On the solutions of the (1+1) and (2+1)-dimensional higher-order Broer-Kaup systems

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Abstract
In this work we study the (1+1) and (2+1)-dimensional higher-order Broer-Kaup (HBK) systems. The HBK systems were obtained from the inner parameter dependent symmetry constraints of the Kadomtsev-Petviashvili (KP) equation. The Bäcklund transformations are used for a reliable treatment of these systems. We obtain multiple soliton solutions for the underlying systems. Moreover, we use a variety of hyperbolic functions methods and trigonometric methods to determine a variety of travelling wave solutions and periodic solutions.

Keywords: Higher-order Broer-Kaup systems; Bäcklund transformations; multiple soliton solutions.

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1 Introduction

In this work, we will examine two systems of higher-order Broer-Kaup (HBK) in (1+1) dimensions and (2+1) dimensions. The (1+1)-dimensional higher-order Broer-Kaup (HBK) system reads

\[ u_t + 4(u_{xx} + u^3 - 3uu_x + 6uv)_x = 0, \]
\[ v_t + 4(v_{xx} + 3u^2v + 3uv_x + 3v^2)_x = 0. \]  

The (2+1)-dimensional Broer-Kaup (HBK) system is given by

\[ u_{xy} + 4(u_{xx} + u^3 - 3uu_x + 3uv)_x + 12(uv)_{xx} = 0, \]
\[ v_x + 4(v_{xx} + 3vv_x + 3uv_x + 3vw)_x = 0, \]
\[ v_x - w_y = 0. \]  

The aforementioned systems (1.1) and (1.2) were obtained from the inner parameter dependent symmetry constraints of the Kadomtsev-Petviashvili (KP) equation [6, 7, 8, 9]. In [9], the Lax pairs and the symmetry constraints were employed to derive these higher-order equations. In [6], the N-fold Darboux transformation were obtained for the (1+1)-dimensional HBK system (1.1). Khalique [7] used the Lie group analysis and the simplest equation method to obtain solitary wave solutions and conservation laws for Eq. (1.1). Lin et. al. [8] used the Painlevé analysis to derive fractal dromion and multi-peakon structures for (1.2). Other approaches were also used to investigate these two systems.

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equations.
The objectives of this work are twofold. First, we seek to investigate the two higher-order systems (1.1) – (1.2) aiming
to determine multiple kink solutions [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] for each system. The second goal is to use a
variety of hyperbolic functions methods and trigonometric methods to determine a variety of travelling wave solutions
and periodic solutions as well.

2 The (1+1)-dimensional higher-order HBK system

We begin our analysis by working on the (1+1)-dimensional higher-order HBK system

\[ u_t + 4(u_{xx} + u^3 - 3uu_x + 6uv)_x = 0, \]
\[ v_t + 4(v_{xx} + 3u^2v + 3uv_x + 3v^2)_x = 0. \]

(2.3)

Substituting

\[ u(x, t) = e^{k_1x - c_1t}, \]

(2.4)

into the linear terms of (2.3) gives the dispersion relations by

\[ c_1 = 4k_1^3, \]

(2.5)

and as a result we obtain the wave variables

\[ \theta_i = k_ix - 4k_i^3 t. \]

(2.6)

We next use the Bäcklund transformation

\[ u(x, t) = (\ln f)_x = \frac{f_x}{f}, \]
\[ v(x, t) = (\ln f)_{xx} = \frac{(f_{xx} - f_x^2)}{f^2}. \]

(2.7)

The last transformation indicates that

\[ v(x, t) = u_x(x, t). \]

(2.8)

Substituting (2.8) into (2.3) converts this system to one equation given by

\[ u_t + 4(u_{xx} + u^3 + 3uu_x)_x = 0. \]

(2.9)

The single soliton solution of (2.9) is assumed to be

\[ u(x, t) = (\ln f)_x = \frac{f_x}{f}, \]

(2.10)

where the auxiliary function \( f(x, t) \), for the single soliton solution, is given by

\[ f(x, t) = 1 + e^{\theta_1} = 1 + e^{k_1x - 4k_1^3 t}. \]

(2.11)

Substituting (2.10) and (2.11) into (2.9) gives the single kink and the single soliton solutions

\[ u(x, t) = \frac{k_1 e^{k_1x - 4k_1^3 t}}{1 + e^{k_1x - 4k_1^3 t}}, \]
\[ v(x, t) = \frac{k_1^2 e^{k_1x - 4k_1^3 t}}{(1 + e^{k_1x - 4k_1^3 t})^2}. \]

(2.12)

respectively.

