MHD oscillating flows of rotating second grade fluids in a porous medium

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Abstract
Exact solutions for the magnetohydrodynamic (MHD) rotating flow of a second grade fluid in a porous medium have been obtained using integral transform technique. A uniform magnetic field has been applied in direction normal to the flow. Expressions for dimensionless velocity have been obtained for oscillating flows. The obtained solutions are expressed as a sum of steady-state and transient solutions and satisfy all imposed initial and boundary conditions. The influence of pertinent parameters on velocity is shown by graphical illustrations. The required time to reach the steady-state for sine and cosine oscillations of the plate is determined.

Keywords: Second grade fluid; MHD oscillating flows; Velocity field; Exact solutions; Porous medium.

1 Introduction

The study of non-Newtonian fluids in a porous medium and rotating frame offers special challenges to mathematicians, numerical analysts, and engineers. Some of these studies are notable and applied in paper, food stuff, personal care product, textile coating, and suspension solutions industries. The non-Newtonian fluids have been mainly classified under the differential, rate, and integral type. Amongst them, the fluids of differential type have received special attention [1]-[12]. The second grade fluids form a subclass of non-Newtonian fluids and are the simplest subclass of differential type fluids which can show the normal stress effects. They were employed to study various problems due to their relatively simple structure. Moreover, one can reasonably hope to obtain exact solutions for this type of fluids. This motivates us to choose the second grade model in this study. Exact solutions are important, as they provide a standard for checking the accuracy of many approximate solutions which can be numerical or empirical. They can also be used as tests for verifying numerical schemes that are developed for studying more complex flow problems. The analysis of the effects of rotation on magnetohydrodynamic (MHD) flows through a porous medium has gained an increasing interest due to the wide range of applications either in geophysics or in engineering such as the optimization of the solidification process of metals and metal alloys and the control underground spreading of chemical wastes and pollutants. MHD is the study of the interaction of conducting fluids with electromagnetic phenomena.

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The flow of an electrically conducting fluid in the presence of magnetic field is of great importance in various areas of technology and engineering such as MHD power generation and MHD pumps. Therefore, several researchers have discussed flows of second grade fluids in different configurations, but there are few attempts which include effects of rotation and MHD (for instances, see studies in [13]-[24] and references therein).

To the best of authors’ knowledge, so far no study has been reported to analyze the unsteady MHD flow of a rotating second grade fluid in a porous medium past an oscillating plate. Therefore, it is here proposed to make such an attempt. The objective of the current study is to establish exact solutions for the velocity field corresponding to sine oscillation for a second grade fluid in such a motion. The fluid is magnetohydrodynamic (MHD) in the presence of an applied magnetic field, occupies a half porous space which is bounded by a rigid and nonconducting plate, and the whole system is rotating.

2 Statement of the problem

Let us consider an incompressible rotating second grade fluid bounded by a rigid plate at \( z = 0 \). The fluid is electrically conducting and fills the porous region \( z > 0 \). The \( z \)-axis is taken to be normal to the plate. Initially, the fluid and plate are at rest. At time \( t = 0^- \), the fluid and the plate start rotation around the \( z \)-axis as a solid body with constant angular velocity \( \Omega \). The plate also begins to oscillate in its plane. A uniform transverse magnetic field of strength \( B_o \) is applied parallel to the axis of rotation. It is assumed that the induced magnetic field, the external electric field and the electric field due to polarization of charges are negligible. The motion of the fluid is governed by the following partial differential equation [17],[22]

\[
\frac{\partial F}{\partial t} + (2i\Omega + \frac{\sigma B_o^2}{\rho})F = \nu \frac{\partial^2 F}{\partial z^2} + \frac{\alpha}{\rho} \frac{\partial^3 F}{\partial z^3 \partial t} - \frac{\nu \phi}{\kappa} (1 + \alpha \frac{\partial}{\partial t})F, \tag{2.1}
\]

where \( F(z,t) = u(z,t) + i v(z,t) \) is the complex velocity of the fluid, \( u \) and \( v \) are the velocity components along \( x \) and \( y \) axes respectively, \( \rho \) is the fluid density, \( \mu \) is the dynamic viscosity, \( \nu \) the kinematic viscosity, \( \alpha \) is the normal stress module, \( \sigma \) is the electrical conductivity of the fluid, \( \phi \) (\( 0 < \phi < 1 \)) is the porosity and \( k > 0 \) is the permeability of the porous medium.

