article

On a new class of integrals involving Bessel functions of the first kind

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Abstract

In recent years, several integral formulas involving a variety of special functions have been developed by many authors. Also many integral formulas containing the Bessel function $J_\nu(z)$ have been presented. Very recently, Rakha *et al.* presented some generalized integral formulas involving the hypergeometric functions. In this sequel, here, we aim at establishing two generalized integral formulas involving a Bessel functions of the first kind, which are expressed in terms of the generalized Wright hypergeometric function. Some interesting special cases of our main results are also considered.

Keywords: Gamma function, Hypergeometric function ${}_2F_1$, Generalized hypergeometric function ${}_pF_q$, Generalized (Wright) hypergeometric functions ${}_p\Psi_q$, generalized hypergeometric series, cosine and sine trigonometric functions, Bessel function of the first kind, Lavoie-Trottier integral formula.

1 Introduction and Preliminaries

Many important functions in applied sciences are defined via improper integrals or series (or infinite products). The general name of these important functions are called special functions. Bessel functions are important special functions and their closely related ones are widely used in physics and engineering; therefore, they are of interest to physicists and engineers as well as mathematicians. In recent years, numerous integral formulas involving a variety of special functions have been developed by many authors (see, e.g., [3, 7, 11]; for a very recent work, see also [12]). Also many integral formulas associated with the Bessel functions of several kinds have been presented (see, e.g., [3, 5, 6, 17]; see also [4]). Those integrals involving Bessel functions are not only of great interest to the pure mathematics, but they are often of extreme importance in many branches of theoretical and applied physics and engineering. Several methods for evaluating infinite or finite integrals involving Bessel functions have been known (see, e.g.,[1] and [11]). However, these methods usually work on a case-by-case basis.

Very recently, Rakha *et al.* [12] gave certain interesting new class of integral formulas involving the hypergeometric function, which are expressed in terms of the gamma functions. In the present sequel to the aforementioned investigations, we present two generalized integral formulas involving a Bessel functions of the first kind, which are expressed in terms of the generalized Wright hypergeometric function. Some interesting special cases and (potential) usefulness

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of our main results are also considered and remarked, respectively.
For our purpose, we begin by recalling some known functions and earlier works. The Bessel function of the first kind $J_\nu(z)$ is defined for $z \in \mathbb{C}\setminus\{0\}$ and $\nu \in \mathbb{C}$ with $\Re(\nu) > -1$ by the following series (see, for example, [11, p. 217,Entry 10.2.2] and [17, p. 40, Eq. (8)]):

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \tag{1.1}$$

where $\mathbb{C}$ denotes the set of complex numbers and $\Gamma(z)$ is the familiar Gamma function (see [14, Section 1.1]).
An interesting further generalization of the generalized hypergeometric series ${}_pF_q$ (1.5) is due to Fox [9] and Wright [18, 19, 20] who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [16, p. 21])

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p);\\ (\beta_1, B_1), \ldots, (\beta_q, B_q);\end{matrix}\, z\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j k)} \frac{z^k}{k!}, \tag{1.2}$$

where the coefficients $A_1, \ldots, A_p$ and $B_1, \ldots, B_q$ are positive real numbers such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0. \tag{1.3}$$

A special case of (1.2) is

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1, 1), \ldots, (\alpha_p, 1);\\ (\beta_1, 1), \ldots, (\beta_q, 1);\end{matrix}\, z\right] = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)}\, {}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p;\\ \beta_1, \ldots, \beta_q;\end{matrix}\, z\right], \tag{1.4}$$

where ${}_pF_q$ is the *generalized hypergeometric series* defined by (see [14, Section 1.5])

$$\begin{aligned}{}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p;\\ \beta_1, \ldots, \beta_q;\end{matrix}\, z\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}\\ &= {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z),\end{aligned} \tag{1.5}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [14, p. 2 and pp. 4–6]):

$$\begin{aligned}(\lambda)_n :&= \begin{cases} 1 & (n=0)\\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \ldots\})\end{cases}\\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C}\setminus\mathbb{Z}_0^-)\end{aligned} \tag{1.6}$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers.
For our present investigation, we also need to recall the following Lavoie and Trottier integral formula [10]:

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} dx\\ &\qquad = \left(\frac{2}{3}\right)^{2\alpha} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad (\Re(\alpha) > 0 \text{ and } \Re(\beta) > 0).\end{aligned} \tag{1.7}$$

2 Main Results

In this section, we establish two generalized integral formulas, which are expressed in terms of the generalized (Wright) hypergeometric functions (1.4), by inserting the Bessel function of the first kind (1.1) with suitable arguments into the integrand of (1.7).

Theorem 2.1. *The following integral formula holds true: For $\rho,\ j,\ \nu \in \mathbb{C}$ with $\Re(\nu) > -1$, $\Re(\rho+j) > 0, \Re(\rho+\nu) > 0$ and $x > 0$,*

$$\begin{aligned} &\int_0^1 x^{\rho+j-1}\,(1-x)^{2\rho-1}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-1} J_\nu\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx \\ &= \left(\frac{2}{3}\right)^{2(\rho+j)}\left(\frac{y}{2}\right)^\nu \Gamma(\rho+j)\cdot{}_1\Psi_2\left[\begin{matrix} (\rho+\nu,1); \\ (\nu+1,1),(2\rho+j+\nu,2); \end{matrix}\ -\frac{y^2}{4}\right]. \end{aligned} \tag{2.1}$$

Proof. By applying (1.1) to the integrand of (2.1) and then interchanging the order of integral sign and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

$$\begin{aligned} &\int_0^1 x^{\rho+j-1}\,(1-x)^{2\rho-1}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-1} J_\nu\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx \\ &= \sum_{k=0}^{\infty}(-1)^k\frac{(y/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}\int_0^1 x^{\rho+j-1}\,(1-x)^{2(\rho+\nu+2k)-1} \\ &\qquad\qquad \left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho+\nu+2k-1}dx. \end{aligned} \tag{2.2}$$

In view of the conditions given in Theorem 2.1, since

$$\Re(\nu) > -1,\ \Re(\rho+\nu+2k) > \Re(\rho+\nu) > 0, \Re(\rho+j) > 0\ (k \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}),$$

we can apply the integral formula (1.7) to the integral in (2.2) and obtain the following expression:

$$\begin{aligned} &\int_0^1 x^{\rho+j-1}\,(1-x)^{2\rho-1}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-1} J_\nu\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx \\ &= \left(\frac{2}{3}\right)^{2(\rho+j)}\left(\frac{y}{2}\right)^\nu \Gamma(\rho+j)\cdot\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(\rho+\nu+2k)}{k!\,\Gamma(1+\nu+k)\,\Gamma(2\rho+j+\nu+2k)}\left(\frac{y}{2}\right)^{2k}, \end{aligned}$$

which, upon using (1.2), yields (2.1). This completes the proof of Theorem 2.1. □

Theorem 2.2. *The following integral formula holds true: For $\rho,\ j,\ \nu \in \mathbb{C}$ with $\Re(\nu) > -1$, $\Re(\rho+j) > 0, \Re(\rho+\nu) > 0$ and $x > 0$,*

$$\begin{aligned} &\int_0^1 x^{\rho-1}\,(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{(\rho+j)-1} J_\nu\left(yx\left(1-\frac{x}{3}\right)^2\right)dx \\ &= \left(\frac{2}{3}\right)^{2(\rho+\nu)}\left(\frac{y}{2}\right)^\nu \Gamma(\rho+j)\cdot{}_1\Psi_2\left[\begin{matrix} (\rho+\nu,2); \\ (\nu+1,1),(2\rho+j+\nu,2); \end{matrix}\ -\left(\frac{4y}{9}\right)^2\right]. \end{aligned} \tag{2.3}$$