For the two soliton solutions we set the auxiliary function

\[ f(x, t) = 1 + e^{k_1x - 4k_1^3 t} + e^{k_2x - 4k_2^3 t}. \]

(2.13)
Using (2.13) in (2.10) and substituting the result in (2.9), we obtain the two kink and the two soliton solutions by

\begin{align}
  u(x,t) &= \frac{k_1 e^{k_1 x - 4k_2 t} + k_2 e^{k_2 x + 4k_3 t}}{1 + e^{k_1 x - 4k_2 t} + e^{k_2 x + 4k_3 t}}, \\
v(x,t) &= \frac{k_1 e^{k_1 x - 4k_2 t} + k_2 e^{k_2 x + 4k_3 t}}{1 + e^{k_1 x - 4k_2 t} + e^{k_2 x + 4k_3 t}} - \left( \frac{k_1 e^{k_1 x - 4k_2 t} + k_2 e^{k_2 x + 4k_3 t}}{1 + e^{k_1 x - 4k_2 t} + e^{k_2 x + 4k_3 t}} \right)^2, \quad (2.14)
\end{align}

respectively.

For the three soliton solutions, we set

\begin{align}
f(x,t) &= 1 + e^{k_1 x - 4k_2 t} + e^{k_2 x - 4k_3 t} + e^{k_3 x - 4k_4 t}, \quad (2.15)
\end{align}

Proceeding as before, we find the three kink and the three soliton solutions are given by

\begin{align}
  u(x,t) &= \frac{\sum_{i=1}^N k_i e^{k_i x - 4k_j t}}{1 + \sum_{i=1}^N k_i e^{k_i x - 4k_j t}}, \\
v(x,t) &= \frac{\sum_{i=1}^N k_i^2 e^{k_i x - 4k_j t}}{1 + \sum_{i=1}^N k_i e^{k_i x - 4k_j t}} - \left( \frac{\sum_{i=1}^N k_i e^{k_i x - 4k_j t}}{1 + \sum_{i=1}^N k_i e^{k_i x - 4k_j t}} \right)^2. \quad (2.17)
\end{align}

### 3 Other solutions: the hyperbolic functions methods

In this section we will apply other approaches in order to determine more travelling wave solutions. The schemes that will be used depend mainly on the hyperbolic functions.

#### 3.1 The tanh method

The tanh method admits the use of the solution

\begin{align}
u(x,t) = \alpha + \beta \tanh(kx - \omega t), \quad (3.18)
\end{align}

as a solution of (2.9). To determine \(\alpha, \beta\) and the wave speed \(\omega\), we substitute (3.18) into (2.9), collect the coefficients of \(\tanh^i, i = 0, 1, 2\), and equate it to zero we obtain the two set of solutions

\begin{align}
\alpha &= 0, \\
\beta &= 2k, \\
\omega &= 16k^3, \quad (3.19)
\end{align}

and

\begin{align}
\beta &= k, \\
\omega &= 4k^3 + 12k\alpha^2, \quad (3.20)
\end{align}

where \(\alpha\) is left as a free parameter in the last set.

This gives the solitary waves solutions of Eq. (2.9) by

\begin{align}
u(x,t) = 2k \tanh \left(kx - 16k^3 t \right), \quad (3.21)
\end{align}
and
\[ u(x,t) = \alpha + k \tanh \left( kx - (4k^3 + 12ka^2)i \right). \] (3.22)

Replacing \( \tanh \) by \( \coth \) in (3.18), and proceeding as before we obtain the singular solutions.