3 Initial and boundary conditions

The appropriate initial and boundary conditions are

\[
F(z,0) = 0, \quad z > 0, \quad F(0,t) = V_o \sin(\omega t) \quad \text{or} \quad V_o \cos(\omega t), \quad \text{for} \quad t > 0,
\]

\[
F(z,t) \to 0 \quad \text{as} \quad z \to \infty, \tag{3.3}
\]

where \( V_o \) is the amplitude and \( \omega \) is the frequency of the oscillations.

Introducing the dimensionless variables

\[
z^* = \frac{z \sqrt{\omega}}{\sqrt{\nu}}; \quad t^* = \omega t; \quad F^* = \frac{F}{V_o}; \quad \Omega^* = \frac{\Omega}{\omega}, \tag{3.4}
\]

into Eq. (2.1) and neglecting the star notation, we find that

\[
\frac{\partial^2 F}{\partial z^2} + \frac{\alpha}{\nu} \frac{\partial^3 F}{\partial z^3 \partial t} - a_0 \frac{\partial F}{\partial t} - b_0 F = 0, \tag{3.5}
\]

\[
F(z,0) = 0; \quad z > 0, \tag{3.6}
\]

\[
F(0,t) = \sin t \quad \text{or} \quad \cos t \quad \text{for} \quad t > 0; \quad F(z,t) \to 0 \quad \text{as} \quad z \to \infty, \tag{3.7}
\]

where

\[
\alpha = \frac{\alpha \omega}{\rho \nu}; \quad a_0 = 1 + \frac{\alpha}{K}; \quad b_0 = \frac{1}{K} + 2i\Omega + M^2; \quad \frac{1}{K} = \frac{\phi v}{k \omega}; \quad M^2 = \frac{\sigma B_o^2}{\rho \omega}. \tag{3.8}
\]
4 Calculation of the Velocity field

In order to determine the solution of the initial-boundary value problem (3.5)-(3.7) we shall use the Laplace transform. All calculi will be presented for the sine oscillations only. Applying the Laplace transform to Eqs. (3.5) and (3.7), and bearing in mind Eq. (3.6) we obtain

\[
\frac{d^2\tilde{F}(s, q)}{ds^2} = \frac{a_0q + b_0}{\alpha q + 1}\tilde{F}(s, q) = 0,
\]

(4.9)

\[
\tilde{F}(0, q) = \frac{1}{q^2 + 1}; \quad \tilde{F}(s, q) \rightarrow 0 \text{ as } z \rightarrow \infty.
\]

(4.10)

The solution of Eq. (4.9) satisfying the conditions (4.10) is

\[
\tilde{F}(z, q) = \frac{1}{q^2 + 1} \exp \left[ -z \sqrt{\frac{a_0q + b_0}{\alpha q + 1}} \right]; \quad \beta = \frac{1}{\alpha}, \quad z \geq 0,
\]

(4.11)

and the complex velocity \( F_i(z, t) = \mathcal{L}^{-1}\{\tilde{F}(z, q)\} \) can be written as a convolution product

\[
F_i(z, t) = (F_1 * F_2)(t) = \int_0^t F_1(t - s) F_2(z, s) ds,
\]

(4.12)

where \( F_1(t) = \sin t \) and \( F_2(z, t) \) are the inverse Laplace transforms of

\[
\tilde{F}_1(q) = \frac{1}{q^2 + 1}, \quad \text{respectively} \quad \tilde{F}_2(q) = \exp(-z \sqrt{\frac{a_0q + b_0}{\alpha q + 1}}), \quad w(q) = \frac{a_0q + b_0}{q + \beta}.
\]