Proof. It is easy to see that a similar argument as in the proof of Theorem 2.1 will establish the integral formula (2.3). Therefore, we omit the details of the proof of this theorem. □

Next we consider other variations of Theorems 2.1 and 2.2. In fact, we establish some integral formulas for the Bessel function $J_\nu(z)$ expressed in terms of the generalized hypergeometric function ${}_pF_q$. To do this, we recall the well-known Legendre duplication formula for the Gamma function Γ:

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)\quad\left(z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \cdots\right), \tag{2.4}$$

which is equivalently written in terms of the Pochhammer symbol (1.6) as follows (see, for example, [14, p. 6]):

$$(\lambda)_{2n} = 2^{2n}\left(\frac{1}{2}\lambda\right)_n\left(\frac{1}{2}\lambda+\frac{1}{2}\right)_n\quad(n \in \mathbb{N}_0). \tag{2.5}$$

Now we are ready to state the following two corollaries.

Corollary 2.1. *Let the condition of Theorem 2.1 be satisfied and $\rho+j, \rho+\nu \in \mathbb{C}\setminus\mathbb{Z}_0^-$. Then the following integral formula holds true:*

$$\begin{aligned}&\int_0^1 x^{\rho+j-1}\,(1-x)^{2\rho-1}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-1} J_\nu\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx\\&=\left(\frac{2}{3}\right)^{2(\rho+j)}\left(\frac{y}{2}\right)^{\nu}\frac{\Gamma(\rho+j)\Gamma(\rho+\nu)}{\Gamma(2\rho+j+\nu)\Gamma(\nu+1)}\\&\quad\cdot{}_2F_3\left[\begin{matrix}\left(\frac{\rho+\nu}{2}\right),\left(\frac{\rho+\nu+1}{2}\right);\\(\nu+1),\left(\frac{2\rho+\nu+j}{2}\right),\left(\frac{2\rho+\nu+j+1}{2}\right);\end{matrix}\;-\frac{y^2}{4}\right].\end{aligned}\tag{2.6}$$

Corollary 2.2. *Let the condition of Theorem 2.2 be satisfied and $\rho+j, \rho+\nu \in \mathbb{C}\setminus\mathbb{Z}_0^-$. Then the following integral formula holds true:*

$$\begin{aligned}&\int_0^1 x^{\rho-1}\,(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{(\rho+j)-1} J_\nu\left(yx\left(1-\frac{x}{3}\right)^2\right)dx\\&=\left(\frac{2}{3}\right)^{2(\rho+\nu)}\left(\frac{y}{2}\right)^{\nu}\frac{\Gamma(\rho+j)\Gamma(\rho+\nu)}{\Gamma(2\rho+j+\nu)\Gamma(\nu+1)}\\&\quad\cdot{}_2F_3\left[\begin{matrix}\left(\frac{\rho+\nu}{2}\right),\left(\frac{\rho+\nu+1}{2}\right);\\(\nu+1),\left(\frac{2\rho+\nu+j}{2}\right),\left(\frac{2\rho+\nu+j+1}{2}\right);\end{matrix}\;-\left(\frac{4y}{9}\right)^2\right].\end{aligned}\tag{2.7}$$

Proof. By writing the right-hand side of Equation (2.1) in the original summation and applying (2.5) to the resulting summation, after a little simplification, we find that, when the last resulting summation is expressed in terms of ${}_pF_q$ in (1.5), this completes the proof of Corollary 2.1. Similarly, it is easy to see that a similar argument as in proof of Corollary 2.1 will establish the integral formula (2.7). Therefore, we omit the details of the proof of Corollary 2.2. □

3 Special Cases

In this section, we derive certain new integral formulas for the cosine and sine functions involving in the integrand (2.1) and (2.3), respectively. To do this, we recall the following known formula (see, for example, [8, p. 79, Eq. (15)]):

$$J_{-1/2}(z)=\sqrt{\frac{2}{\pi z}}\cos z.\tag{3.1}$$

By applying the expression in (3.1) to (2.1), (2.3), (2.6) and (2.7), we obtain four integral formulas in Corollaries 3.1, 3.2, 3.3 and 3.4, respectively.