This gives the solitary waves solutions of Eq. (2.9) by
\[ u(x,t) = 2k \coth \left( kx - 
\frac{16k^3}{k^3 \lambda} t \right), \] (3.23)

\[ u(x,t) = \alpha + k \coth \left( kx - (4k^3 + 12ka^2)i \right). \] (3.24)

Recall that \( v(x,t) = u_x(x,t) \).

3.2 The rational tanh method

The rational tanh method admits the use of the solution in the form
\[ u(x,t) = \frac{\tanh(kx - \lambda t)}{\lambda + \mu \tanh(kx - \lambda t)}, \] (3.25)
as a solution of the equation (2.9). To determine \( \lambda, \mu \) and the wave speed \( \nu \), we substitute (3.25) into (2.9), collect the coefficients of \( \tanh^i, i = 0, 1, 2 \), equate these coefficients to zero and solve the resulting equations we obtain
\[ \mu = \pm \frac{\sqrt{k(k\lambda - 1)}}{k}, k\lambda(k\lambda - 1) > 0, \]
\[ \nu = \frac{4k^3(4k\lambda - 3)}{\lambda}, \] (3.26)

where \( \lambda \) is left as a free parameter. This gives the travelling wave solution of equation (2.9) by
\[ u(x,t) = \frac{\tanh(kx - \frac{4k^3(4k\lambda - 3)}{\lambda} t)}{\lambda \pm \frac{\sqrt{k(k\lambda - 1)}}{k} \tanh(kx - \frac{4k^3(4k\lambda - 3)}{\lambda} t)}. \] (3.27)

We can also show that
\[ u(x,t) = \frac{\coth(kx - \frac{4k^3(4k\lambda - 3)}{\lambda} t)}{\lambda \pm \frac{\sqrt{k(k\lambda - 1)}}{k} \coth(kx - \frac{4k^3(4k\lambda - 3)}{\lambda} t)}, \] (3.28)

are another travelling wave solutions of equation (2.9). Recall that \( v(x,t) = u_x(x,t) \).

3.3 The rational sinh-cosh method

The rational sinh-cosh method admits the use of the solution in the form
\[ u(x,t) = \frac{\alpha}{1 + \beta \sinh(kx - \omega t) + \gamma \cosh(kx - \omega t)}, \] (3.29)
as a solution of equation (2.9). To determine \( \alpha, \beta \) and the wave speed \( \omega \), we substitute (3.29) into (2.9), collect the coefficients of \( \sinh^i, \cosh^i, i = 0, 1 \), equate these coefficients to zero and solve the resulting equations we obtain
\[ \omega = \frac{4k^3}{\lambda}, \]
\[ \beta = k, \]
\[ \gamma = \beta^2, \] (3.30)

where \( \beta \) is left as a free parameter. This gives the travelling wave solution by
\[ u(x,t) = \frac{k}{1 + \beta \sinh(kx - 4k^3 t) + \beta^2 \cosh(kx - 4k^3 t)}. \] (3.31)
4 Other solutions: the trigonometric functions methods

In this section we will apply other approaches in order to determine more travelling wave solutions. The schemes that will be used depend on the trigonometric functions.

4.1 The tan method

The tan method admits the use of the solution
\[
u(x,t) = \alpha + \beta \tanh(kx - \omega t),
\]
as a solution of (2.9). To determine \(\alpha, \beta\) and the wave speed \(\omega\), we proceed as before to obtain the two set of solutions
\[
\begin{align*}
\alpha &= 0, \\
\beta &= -2k, \\
\omega &= -16k^3,
\end{align*}
\]
and
\[
\begin{align*}
\beta &= -k, \\
\omega &= -4k^3 + 12k\alpha^2,
\end{align*}
\]
where \(\alpha\) is left as a free parameter in the last set. This gives the solitary waves solutions of Eq. (2.9) by
\[
\begin{align*}
u(x,t) &= 2k \tan(kx + 16k^3 t), \\
u(x,t) &= a + k \cot(kx - (4k^3 + 12k\alpha^2) t).
\end{align*}
\]
Replacing tan by cot, and proceeding as before we obtain the singular solutions
\[
\begin{align*}
u(x,t) &= 2k \cot(kx + 16k^3 t), \\
u(x,t) &= a + k \cot(kx - (4k^3 + 12k\alpha^2) t).
\end{align*}
\]
Recall that \(v(x,t) = u_x(x,t)\).