In order to find the inverse Laplace transform of the function \( \tilde{F}_2(z, q) \) we use Eqs. (A1) and (A2) from Appendix and obtain

\[
F_2(z, t) = \int_0^\infty f(z, u) g(u, t) du = \frac{z}{2\sqrt{\alpha \pi}} \int_0^\infty \frac{1}{u \sqrt{u}} \exp(-\frac{z^2}{4au}) g(u, t) du,
\]

(4.13)

where

\[
g(u, \tau) = \mathcal{L}^{-1}\{\exp(-uw(q))\} = \exp(-ua_0) \mathcal{L}^{-1}\{[1 + \exp(\frac{ua_1}{q + \beta}) - 1]
\]

\[
= \exp(-ua_0)[\delta(t) + \sqrt{\frac{a_1 u}{2}} e^{-\beta t} I_1(2\sqrt{a_1 u})],
\]

(4.14)

\( a_1 = a_0 \beta - b_0, I_1(\cdot) \) is the modified Bessel function of the second kind of order one and \( \delta(\cdot) \) is the Dirac delta function. Hence Eqs. (4.13) and (4.14) implies

\[
F_2(z, t) = \frac{z}{2\sqrt{\alpha \pi}} \int_0^\infty \frac{1}{u \sqrt{u}} \exp(-\frac{z^2}{4au}) g(u, t) du + \frac{z \sqrt{a_1}}{2\sqrt{\alpha \pi}} \delta(t) + \sqrt{\frac{a_1 u}{2}} e^{-\beta t} I_1(2\sqrt{a_1 u}).
\]

(4.15)

Introducing Eq. (4.15) into Eq. (4.12) we obtain the expression for \( F_i(z, t) \),

\[
F_i(z, t) = \frac{z \sin t}{2\sqrt{\alpha \pi}} \int_0^\infty \frac{1}{u \sqrt{u}} \exp(-\frac{z^2}{4au} - ua_0) du + \frac{z \sqrt{a_1}}{2\sqrt{\alpha \pi}} (t-s) I_1(2\sqrt{a_1 u}) \sin(t-s) du.
\]

(4.16)
Using (A₃) from Appendix we obtain a simpler form of the solution, namely

\[ F_s(z,t) = e^{-z\sqrt{a_0/\alpha}} \sin t + \frac{z\sqrt{a_0}}{2\sqrt{\alpha \pi}} \int_0^t \int_0^\infty \frac{1}{u\sqrt{s}} \exp\left(-\frac{z^2}{4\alpha u} - u\omega_o - \frac{s}{\alpha}\right) I_1(2\sqrt{a_1 u s}) \sin(t-s) ds du. \]  

(4.17)

The starting solution (4.17) holds for both small and large times. Following [24], we can write our solution as a sum of the steady-state

\[ F_s(z,t) = e^{-z\sqrt{a_0/\alpha}} \sin t \]

\[ + \frac{z\sqrt{a_0}}{2\sqrt{\alpha \pi}} \int_0^\infty \frac{1}{u\sqrt{s}} \exp\left(-\frac{z^2}{4\alpha u} - u\omega_o - \frac{s}{\alpha}\right) I_1(2\sqrt{a_1 u s}) \sin(t-s) ds, \]

(4.18)

and transient solution

\[ F_t(z,t) = -\frac{z\sqrt{a_1}}{2\sqrt{\alpha \pi}} \int_0^t \int_0^\infty \frac{1}{u\sqrt{s}} \exp\left(-\frac{z^2}{4\alpha u} - u\omega_o - \frac{s}{\alpha}\right) I_1(2\sqrt{a_1 u s}) \sin(t-s) ds. \]

(4.19)

Similar computations lead to the starting solution

\[ F_c(z,t) = e^{-z\sqrt{a_0/\alpha}} \cos t + \frac{z\sqrt{a_0}}{2\sqrt{\alpha \pi}} \int_0^t \int_0^\infty \frac{1}{u\sqrt{s}} \exp\left(-\frac{z^2}{4\alpha u} - u\omega_o - \frac{s}{\alpha}\right) I_1(2\sqrt{a_1 u s}) \cos(t-s) ds, \]