Corollary 3.1. *The following integral formula holds true: $\rho, j \in \mathbb{C}$ with $\Re(2\rho+j) > \frac{1}{2}, \Re(\rho) > \frac{1}{2}$ and $x > 0$,*

$$\begin{aligned}&\int_0^1 x^{\rho+j-1}\,(1-x)^{2(\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-\frac{3}{2}}\cdot\cos\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx\\&=\sqrt{\pi}\left(\frac{2}{3}\right)^{2(\rho+j)}\Gamma(\rho+j)\cdot{}_1\Psi_2\left[\begin{matrix}\left(\rho-\frac{1}{2},1\right);\\\left(\frac{1}{2},1\right),\left(2\rho+j-\frac{1}{2},2\right);\end{matrix}\;-\frac{y^2}{4}\right].\end{aligned}\tag{3.2}$$

Corollary 3.2. *The following integral formula holds true:* $\rho,\, j \in \mathbb{C}$ *with* $\Re(2\rho + j) > \frac{1}{2}, \Re(\rho) > \frac{1}{2}$ *and* $x > 0$,

$$\int_0^1 x^{\rho-1}\,(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{(\rho+j)-1}\cos\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$
$$= \sqrt{2\pi}\left(\frac{2}{3}\right)^{2\rho-1}\Gamma(\rho+j)\cdot {}_1\Psi_2\left[\begin{matrix} \left(\rho-\frac{1}{2},1\right); \\ \left(\frac{1}{2},1\right),\left(2\rho+j-\frac{1}{2},2\right); \end{matrix}\; -\left(\frac{4y}{9}\right)^2\right]. \tag{3.3}$$

If we employ the same method as in getting (2.6) *and* (2.7) *to* (3.2) *and* (3.3)*, we obtain the following two corollaries.*

Corollary 3.3. *Let the condition of Corollary 3.1 be satisfied and* $\rho + j, \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-$. *Then the following integral formula holds true.*

$$\int_0^1 x^{\rho+j-1}\,(1-x)^{2(\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-\frac{3}{2}}\cdot\cos\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$
$$= \left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)\Gamma(\frac{2\rho-1}{2})}{\Gamma(\frac{4\rho+2j-1}{2})}\,{}_2F_3\left[\begin{matrix} \left(\frac{2\rho-1}{4}\right),\left(\frac{2\rho+1}{4}\right); \\ \left(\frac{1}{2}\right),\left(\frac{4\rho+2j-1}{4}\right),\left(\frac{4\rho+2j+1}{4}\right); \end{matrix}\; -\frac{y^2}{4}\right]. \tag{3.4}$$

Corollary 3.4. *Let the condition of Corollary 3.2 be satisfied and* $\rho + j, \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-$. *Then the following integral formula holds true.*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{(\rho+j)-1}\cos\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$
$$= \sqrt{2}\left(\frac{2}{3}\right)^{2(\rho-1)}\frac{\Gamma(\rho+j)\Gamma(\frac{2\rho-1}{2})}{\Gamma(\frac{4\rho+2j-1}{2})}$$
$$\cdot\,{}_2F_3\left[\begin{matrix} \left(\frac{2\rho-1}{4}\right),\left(\frac{2\rho+1}{4}\right); \\ \left(\frac{1}{2}\right),\left(\frac{4\rho+2j-1}{4}\right),\left(\frac{4\rho+2j+1}{4}\right); \end{matrix}\; -\left(\frac{4y}{9}\right)^2\right]. \tag{3.5}$$

By recalling the following formula (see, for example, [8, p. 79, Eq. (14)]):

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z, \tag{3.6}$$

and applying this formula to (2.1), (2.3), (2.6) and (2.7) we obtain four more integral formulas in Corollaries 3.5 to 3.8, respectively.