4.2 The rational tan method

The rational tanh method admits the use of the solution in the form
\[
u(x,t) = \tan(kx - \nu t) \frac{1}{\lambda + \mu \tan(kx - \nu t)},
\]
as a solution of the equation (2.9). Substituting (3.25) into (2.9), and proceeding as before we obtain two sets of solutions given by
\[
\begin{align*}
\lambda &= -\frac{1}{2\xi}, \\
\mu &= \pm \frac{i}{2\xi}, \\
\nu &= 8k^3,
\end{align*}
\]
and
\[
\begin{align*}
\mu &= \pm \frac{\sqrt{-k\lambda(k\lambda + 1)}}{k}, \quad k\lambda(k\lambda + 1) < 0, \\
\nu &= -\frac{4k^2(4k\lambda + 3)}{\lambda},
\end{align*}
\]
where \(\lambda\) is left as a free parameter for the last set. This gives the travelling wave solutions of the equation (2.9) by
\[
\begin{align*}
u(x,t) &= \frac{2k\tan(kx - 8\nu^3 t)}{1 \pm \tan(kx - 8\nu^3 t)}.
\end{align*}
\]
and

\[ u(x,t) = \frac{k \tan(kx + \frac{4k^2(4k\lambda + 3)}{\lambda} t)}{k\lambda \pm \sqrt{-k\lambda(k\lambda + 1)} \tan(kx + \frac{4k^2(4k\lambda + 3)}{\lambda} t)} \]  (4.43)

We can also show that

\[ u(x,t) = \frac{2k \tan(kx - 8k^3 t)}{1 \pm \tan(kx - 8k^3 t)} \]  (4.44)

and

\[ u(x,t) = \frac{k \cot(kx + \frac{4k^2(4k\lambda + 3)}{\lambda} t)}{k\lambda \pm \sqrt{-k\lambda(k\lambda + 1)} \cot(kx + \frac{4k^2(4k\lambda + 3)}{\lambda} t)} \]  (4.45)

are another travelling wave solutions of the equation (2.9). Recall that \( v(x,t) = u_x(x,t) \).

4.3 The rational cos-sin method

The rational cos-sin method admits the use of the solution in the form

\[ u(x,t) = \frac{\cos(kx + ry - \omega t)}{\beta \cos(kx + ry - \omega t) + \gamma \sin(kx + ry - \omega t)}, \]  (4.46)

as a solution of the equation (2.9). Proceeding as before we obtain

\[ \omega = -\frac{4k^2(4k\beta - 3)}{\beta}, \]

\[ \gamma = \pm \sqrt{-k\beta(2k\beta - 1)}, k\beta(2k\beta - 1) < 0, \]  (4.47)

where \( \beta \) is left as free parameter. Substituting these results in (4.46) gives \( u(x,t) \), noting that \( v(x,t) = u_x(x,t) \).

5 The (2+1)-dimensional higher-order HBK system

We next continue our work on the (2+1)-dimensional higher-order HBK system

\[ u_{yr} + 4(u_{xx} + u^3 - 3uu_x + 3uv)_x + 12(uv)_{xx} = 0, \]

\[ v_{x} + 4(v_{xx} + 3vv_x + 3uv_x + 3vw)_x = 0, \]

\[ v_x - w_y = 0. \]  (5.48)

Substituting

\[ u(x,y,t) = e^{kx + ry - \omega t}, \]  (5.49)

into the linear terms of (5.48), the dispersion relation

\[ c_i = 4k^3, \]  (5.50)

and the wave variable

\[ \theta_i = k_i x + r_i y - 4k_i^3 t, \]  (5.51)

follow immediately.