(4.20)

corresponding to the cosine oscillations of the boundary. It can also be written as a sum of steady-state and transient solutions

\[ F_c(z,t) = e^{-z\sqrt{a_0/\alpha}} \cos t \]

\[ + \frac{z\sqrt{a_1}}{2\sqrt{\alpha \pi}} \int_0^\infty \frac{1}{u\sqrt{s}} \exp\left(-\frac{z^2}{4\alpha u} - u\omega_o - \frac{s}{\alpha}\right) I_1(2\sqrt{a_1 u s}) \cos(t-s) ds, \]

(4.21)

and

\[ F_d(z,t) = -\frac{z\sqrt{a_1}}{2\sqrt{\alpha \pi}} \int_0^t \int_0^\infty \frac{1}{u\sqrt{s}} \exp\left(-\frac{z^2}{4\alpha u} - u\omega_o - \frac{s}{\alpha}\right) I_1(2\sqrt{a_1 u s}) \cos(t-s) ds. \]

(4.22)

Looking to Eqs. (4.18) and (4.21), it is clearly seen that the two steady-state solutions differ by a phase shift. However, this property is not hold for transient solutions. This is the reason that we separately gave both starting solutions.

By making \( \Omega \to 0 \) in Eqs. (4.17) and (4.20) we recover the solutions obtained by Ali et al. [24], (Eqs. (15) and (16)).

By making \( M = 0 \) (hydrodynamic fluid), \( 1/K = 0 \) (non porous space) and \( \Omega \to 0 \) in Eqs. (4.17) and (4.20) we recover the solutions

\[ F_s(z,t) = e^{-z\sqrt{a_0/\alpha}} \sin t \]

\[ + \frac{z\sqrt{a_0}}{2\sqrt{\alpha \pi}} \int_0^\infty \frac{1}{u\sqrt{s}} \exp\left(-\frac{z^2}{4\alpha u} - u\omega_o - \frac{s}{\alpha}\right) I_1(2\sqrt{a_1 u s}) \sin(t-s) ds, \]

(4.23)

\[ F_c(z,t) = e^{-z\sqrt{a_0/\alpha}} \cos t \]

\[ + \frac{z\sqrt{a_1}}{2\sqrt{\alpha \pi}} \int_0^\infty \frac{1}{u\sqrt{s}} \exp\left(-\frac{z^2}{4\alpha u} - u\omega_o - \frac{s}{\alpha}\right) I_1(2\sqrt{a_1 u s}) \cos(t-s) ds, \]

(4.24)

obtained by Nazar et al. [25], (Eqs. (13) and (14)) by using a different technique.

Generally, the starting solutions for unsteady motions of fluids are important for those who want to eliminate the transients from their experiments. In order to do that they need the required time to get the steady-state. After that time, which can be graphically determined using the obtained solutions, the fluid flows according to the steady-state solutions.
5 Numerical results and conclusions

In this note, exact solutions for MHD rotating flow of a second grade fluid through a porous medium have been established using the Laplace transform. A uniform magnetic field has been applied parallel to the rotation axis and expressions for the dimensionless velocity have been determined for oscillating flows. The solutions that have been obtained are presented as a sum between steady-state and transient solutions. They describe the motion of the fluid some time after its initiation. After that time, when the transients disappear these solutions tend to the steady-state solutions that are periodic in time and independent of the initial conditions. However, they satisfy the governing equation and boundary conditions. Furthermore, as it was to be expected, the steady-state solutions (4.18) and (4.21) differ by a phase shift. This is not true for the transient solutions. This is the reason that we separately presented the starting solutions for both sine and cosine oscillations of the plate.

Now, in order to reveal some relevant physical aspects of obtained results the diagrams of the dimensionless velocity $F_s(z, t)$ are depicted against $z$ for different values of the involved parameters ($\alpha, M, K, \Omega$). Figs. (1 - 4) corresponding to the sine oscillations, show the effects of involved parameters on the fluid motion. Furthermore, in each figure, (a) and (b) show the real and imaginary parts of the velocity respectively. The required time to reach the steady-state corresponding to the sine and cosine oscillations of the plate is determined by Figs (5 - 8).