Corollary 3.5. *The following integral formula holds true:* $\rho,\, j \in \mathbb{C}$ *with* $\Re(2\rho + j) > \frac{1}{2}, \Re(\rho) > -\frac{1}{2}$ *and* $x > 0$,

$$\int_0^1 x^{\rho+j-1}\,(1-x)^{2(\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-\frac{3}{2}}\sin\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$
$$= \left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\sqrt{(\pi)}\Gamma(\rho+j)y}{2}\cdot {}_1\Psi_2\left[\begin{matrix} \left(\rho+\frac{1}{2},1\right); \\ \left(\frac{3}{2},1\right),\left(2\rho+j+\frac{1}{2},2\right); \end{matrix}\; -\frac{y^2}{4}\right]. \tag{3.7}$$

Corollary 3.6. *The following integral formula holds true:* $\rho,\, j \in \mathbb{C}$ *with* $\Re(2\rho+j) > \frac{1}{2}, \Re(\rho) > -\frac{1}{2}$ *and* $x > 0$*,*

$$\int_0^1 x^{\rho-\frac{3}{2}}\,(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2(\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j)-1}\sin\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$
$$= \frac{y}{3}\left(\frac{2}{3}\right)^{2\rho}\sqrt{\pi}\,\Gamma(\rho+j)\cdot{}_1\Psi_2\left[\begin{matrix}\left(\rho+\frac{1}{2},1\right); \\ \left(\frac{3}{2},1\right),\left(2\rho+j+\frac{1}{2},2\right);\end{matrix}\; -\left(\frac{4y}{9}\right)^2\right]. \tag{3.8}$$

Corollary 3.7. *Let the condition of Corollary 3.1 be satisfied and* $\rho+j, \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-$*. Then the following integral formula holds true.*

$$\int_0^1 x^{\rho+j-1}\,(1-x)^{2(\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-\frac{3}{2}}\sin\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$
$$= \left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)\Gamma(\frac{2\rho+1}{2})}{\Gamma(\frac{4\rho+2j+1}{2})}\,{}_2F_3\left[\begin{matrix}\left(\frac{2\rho+1}{4}\right),\left(\frac{2\rho+3}{4}\right); \\ \left(\frac{3}{2}\right),\left(\frac{4\rho+2j+1}{4}\right),\left(\frac{4\rho+2j+3}{4}\right);\end{matrix}\; -\frac{y^2}{4}\right]. \tag{3.9}$$

Corollary 3.8. *Let the condition of Corollary 3.2 be satisfied and* $\rho+j, \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-$*. Then the following integral formula holds true.*

$$\int_0^1 x^{\rho-\frac{3}{2}}\,(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2(\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j)-1}\sin\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$
$$= \frac{2y}{3}\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\rho+j)\Gamma(\frac{2\rho+1}{2})}{\Gamma(\frac{4\rho+2j+1}{2})}\,{}_2F_3\left[\begin{matrix}\left(\frac{2\rho+1}{4}\right),\left(\frac{2\rho+3}{4}\right); \\ \left(\frac{3}{2}\right),\left(\frac{4\rho+2j+1}{4}\right),\left(\frac{4\rho+2j+3}{4}\right);\end{matrix}\; -\left(\frac{4y}{9}\right)^2\right]. \tag{3.10}$$

4 Concluding Remark

Here we briefly consider another variation of the results derived in the preceding sections. Bessel functions are important special functions that arise widely in science and engineering. Certain special cases of integrals involving the Bessel functions of the first kind $J_\nu(z)$ of the type (2.1) have been investigated in the literature by a number of authors with different arguments (see, for example, [2, 13] and [15]). Therefore, the results presented in this paper are easily converted in terms of a similar type of new interesting integrals with different arguments after some suitable parametric replacements.

Acknowledgement

The authors are grateful to the anonymous referees for their comments which substantially improved the quality of this paper.

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