We next use the Bäcklund transformations of the system (5.48)

\[ u(x,y,t) = (\ln f)_x = \frac{f_y}{f}, \]

\[ v(x,y,t) = (\ln f)_y = \frac{f_{xy} - f_x f_y}{f^2}, \]

\[ w(x,y,t) = (\ln f)_{xx}, \]  (5.52)

obtained by using the standard Painlevé truncation expansion. This in turn gives the relations

\[ v(x,y,t) = u_y(x,y,t), \]

\[ w(x,y,t) = u_x(x,y,t). \]  (5.53)
Using (5.53) converts (5.48) into a single equation given by
\[
\left\{ u_t + 4u_{xxx} + 12u_x^2 + 12u_{xx}^2 + 12uu_{xx} \right\}_t = 0. \tag{5.54}
\]
Using the first equation of (5.52), that corresponds to the Cole-Hopf transformation, where the auxiliary function \( f(x,y,t) \), for the single soliton, is given by
\[
f(x,y,t) = 1 + e^{k_1x + r_1y - 4k_1^2t}. \tag{5.55}
\]
We obtain the single kink and the single soliton solutions
\[
\begin{align*}
u(x,y,t) & = \frac{k_1 e^{k_1x + r_1y - 4k_1^2t}}{1 + e^{k_1x + r_1y - 4k_1^2t}}, \\
v(x,y,t) & = \frac{k_1 e^{k_1x + r_1y - 4k_1^2t}}{1 + e^{k_1x + r_1y - 4k_1^2t}}, \\
w(x,y,t) & = \frac{k_1 e^{k_1x + r_1y - 4k_1^2t}}{1 + e^{k_1x + r_1y - 4k_1^2t}},
\end{align*}
\tag{5.56}
\]
respectively.
For the two soliton solutions we use the auxiliary functions
\[
f(x,y,t) = 1 + e^{k_1x + r_1y - 4k_1^2t} + e^{k_2x + r_2y - 4k_2^2t}. \tag{5.57}
\]
Using (5.57) in (5.52) and substituting the result in (5.54), we obtain the two kink and the two soliton solutions by
\[
\begin{align*}
u(x,y,t) & = \frac{k_1 e^{k_1x + r_1y - 4k_1^2t} + k_2 e^{k_2x + r_2y - 4k_2^2t}}{1 + e^{k_1x + r_1y - 4k_1^2t} + e^{k_2x + r_2y - 4k_2^2t}}, \\
v(x,y,t) & = \frac{k_1 e^{k_1x + r_1y - 4k_1^2t} + k_2 e^{k_2x + r_2y - 4k_2^2t}}{1 + e^{k_1x + r_1y - 4k_1^2t} + e^{k_2x + r_2y - 4k_2^2t}} - \frac{(k_1 e^{k_1x + r_1y - 4k_1^2t} + k_2 e^{k_2x + r_2y - 4k_2^2t})^2}{1 + e^{k_1x + r_1y - 4k_1^2t} + e^{k_2x + r_2y - 4k_2^2t}}, \\
w(x,y,t) & = \frac{k_1 e^{k_1x + r_1y - 4k_1^2t} + k_2 e^{k_2x + r_2y - 4k_2^2t}}{1 + e^{k_1x + r_1y - 4k_1^2t} + e^{k_2x + r_2y - 4k_2^2t}} - \frac{(k_1 e^{k_1x + r_1y - 4k_1^2t} + k_2 e^{k_2x + r_2y - 4k_2^2t})^2}{1 + e^{k_1x + r_1y - 4k_1^2t} + e^{k_2x + r_2y - 4k_2^2t}},
\end{align*}
\tag{5.58}
\]
respectively.
For the three kink and soliton solutions, we proceed as before to obtain
\[
\begin{align*}
u & = \frac{k_1 e^{k_1x + r_1y - 4k_1^2t} + k_2 e^{k_2x + r_2y - 4k_2^2t} + k_3 e^{k_3x + r_3y - 4k_3^2t}}{1 + e^{k_1x + r_1y - 4k_1^2t} + e^{k_2x + r_2y - 4k_2^2t} + e^{k_3x + r_3y - 4k_3^2t}}, \\
v & = \frac{k_1 e^{k_1x + r_1y - 4k_1^2t} + k_2 e^{k_2x + r_2y - 4k_2^2t} + k_3 e^{k_3x + r_3y - 4k_3^2t}}{1 + e^{k_1x + r_1y - 4k_1^2t} + e^{k_2x + r_2y - 4k_2^2t} + e^{k_3x + r_3y - 4k_3^2t}} - \frac{(k_1 e^{k_1x + r_1y - 4k_1^2t} + k_2 e^{k_2x + r_2y - 4k_2^2t} + k_3 e^{k_3x + r_3y - 4k_3^2t})^2}{1 + e^{k_1x + r_1y - 4k_1^2t} + e^{k_2x + r_2y - 4k_2^2t} + e^{k_3x + r_3y - 4k_3^2t}}, \\
w & = \frac{k_1 e^{k_1x + r_1y - 4k_1^2t} + k_2 e^{k_2x + r_2y - 4k_2^2t} + k_3 e^{k_3x + r_3y - 4k_3^2t}}{1 + e^{k_1x + r_1y - 4k_1^2t} + e^{k_2x + r_2y - 4k_2^2t} + e^{k_3x + r_3y - 4k_3^2t}} - \frac{(k_1 e^{k_1x + r_1y - 4k_1^2t} + k_2 e^{k_2x + r_2y - 4k_2^2t} + k_3 e^{k_3x + r_3y - 4k_3^2t})^2}{1 + e^{k_1x + r_1y - 4k_1^2t} + e^{k_2x + r_2y - 4k_2^2t} + e^{k_3x + r_3y - 4k_3^2t}},
\end{align*}
\tag{5.59}
\]
This shows that the (2+1)-dimensional HBK system of equations (5.48) gives N-kink solutions for \( u(x,y,t), \) and N-soliton solutions for \( v(x,y,t) \) and \( w(x,y,t), \) for finite \( N, \) where \( N \geq 1. \) We point out that the obtained multiple soliton solutions are obtained where the coefficients \( k_i \) and \( r_i \) of the spatial variables \( x \) and \( y \) are free parameters without any.
restriction.