Fig. 1 shows the profiles of velocity $F_s(z, t)$ for different values of the material parameter $\alpha$. It is clearly seen from figures that both real and imaginary parts of velocity are increasing functions of $\alpha$ on the whole flow domain. The influence of the magnetic parameter $M$ and the permeability $K$ on the fluid motion is shown by Figs. 2 and 3. The effects of $M$ on the two components of velocity are opposite. More exactly, the real part of velocity decreases and its imaginary part increases if $M$ increases. Figs. 3 show that both components of velocity are increasing functions of $K$. This result is in accordance with the fact that the drag force reduces if the permeability of the porous medium increases. The influence of the rotating parameter $\Omega$ on the fluid motion is shown in Fig. 4. Both real and imaginary parts of velocity are increasing functions with respect to $\Omega$. In conclusion, the main findings are:

- The real part of velocity and the boundary layer thickness increases if the material parameter $\alpha$ increases.
- The effect of MHD parameter $M$ on the real part of velocity is opposite to that of the permeability parameter $K$. Their effects on the magnitude of the imaginary part of velocity are similar.
- The real part of the complex velocity, as well as the magnitude of its imaginary component, is an increasing function with regard to rotating parameter $\Omega$ of the oscillations.
- The time after which the fluid moves according to the steady-state solutions, as it results from Figs. (5 - 8), is smaller for cosine oscillations in comparison to sine oscillations of the boundary. It is an increasing function with respect to $\alpha, M, K, \Omega$. 
Figure 1: Profiles of the velocity $F_s(z, t)$ given by Eq. (4.9) for $M = 0.5$, $K = 2$, $\Omega = 2$, $t = 1$ and different values of $\alpha$.

Figure 2: Profiles of the velocity $F_s(z, t)$ given by Eq. (4.9) for $K = 2$, $\Omega = 2$, $t = 1$, $\alpha = 1$ and different values of $M$. 
Figure 3: Profiles of the velocity $F_s(z, t)$ given by Eq. (4.9) for $M = 0.5$, $\Omega = 2$, $\alpha = 1$, $t = 0.8$ and different values of $K$.

Figure 4: Profiles of the velocity $F_s(z, t)$ given by Eq. (4.9) for $M = 0.5$, $K = 2$, $\alpha = 1$, $t = 2$ and different values of $\Omega$. 
Figure 5: The required time to reach the steady-state for sine and cosine oscillations of the plate for $M = 0.5$, $K = 2$, $\Omega = 2$, and different values of $\alpha$.

Figure 6: The required time to reach the steady-state corresponding to sine and cosine oscillations of the plate for $K = 1$, $\Omega = 1$, $\alpha = 0.5$ and different values of $M$. 
Figure 7: The required time to reach the steady-state corresponding to sine and cosine oscillations of the plate for \( M = 1, \Omega = 2, \alpha = 0.5 \), and different values of \( K \).

Figure 8: The required time to reach steady-state corresponding to sine and cosine oscillations of the plate for \( M = 1.5, K = 2, \alpha = 0.5 \), and different values of \( \Omega \).
6 Appendix

\[ f(t) = \mathcal{L}^{-1}\left\{\exp\left(-\frac{\xi^2}{4\tau}\right)\right\}; \quad \xi > 0, \quad (A_1) \]

If \( f(t) = \mathcal{L}^{-1}\{F(q)\} \)

then \( \mathcal{L}^{-1}\{F[W(q)]\} = \int_0^\infty f(u)g(u,t)du, \) where \( g(u,t) = \mathcal{L}^{-1}\{\exp(-uW(q))\}, \quad (A_2) \)

\[ \int_0^\infty \exp(-a^2x^2 - \frac{b^2}{4x^2})dx = \frac{\sqrt{\pi}a}{2a}e^{-ab}. \quad (A3) \]

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