Based on (5.59), the general solutions can be set in the form

\[
\begin{align*}
  u(x,y,t) &= \frac{\sum_{i=1}^{N} k_i e^{k_i x + r_i y - 4k_i t}}{1 + \sum_{i=1}^{N} k_i e^{k_i x + r_i y - 4k_i t}}, \\
v(x,y,t) &= \frac{\sum_{i=1}^{N} k_i r_i e^{k_i x + r_i y - 4k_i t}}{1 + \sum_{i=1}^{N} k_i e^{k_i x + r_i y - 4k_i t}}, \\
w(x,y,t) &= \frac{\sum_{i=1}^{N} k_i^2 e^{k_i x + r_i y - 4k_i t}}{1 + \sum_{i=1}^{N} k_i e^{k_i x + r_i y - 4k_i t}}.
\end{align*}
\]

(5.60)

It is interesting to point out that other travelling wave solutions can be obtained by using similar approaches to the schemes that we used before. The tanh method, the rational tanh method and other trigonometric methods can be used in a similar manner to the approach we used earlier.

6 Discussion

We examined two systems of higher-order Broer–Kaup in (1+1) and (2+1) dimensions. We derived multiple kink solutions and multiple soliton solutions for both systems where the coefficients of the spatial variables \(x\) and \(y\) remain free parameters. The Bäcklund transformations are used to conduct this work. Moreover, we used a variety of hyperbolic functions methods and trigonometric methods to determine a variety of travelling wave solutions and periodic solutions.

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